Stochastic resonance in the linear and nonlinear responses
of a bistable system to a periodic field

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The validity of the theory of the linear response (the satisfaction of the standard spectral relations) during a stochastic resonance is demonstrated. A stochastic resonance in relatively strong fields is studied. The reason for the appearance of a stochastic resonance and the region of parameter values in which it is seen are pointed out.

A fundamental new phenomenon in bistable and multistable systems is a stochastic resonance: an increase and a subsequent decay of the signal-to-noise ratio $R$ with increasing level of the external noise. The signal-to-noise ratio here is the ratio of the peak height on the spectral density of the fluctuations, $Q(\omega)$, at the frequency of the external field, $\Omega$, to the value of the spectral density of the fluctuations in the absence of a field, $Q^{(0)}(\Omega)$ (there are also other, essentially equivalent, definitions of a stochastic resonance; see Refs. 1 and 2 and the bibliography there). Stochastic resonances have been observed in ring lasers$^1$ and in numerical and analog simulations of various bistable systems.$^1 \sim 3$ In studies of the stochastic resonance, its nonlinear nature is usually stressed.

In the present letter we show that in a weak periodic field a stochastic resonance can be described completely by the theory of the linear response. It thus becomes simple to see the reason for the occurrence of a stochastic resonance and to determine how $R$ depends on the parameters. For definiteness, an analysis is carried out for a Brownian particle in a two-well potential $U(q)$ (interest in this model has revived because of, in particular, research on Josephson-junction systems$^4$):

$$\ddot{q} + 2\Gamma \dot{q} + U'(q) = A \cos \Omega t + f(t), \quad \langle f(t) f(t') \rangle = 4\Gamma T \delta(t - t').$$

As a result of forced oscillations, which are determined at a small field amplitude $A$ by the linear susceptibility $\chi(\Omega)$, i.e., $\langle q(t) \rangle = A \Re \{\chi(\Omega) \exp(-i\Omega t)\} + \text{const}$, a $\delta$-function peak arises at the field frequency $\Omega$ on the spectra fluctuation density of the system,

$$Q(\omega) = \lim_{\tau \to \infty} (4\pi \tau)^{-1/2} \int_{-\tau}^{\tau} dt e^{i\omega t} q(t)^2.$$

The ratio of its integral intensity (the area under it) to $Q^{(0)}(\Omega)$, i.e., to the value of the spectral fluctuation density at $A = 0$, is evidently
\[ R = \frac{1}{4} A^2 |\chi(\omega)|^2 / Q(0)(\omega) \]  

(3)

\[ Q(\omega) \text{ in (2) is equal to the Fourier transform of the correlation function} \langle q(\tau + t_i) q(t_i) \rangle, \text{ averaged over } t_i; \text{ it corresponds to the spectral density of fluctuations which is ordinarily measured experimentally}. \] 

For a system which is at thermodynamic equilibrium at \( A = 0 \) [or at a quasiequilibrium if the noise \( f(t) \) is of a non-thermal nature], \( \chi(\omega) \) can be expressed in terms of \( Q(0)(\omega) \) in the standard way:

\[ \text{Re} \chi(\omega) = \frac{2}{T} \int_0^\infty d\omega_1 Q(0)(\omega_1) \omega_1^2 (\omega_1^2 - \omega^2)^{-1}, \quad \text{Im} \chi(\omega) = \frac{\pi \omega}{T} Q(0)(\omega). \]  

(4)

The quantitative agreement between the values of \( R \) measured through an analog simulation of system (1), on the one hand, and the values of \( R \) calculated from Eqs. (3) and (4) with the help of the spectral density of fluctuations \( Q(0)(\omega) \) measured for the same system at \( A = 0 \), on the other, is demonstrated by Fig. 1. These results show that the fluctuation-dissipation theorem and the dispersion relations hold under conditions corresponding to a stochastic resonance.

The appearance of a stochastic resonance with a periodic modulation of the probabilities for fluctuational interwell transitions \( W_{ij} \) and of the populations of the potential wells, \( w_i \) (\( i = 1,2 \)), by an alternating field (cf. Ref. 1) [sic]. In the absence of a field, fluctuational transitions at \( T \ll \Delta U_{1,2} \) (\( \Delta U_i \) is the depth of well \( i \)) lead to a narrow peak \( Q_{tr}^{(0)}(\omega) \) in \( Q^{(0)}(\omega) \) (Ref. 7):

\[ Q_{tr}^{(0)}(\omega) = \frac{1}{\pi} w_1 w_2 (q_1 - q_2)^2 W/(W^2 + \omega^2), \quad W = W_{12} + W_{21} \]  

(5)

\[ \tilde{R} \]

\[ \tilde{R} = 6.51 \times 10^{-4} R \] for the potential \( U(q) = -\Omega q^2 + \Omega q^4 + \Omega q^4; \Omega = 0.0695, A = 0.1, \Gamma = 0.125; \Delta U = 1/4. \) —Direct measurements; + —calculated from measured values of \( Q^{(0)}(\omega) \).

\[ \text{FIG. 1. Values of } \tilde{R} = 6.51 \times 10^{-4} R \text{ for the potential } U(q) = -\Omega q^2 + \Omega q^4; \Omega = 0.0695, A = 0.1, \Gamma = 0.125; \Delta U = 1/4. \]
(\(q_{1,2}\) are equilibrium positions). If only the term in (5) is retained in \(Q^{(0)}(\omega)\), it is clear from (3) and (4) that we have

\[
R = \frac{1}{4} \pi a^2 (q_1 - q_2)^2 w_1 w_2 w / T^2 \propto \exp(-\Delta U_{\text{max}}/T), \quad \Delta U_{\text{max}} = \max(\Delta U_1, \Delta U_2).
\]

(6)

In other words, the signal-to-noise ratio \(R\) increases exponentially with increasing \(T\), both for equal well depths and for different well depths.

The range of applicability of (6) and of the existence of a stochastic resonance is found from the condition that \(Q^{(0)}(\omega)\) is close to \(Q^{(0)}_{\text{tr}}\), i.e., the condition that the parts of \(Q^{(0)}(\omega)\) due to vibrations with respect to the equilibrium positions, \(Q^{(0)}_i(\omega) \approx 2/\pi \Gamma T w_i [\omega^2 - \omega_i^2 + 4\Gamma^2 \omega_i^2]^{-1}\), where \(\omega_i = [U''(q_i)]^{1/2}\), \(T \ll \Delta U_i\) are small. Since \(\omega_i \propto \exp(-\Delta U_i / T) \ll 1\), a stochastic resonance is clearly manifested only at very low frequencies \(\Omega \ll \omega_{1,2}, \omega_{1,2}^2 / \Gamma\). In systems with a slight damping, \(\Gamma \ll \omega_{1,2}\) the increment \(\delta Q^{(0)}_i = (4\Gamma^2 + \omega_i^2)^{-1} / 2 \pi w_i [T U''(q_i) \omega_i^{-4}]^2\) becomes important in the region \(\omega \ll 2 \Gamma\), even at comparatively small values of \(T\). This circumstance seriously limits the interval of \(\Omega\) and \(T\) values in which a stochastic resonance is observed, as was verified in the present experiments. It is clear from this discussion that for a given \(\Omega\) and for very small values of \(T\), under the condition \(W \ll \Omega\), the ratio \(R\) decreases with \(T\) (as \(1/T\) at \(\Gamma \gg \omega_{1,2}\)). At large \(T\), an increase in \(R\) then occurs, as was observed in Ref. 1 (cf. Fig. 1).

At low values of \(T\), the response of a bistable system to a low-frequency field \((\Omega \ll \Gamma, \omega_{1,2}^2 / \Gamma)\) is very nonlinear even at small values of \(A\), under the conditions \(\Delta U_{1,2} \gg A |q_{1,2} - q_s| \gg T\) \([q_s\) is the position of a local maximum of \(U(q)\)], since the modulation of the well populations due to the field is strong. Solving the balance equations for the populations, and incorporating the periodic increment in \(\Delta U_i\) in the expression for the transition probability \(W_{ij} \propto \exp(-\Delta U_i / T)\), we can show that the term associated with the transitions, \(\langle q(t) \rangle_{\text{tr}}\), in \(\langle q(t) \rangle\) described by a nearly rectangular crest. In the simplest case of a symmetric potential, \(U(q) = U(-q)\), \(q_2 = -q_1\), we would have

\[
\langle q(t) \rangle_{\text{tr}} \approx 2\bar{q} \sum_{n=-\infty}^{\infty} \left[ \theta(t - \frac{-\pi n}{\Omega}) - \theta(t - \frac{\pi (2n + 1)}{\Omega}) \right] - \bar{q}.
\]

\[
\bar{q} = q_1 \tanh \lambda, \quad \lambda = \left( \frac{2\pi T}{|Aq_1|} \right)^{1/2} \frac{W_{12}}{2 \Omega} \exp \left( \frac{|Aq_1|}{T \Omega} \right) (W_{12} = W_{21}).
\]

(7)

Under the condition \(\lambda \gg 1\) (note that we have \(1 - \tanh \lambda < 0.1\) for \(\lambda > 1.5\)), we have \(\bar{q} \approx q_1\), and \(\bar{q}\) is essentially independent of \(T\). The peak intensity \(Q(\omega)\) in (2) at the frequency \(\Omega\) is thus also nearly independent of \(T\). A stochastic resonance arises in the region \(\Omega < W = 2W_{12}\) (in which the condition \(\lambda \gg 1\) definitely holds). It results from a decrease in \(Q^{(0)}(\Omega)\) with increasing \(T\), as a result of a spreading of the peak in (5): \(Q^{(0)}_{\text{tr}}(\Omega) \propto \exp(\Delta U_i / T)\) at \(\Omega \ll W\). It can be seen from (2) and (7) that the \(Q(\omega)\)
peaks at the overtones of $\Omega$ were found to be small in the regime of a strong nonlinearity in Refs. 2 and 3 because of a numerical reason: The intensity of the peak at the frequency $(2k + 1)\Omega$ is proportional to $(2k + 1)^{-2}$.

It follows from the results of the present study and those of Refs. 6 and 7 that a stochastic resonance should occur both at low frequencies and at frequencies close to the frequency of the strong field in systems with several stable forced-oscillation regimes in an intense periodic field (optically bistable and multistable systems, electrons excited by a field at the cyclotron frequency, etc.).


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