Lectures on

Random combinatorial optimization

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Exercises

Exercise 1: Relationship between the multispins Potts model and unusual random graph distributions.

Random graphs may be generated with distributions different from the usual Erdös-Renyi model. An example is given below. Start from the empty graph on N vertices. Consider all $\binom{N}{K}$ K-uplets of vertices, and for each of them,

- with probability $1 - c/N^{K-1}$, do nothing;
- with probability c/N^{K-1}, add $K(K-1)/2$ edges between all vertices of the K-uplet.

Show that the generating function of the connected components of the above random graphs distribution is related to the partition function of the mean field Potts model with K-spins interactions, whose Hamitonian reads,

$$H = - \sum_{i_1 < i_2 < \ldots < i_K} \delta(S_{i_1}; S_{i_2})\, \delta(S_{i_2}; S_{i_3}) \ldots \delta(S_{i_{K-1}}; S_{i_K}) \quad . \tag{1}$$

Calculate the free energy of this multispins Potts model as a function of K and the connectivity c.

- What is the threshold $c(K)$ at which the random graph obtained via the above process percolates?

- How does the size of its giant component increase with c above the percolation transition?

Exercise 2: The 3-spins ferromagnetic Ising model on Cayley graphs.

Consider the 3-spins ferromagnetic model on a Cayley hyper-lattice with Hamiltonian,

$$H = - \sum_{i<j<k} J_{ijk} \; S_i \, S_j \, S_k \quad , \tag{2}$$

where the couplings J are equal to unity if the spins i, j, k are the corners of some (triangular) plaquette, zero otherwise. Each spin belongs to exactly $k+1$ plaquettes, where k is an integer number. The resulting lattice has a tree-like structure, with plaquettes replacing the bonds of the usual two-spins Ising ferromagnet.

Show that, at high temperature, the system is in a paramagnetic phase with free energy,

$$-\beta f_{para} = \ln 2 + \frac{k+1}{3} \; \ln \cosh \beta \quad , \tag{3}$$

where β is the inverse temperature. Show that, at low temperature, a ferromagnetic phase appears in which spins exert onto each other a non zero effective field h. Show that h fulfills the equation,

$$\tanh(\beta h) = \tanh(\beta) \; (\tanh \beta k h)^2 \quad . \tag{4}$$

What is the free energy of this phase? Calculate numerically the critical temperature of the para/ferro transition for $k = 3$.

Show that there exists another phase, of spin-glass nature, that corresponds to disordered configurations, identical to the low temperature equilibrium phase of the disordered version of Hamiltonian (2) with non zero couplings equal to $+1$ with probability $1/2$, -1 with probability $1/2$. Write down the replicated free energy for the latter model, and propose some variational Ansatz for the RSB order parameter.

What is the expected phase diagram associated to model (2)?

Exercise 3: Packing of hard objects on random graphs.

Consider a random graph G generated from the Erdös-Renyi distribution with average connectivity c, and a set of particles interacting through each other via impenetrable cores. If a particle is located on some vertex of G, it prevents other particles from sitting on adjacent sites (hard core repulsion). The goal is to find the statistical properties of the densest packings on G.

Write down the expression for the grand canonical partition function of the system of particles on G, denoting by μ the chemical potential regulating the density ρ of particles per site. Calculate the average logarithm of the grand canonical partition function using the replica method. Show that, within the replica symmetric approach, the largest density of particles equals,

$$\rho = \frac{1}{2c} \left(2\,W + W^2 \right) \qquad , \tag{5}$$

where W is defined from c through

$$c = W\, e^W \quad . \tag{6}$$

Compute the fractions of sites belonging to the covered (respectively uncovered) backbone of G, that is, covered (resp. uncovered) by particles in all densest packings.

How can the number of densest packings be calculated from the replica approach?

Exercise 4: Balanced random 3-Satisfiability

Consider the following distribution of random Boolean formulas on a set of N Boolean variables x_i, $i = 1, \ldots, N$. Draw randomly a triplet $1 \le i < j < k \le N$ of integers with a flat distribution over the set of possible triplets. Then with probability on half, consider the following set of four clauses,

$$(x_i \vee x_j \vee x_k) \wedge (\bar{x}_i \vee \bar{x}_j \vee x_k) \wedge (x_i \vee \bar{x}_j \vee \bar{x}_k) \wedge (\bar{x}_i \vee x_j \vee \bar{x}_k) \ , \tag{7}$$

and with probability one half,

$$(\bar{x}_i \vee \bar{x}_j \vee \bar{x}_k) \wedge (x_i \vee x_j \vee \bar{x}_k) \wedge (\bar{x}_i \vee x_j \vee \bar{x}_k) \wedge (\bar{x}_i \vee x_j \vee \bar{x}_k) \ . \tag{8}$$

Repeat the process $M = \gamma\, N$ times (all choices of triplets are uncorrelated), and consider the logical AND of all $4M$ clauses.

Derive the replicated free energy of the above Satisfiability problem at zero temperature. Show that, within the replica symmetric assumption, the SAT/UNSAT threshold is located at $\gamma_c = 0.918$. Compute the typical number of satisfying assignments in the satisfiable phase. What is the size of the backbone at the transition?

Exercise 5: Analysis of algorithms on random structures.

Consider a random graph G generated from the Erdös-Renyi distribution with average connectivity c, and an integer k. The core G' of G is the subgraph of G obtained from recursive deletions of all edges touching vertices with degree $\leq k-1$. For $k=1$, G' is identical to G. The question is to determine the location of the giant component and its size at criticality.

Call $\Phi_j(T)$ the average value of the number of vertices with degree j after T steps of the algorithm, that is, the deletions of edges arriving onto T vertices of degrees $\leq k-1$. Write some evolution equations for the Φ_j between times T and $T+1$ (consider separately the cases $j=0$, $1 \leq j \leq k-1$ and $j \geq k$). Show that these equations are compatible with the scaling hypothesis $\Phi_j(T) = N\varphi_j(T/N)$ in the large N limit and write down the set of coupled ordinary differential equations satisfied by the φ_j. Deduce from the solution of these equations the critical connectivity for core percolation. How does the order of the transition depend on k?