

Solutions to assigned problems and sample quiz problems for Part 4

Sample Quiz Problems

Quiz Problem 1. Write down the BCS gap equation at finite temperature and compare it to the zero temperature expression. Explain the origin of the differences between the two expressions.

Quiz Problem 2. Make a plot of the BCS gap as a function of temperature. Write down the scaling behavior of the gap on approach to the critical temperature.

Quiz Problem 3. Starting from the quantum mechanical expression for the current, show that in neutral superfluids circulation is quantized. Give an expression of the quantum of circulation.

Quiz Problem 4. Starting from the quantum mechanical expression for the current in the presence of a vector potential, show that flux is quantized. Give an expression for the flux quantum.

Quiz Problem 5. Describe the physical meaning of the healing length (ξ) in superfluids and superconductors. By considering the linearized Ginzburg-Landau equation in zero field find a solution describing the attenuation of superconducting pair density near the surface of a superconductor.

Quiz Problem 6. Using either London's original argument or starting with the Ginzburg-Landau equation, derive the London differential equation describing the penetration of parallel magnetic field into a superconducting surface. Show that it has the solution $B(x) = B_0 e^{-x/\lambda}$, where λ is the London penetration depth.

Quiz Problem 7. What is the mixed phase of a type II superconductor? Give a physical reasoning to explain why the mixed phase of a type II superconductor can have, at sufficiently high external field, a lower free energy than the Meissner state.

Quiz Problem 8. Using London's theory, find an expression for the lower critical field of a type II superconductor.

Quiz Problem 9. Using G-L theory find an expression for the upper critical field of a type II superconductor.

Quiz Problem 10. Write down the Langevin equation for motion of a particle in the presence of random forces. Describe the physical meaning of the random forces and the meaning of Gaussian white noise.

Quiz problem 11. Write down the non-conserved order parameter relaxation equation for the Ising model within Landau theory and show that near the critical point the relaxation time diverges.

Quiz problem 12. Explain the difference between conserved dynamics and non-conserved dynamics. Give an example of the two cases.

Quiz problem 13. Sketch the phase diagrams of the van der Waals gas and the ferromagnetic Ising model, indicating the spinodal lines. How are the spinodal lines defined in the two cases.

Quiz problem 14. Explain the physical basis for the concept of a critical droplet in the homogeneous nucleation of a new stable phase from a metastable phase. Find an expression for the critical droplet size for an Ising model in a magnetic field at low temperature.

Quiz problem 15. Derive the Vlasov equation for the time evolution of the single particle distribution function $f(t, \vec{r}, \vec{v})$ for a collisionless system.

Assigned problems

Assigned Problem 1. By doing a variation with respect to ψ^* of the Ginzburg-Landau free energy (Eq. (64)) of the notes, derive the Ginzburg-Landau equation (65).

Solution

Within Ginzburg-Landau theory, the Gibbs' free energy for a superconductor is given by,

$$g_{GL} = \int d^3r \left(\frac{1}{2m} |(i\hbar\nabla + qA)\psi(\vec{r})|^2 + a(T)|\psi(\vec{r})|^2 + \frac{b(T)}{2} |\psi(\vec{r})|^4 + \frac{B^2}{2\mu_0} + \frac{\mu_0 H^2}{2} - B \cdot H \right). \quad (1)$$

Expanding this expression (and dropping function arguments), we have,

$$g_{GL} = \int d^3r \left(\frac{1}{2m} [(-i\hbar\nabla + qA)\psi^* \cdot (i\hbar\nabla + qA)\psi] + a\psi^*\psi + \frac{b}{2}(\psi^*)^2\psi^2 + \frac{(\nabla \wedge A)^2}{2\mu_0} + \frac{\mu_0 H^2}{2} - (\nabla \wedge A) \cdot H \right). \quad (2)$$

Expanding the first term yields,

$$g_{GL} = \int d^3r \left(\frac{1}{2m} \sum_{\alpha=1}^3 [\hbar^2 \frac{\partial\psi^*}{\partial x_\alpha} \frac{\partial\psi}{\partial x_\alpha} + i\hbar q A_\alpha (\psi^* \frac{\partial\psi}{\partial x_\alpha} - \psi \frac{\partial\psi^*}{\partial x_\alpha}) + q^2 A_\alpha^2 \psi^* \psi] \right. \\ \left. + a\psi^*\psi + \frac{b}{2}(\psi^*)^2\psi^2 + \frac{(\nabla \wedge A)^2}{2\mu_0} + \frac{\mu_0 H^2}{2} - (\nabla \wedge A) \cdot H \right) \quad (3)$$

Using $\vec{r} = (x_1, x_2, x_3)$, the Euler-Lagrange equation for a variation with respect to ψ^* is,

$$\frac{\delta g_{GL}}{\delta \psi^*} = \frac{\partial g_{GL}}{\partial \psi^*} - \sum_{\alpha} \frac{\partial}{\partial x_\alpha} \left(\frac{\partial g_{GL}}{\partial (\frac{\partial \psi^*}{\partial x_\alpha})} \right) \quad (4)$$

Carring out this variation yields,

$$\frac{\delta g_{GL}}{\delta \psi^*} = \frac{1}{2m} \sum_{\alpha} (i\hbar q A_\alpha \frac{\partial\psi}{\partial x_\alpha} + q^2 A_\alpha^2 \psi) + a\psi + b|\psi|^2\psi - \frac{1}{2m} \sum_{\alpha} \frac{\partial}{\partial x_\alpha} (\hbar^2 \frac{\partial\psi}{\partial x_\alpha} - i\hbar q A_\alpha \psi). \quad (5)$$

This is equivalent to,

$$\frac{\delta g_{GL}}{\delta \psi^*} = \frac{1}{2m} (i\hbar\nabla + qA)^2 \psi + a\psi + b|\psi|^2\psi \quad (6)$$

which is the GL equation in an applied field, as required. Another useful form is found by using the London gauge where $\nabla \cdot A = 0$ to find,

$$\frac{\delta g_{GL}}{\delta \psi^*} = \frac{1}{2m} \sum_{\alpha} (-\hbar^2 \frac{\partial^2 \psi}{\partial x_\alpha^2} + 2i\hbar q A_\alpha \frac{\partial\psi}{\partial x_\alpha} + q^2 A_\alpha^2 \psi) + a\psi + b|\psi|^2\psi = 0 \quad (7)$$

Assigned Problem 2. By doing a variation with respect to the vector potential A of the Ginzburg-Landau free energy (Eq. (64)) of the notes, derive the expression for the current (66).

Solution

The Euler-Lagrange equation for a variation with respect to one component A_i of the vector potential is,

$$\frac{\delta g_{GL}}{\delta A_i} = \frac{\partial g_{GL}}{\partial A_i} - \sum_{\alpha} \frac{\partial}{\partial x_\alpha} \left(\frac{\partial g_{GL}}{\partial (\frac{\partial A_i}{\partial x_\alpha})} \right) \quad (8)$$

We then have,

$$\frac{\delta g_{GL}}{\delta A_i} = i \frac{\hbar q}{2m} (\psi^* \frac{\partial\psi}{\partial x_i} - \psi \frac{\partial\psi^*}{\partial x_i}) + \frac{q^2}{m} A_i \psi^* \psi - \sum_{\alpha} \frac{1}{2\mu_0} \frac{\partial}{\partial x_\alpha} \left(\frac{\partial (\nabla \wedge A)^2}{\partial (\frac{\partial A_i}{\partial x_\alpha})} \right) \quad (9)$$

The term $B \cdot H$ does not contribute as it is linear in the derivatives of A and hence is zero after the application of the second term in the Euler-Lagrange equation. The expansion of $(\nabla \wedge A)^2$ is,

$$(\nabla \wedge A) \cdot (\nabla \wedge A) = \left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} \right)^2 + \left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1} \right)^2 + \left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} \right)^2 \quad (10)$$

Using this expression to evaluate the last term in the case $A_i = A_1$, yields

$$\sum_{\alpha} \frac{\partial}{\partial x_{\alpha}} \left(\frac{\partial(\nabla \wedge A)^2}{\partial(\frac{\partial A_1}{\partial x_{\alpha}})} \right) = 2 \left(-\frac{\partial^2 A_2}{\partial x_1 \partial x_2} + \frac{\partial^2 A_1}{\partial x_2^2} + \frac{\partial^2 A_1}{\partial x_3^2} - \frac{\partial^2 A_3}{\partial x_1 \partial x_3} \right) \quad (11)$$

Similar expressions hold for A_2 and A_3 . Now note that $\nabla \wedge (\nabla \wedge A) = \nabla(\nabla \cdot A) - \nabla^2 A$. The first element of this expression is,

$$(\nabla(\nabla \cdot A) - \nabla^2 A)_1 = \frac{\partial^2 A_1}{\partial x_1^2} + \frac{\partial^2 A_2}{\partial x_1 x_2} + \frac{\partial^2 A_3}{\partial x_1 x_3} - \frac{\partial^2 A_1}{\partial x_1^2} - \frac{\partial^2 A_1}{\partial x_2^2} - \frac{\partial^2 A_1}{\partial x_3^2}$$

We thus find the identity,

$$\frac{\delta(\nabla \wedge A)^2}{\delta A} = 2\nabla \wedge (\nabla \wedge A) = 2[\nabla(\nabla \cdot A) - \nabla^2 A] \quad (12)$$

We also have,

$$\nabla \wedge (\nabla \wedge A) = \nabla \wedge B = \mu_0 J \quad (13)$$

Substituting these results into the equation above yields the simple result,

$$-\sum_{\alpha} \frac{1}{2\mu_0} \frac{\partial}{\partial x_{\alpha}} \left(\frac{\partial(\nabla \wedge A)^2}{\partial(\frac{\partial A_i}{\partial x_{\alpha}})} \right) = j_s$$

Using this in equation (9) for we find,

$$j_s = -i \frac{\hbar q}{2m} (\psi^* \nabla \psi - \psi \nabla \psi^*) - \frac{q^2}{m} A_i \psi^* \psi \quad (14)$$

as required.

Assigned Problem 3. Show that minimizing the Helmholtz free energy within London theory,

$$f_1 = \frac{1}{2\mu_0} \int_0^{\infty} (B^2 + \lambda^2 \mu_0^2 J^2) 2\pi r dr \quad (15)$$

gives the London equation.

Solution

We apply the Euler-Lagrange equations to,

$$f_{London} = \frac{1}{2\mu_0} \int dV [\lambda^2 (\nabla \wedge B)^2 + B^2]. \quad (16)$$

Using the Euler-Lagrange equation with the identities

$$\frac{\delta(\nabla \wedge B)^2}{\delta B} = 2\nabla \wedge (\nabla \wedge B) = 2[\nabla(\nabla \cdot B) - \nabla^2 B] \quad (17)$$

and using $\nabla \cdot B = 0$, yields the London equation $\lambda^2 \nabla^2 B = B$.

Assigned Problem 4. Using London theory, find the Helmholtz free energy per unit length of a vortex in a superconductor in an external field H .

Solution. The *energy*(per unit length) of an isolated vortex is given by,

$$\epsilon_1 = \int_0^{\infty} \left(\frac{B^2}{2\mu_0} + \frac{1}{2} \rho_s v_s^2 \right) 2\pi r dr \quad (18)$$

The first term is the field energy and the second is the kinetic energy of the superconducting electrons (see Eq. (4)). The supercurrent is related to the velocity by,

$$J = n_b q v_s \quad (19)$$

and using Maxwells equation, we then write,

$$\epsilon_1 = \frac{1}{2\mu_0} \int_0^\infty (B^2 + \lambda^2 \mu_0^2 J^2) 2\pi r dr \quad (20)$$

which explicitly shows the contributions of the field and the current. In the large λ/ξ limit, this is dominated by the regime $\xi \leq r \leq \lambda$, so we find an approximate value of the vortex energy by using,

$$\begin{aligned} \epsilon_1 &\approx \frac{1}{2\mu_0} \left(\frac{\phi_0}{2\pi\lambda^2}\right)^2 2\pi \int_\xi^\lambda \left[r \left(\text{Ln}\left(\frac{\lambda}{r}\right)\right)^2 + r \left(\frac{\lambda}{r}\right)^2\right] dr \\ &= \frac{\phi_0^2}{4\pi\mu_0\lambda^2} \int_{\xi/\lambda}^1 \left[x \left(\text{Ln}(x)\right)^2 + \frac{1}{x}\right] dx \\ &= \frac{\phi_0^2}{4\pi\mu_0\lambda^2} \left[\frac{x^2}{2} \left(\frac{1}{2} - \text{Ln}(x) + \text{Ln}(x)^2\right) + \text{Ln}(x)\right] \Big|_{\frac{\xi}{\lambda}}^1 \\ &\approx \frac{\phi_0^2}{4\pi\mu_0\lambda^2} \left[\text{Ln}\left(\frac{\lambda}{\xi}\right)\right] \end{aligned} \quad (21)$$

Notice that the energy cost of forming the vortex is dominated by the kinetic energy of the superconducting electrons (the logarithmic term). The energy cost due to the magnetic field is relatively small.

Assigned Problem 5. Using the Gibb's free energy within London theory, demonstrate that the triangular lattice vortex array has lower energy than the square lattice array, for a fixed applied field, $H > H_{c1}$, which is close to H_{c1} .
Solution

The energy to add the N^{th} vortex to a vortex lattice is given by the energy of the added vortex plus the vortex-vortex interactions. Within London theory, the Gibbs free energy (or chemical potential) to add the N^{th} vortex to a vortex array is given by,

$$\delta g = (\epsilon_1 - \phi_0 H) + \sum_j \frac{B_0 \phi_0}{\mu_0} K_0(|\vec{r} - \vec{r}_j|/\lambda) \quad (22)$$

Using $H_{c1} = \epsilon_1/\phi_0$, the relation between the applied field and the lattice spacing is found by setting $\delta g = 0$, which yields,

$$H - H_{c1} = \frac{B_0}{\mu_0 N} \sum_{ij} K_0(|\vec{r}_i - \vec{r}_j|/\lambda) \quad (23)$$

This fixes the lattice spacing, a , of a flux array. This spacing is different for different lattice structures. Associated with the lattice spacing there is a total number of vortices inside the superconductor. This total number of vortices, at fixed a , is larger for the triangular lattice, N_{tr} than it is for the square lattice, $N_{tr} > N_{sq}$. In fact the highest packing possible is for the triangular lattice case.

In setting up a vortex lattice, the gain in Gibb's free energy is given by,

$$\delta G = N(\epsilon_1 - \phi_0 H) + \sum_{ij} \frac{B_0 \phi_0}{2\mu_0} K_0(|\vec{r}_i - \vec{r}_j|/\lambda) \quad (24)$$

Notice that there is a factor of two in the second term (compared to Eq. (33)), which is the energy cost of vortex-vortex repulsion, due to the fact that the energy cost in setting up the flux lattice is small for the first vortices added to the lattice. Using Eq. (34), this can be rewritten as,

$$\phi_0 \frac{\delta G}{N} = (H_{c1} - H) + \frac{1}{2}(H - H_{c1}); \quad \text{so that} \quad \frac{\delta G}{N} = \frac{1}{2\phi_0}(H_{c1} - H) \quad (25)$$

This is the energy gain per vortex in setting up the lowest energy vortex lattice. Notice that if we consider only nearest neighbor interactions, the energy per vortex is *the same* no matter what the vortex lattice structure. However the *number* of vortices which can be added to the superconductor does depend on the structure of the final vortex lattice. Since the number of vortices in the case of a square lattice is smaller than the number added for the triangular lattice, the total energy is lowest for the case of a triangular lattice. Notice that this result is general, the triangular lattice is the true ground state because it has the highest possible vortex density.

Assigned Problem 6. Go through the calculations to show that the kinetic energy and angular momentum per unit length of a quantum of circulation in a neutral superfluid are given by equations (40) and (41) of the notes.

Solution We have,

$$\epsilon_{kin} = \int_{\xi}^b \frac{1}{2} dm(r) v_s(r)^2 = \int_{\xi}^b \frac{1}{2} \rho_s (2\pi r dr) \frac{(\kappa_0)^2}{(2\pi r)^2} = \frac{\kappa_0^2 \rho_s}{4\pi} \ln(b/\xi) \quad (26)$$

and

$$l_{kin} = \int_{\xi}^b r^2 dm(r) \frac{v_s(r)}{r} = \int_{\xi}^b \rho_s (2\pi r^3 dr) \frac{\kappa_0}{2\pi r} = \frac{\kappa_0 \rho_s}{2} (b^2 - \xi^2) \quad (27)$$

Assigned Problem 7. Considering only the kinetic energy of the supercurrent and using arguments like those of Problem 6, show that the supercurrents circulating around a flux quantum in a superconductor have energy approximately given by Eq. (55) of the notes.

Solution Using the same procedure as for the neutral superfluid, with $v_s = m j_s / (q \rho_s)$ with current given by,

$$j_s(r) = j_0 \frac{\lambda}{r}; \quad j_0 = \frac{\phi_0}{2\pi \mu_0 \lambda^3} \quad (28)$$

we find,

$$\epsilon_{kin} = \int_{\xi}^{\lambda} \frac{1}{2} dm(r) v_s(r)^2 = \int_{\xi}^{\lambda} \frac{1}{2} \rho_s (2\pi r dr) \left(\frac{m \phi_0}{2\pi q \rho_s \mu_0 \lambda^3} \right)^2 \frac{\lambda^2}{r^2} = \frac{\phi_0^2}{4\pi \mu_0 \lambda^2} \ln(\lambda/\xi) \quad (29)$$

where we used,

$$\lambda = \left(\frac{m^2}{\rho_s q^2 \mu_0} \right)^{1/2} \quad (30)$$

Assigned Problem 8. Derive the drift diffusion equation

$$\vec{J}_n = en\mu_n \vec{E} + eD_n \frac{\partial n}{\partial x}; \quad \vec{J}_p = ep\mu_p \vec{E} - eD_p \frac{\partial p}{\partial x} \quad (31)$$

from the Boltzmann equation using the relaxation time approximation.

Solution

Same as the solution for \vec{J}_n with a change in sign in the force and in the definition of the current.

Assigned Problem 9. The collision integral (Eq. (113) of the notes) is zero when,

$$f(\vec{p}_1)f(\vec{p}_2) = f(\vec{p}_3)f(\vec{p}_4). \quad (32)$$

Show that the Maxwell Boltzmann distribution

$$f(t, \vec{r}, \vec{v}) = Ce^{-\alpha(\vec{v}-\vec{V})^2} \quad (33)$$

satisfies this equation, for the case of elastic scattering.

Solution

Substituting into the equation, cancelling C factors and taking a logarithm of both sides, then cancelling α factors gives,

$$(\vec{v}_1 - \vec{V})^2 + (\vec{v}_2 - \vec{V})^2 = (\vec{v}_3 - \vec{V})^2 + (\vec{v}_4 - \vec{V})^2 \quad (34)$$

Expanding and rearranging gives,

$$|\vec{v}_1|^2 + |\vec{v}_2|^2 - (|\vec{v}_3|^2 + |\vec{v}_4|^2) = 2[(\vec{v}_1 + \vec{v}_2) - (\vec{v}_3 + \vec{v}_4)] \cdot \vec{V} \quad (35)$$

The left hand side is zero for elastic scattering, while the right hand side is zero by momentum conservation, hence the Maxwell-Boltzmann form leads to zero for the collision integral in this case.
