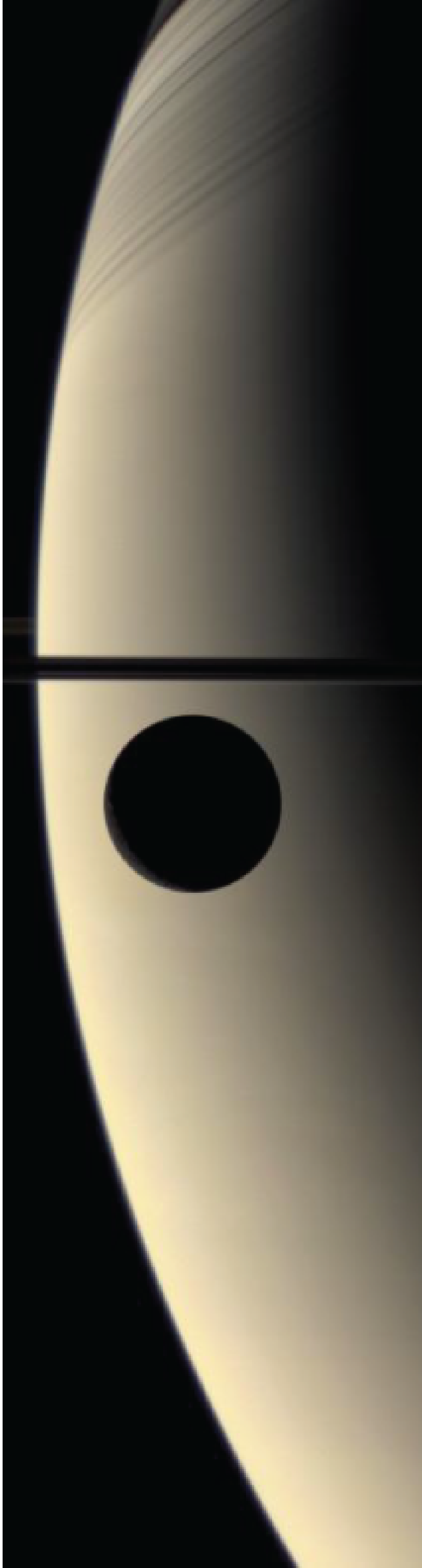
EDWARD BROWN

PLANETS AND TELESCOPES



About the cover: an image of Rhea occulting Saturn, as captured by the Cassini spacecraft.
Credit: Cassini Imaging Team, SSI, JPL, ESA, NASA

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Preface

These notes were written while teaching a sophomore-level astronomy course, “Planets and Telescopes” at Michigan State University during Spring Semesters of 2015 and 2016. The background required is introductory calculus and freshman-level physics.

In the first year, the main text was Lissauer and de Pater, Bennett et al.[1]; in the second year, we switched to Ryden and Peterson[2] and Taylor[3] and increased the amount of time spent on basics of astronomical observation and statistical analysis. Some of the notes and exercises on statistics are written in the form of Jupyter Notebooks; these are in the folder `statistics/notebooks`.

The text layout uses the `tufte-book` (`https://tufte-latex.github.io/tufte-latex/`) LaTeX class: the main feature is a large right margin in which the students can take notes; this margin also holds small figures and sidenotes. Exercises are embedded throughout the text. These range from “reading exercises” to longer, more challenging problems. Because the exercises are embedded with the text, a list of exercises is provided in the frontmatter to help with locating material.

In the course, about three weeks were spent covering the material in Appendix C, “Probability and Statistics”. This was done between covering Chapter 2, “Light and Telescopes” and Chapter 4, “Detection of Exoplanets”. This ordering was driven by the desire to keep the lectures and labs synchronized as much as possible. In Chapter 1, “Coordinates”, several of the exercises refer to the night sky as viewed from mid-Michigan in late January.

I am grateful for many conversations with, and critical feedback from, Prof. Laura Chomiuk, who taught the lab section of this course, graduate teaching assistants Laura Shishkovsky and Alex Deibel, and undergraduate learning assistants Edward Buie III, Andrew Bundas, Claire Kopenhafer, Pham Nguyen, and Huei Sears.

These notes are being continuously revised; to refer to a specific version of the notes, please use the eight-character stamp labeled “git version” on the front page.

[1] Jack J. Lissauer and Imke de Pater. *Fundamental Planetary Science: Physics, Chemistry and Habitability*. Cambridge University Press, 2013; and Jeffrey O. Bennett, Megan O. Donahue, Nicholas Schneider, and Mark Voit. *The Cosmic Perspective*. Addison-Wesley, 7th edition, 2013

[2] Barbara Ryden and Bradley M. Peterson. *Foundations of Astrophysics*. Addison-Wesley, 2010

[3] John R. Taylor. *An Introduction to Error Analysis*. University Science Books, Sausalito, CA, 2nd edition, 1997

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1
Coordinates: Specifying Locations on the Sky

1.1 Declination and right ascension

To talk about events in the sky, we need to specify where they are located. To specify where they are located, we need a point of reference. This is a bit tricky: we are riding on the Earth, which rotates and orbits the Sun; the Sun orbits the Milky Way; the Milky Way moves through the Local Group; and on top of all this the universe is expanding.

The primary criterion for choosing a coordinate system is convenience. We want a system that is easy to use and that describes the sky straightforwardly. As viewed from Earth, we appear to be at the center of a great sphere, with celestial objects lying on its surface. This is similar to describing locations on the Earth, for which we use two angles: latitude, which measures the angle north or south from the equator; and longitude, which measures the angle east or west from the prime meridian. Likewise, to describe the apparent position of objects as viewed from Earth, we also need two angles.

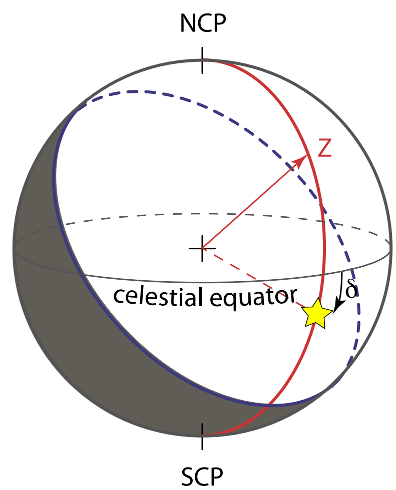


Figure 1.1: The meridian (red) passing through our zenith (Z). Our vantage point is from the center of the sphere. Also shown are the north celestial pole (NCP), south celestial pole (SCP) and the celestial equator (CE). The shaded region are points below our horizon; objects in that region are not visible from our location. A star with negative declination δ is shown as well.

First, let's describe our measurement of position on the sky. The local gravitational acceleration $\boldsymbol{g}$ specifies the *local vertical*; this picks out a point on the celestial sphere, our *zenith*. Our *horizon* is then defined by points that lie 90° from this zenith, measured along a great circle passing through the zenith. The zeniths above the north and south poles define the *north and south celestial poles*. A great circle connecting the celestial poles and our zenith defines our *meridian* (see Fig. 1.1). As the Earth rotates, celestial objects appear to move westward on circles about the celestial poles.

Midway between the north and south celestial poles lies the celestial equator. For any star, you specify its declination δ as the angle north (positive) or south (negative) of the celestial equator along that star's meridian. For example, Betelgeuse, the red star in the shoulder of Orion, has a declination $\delta = 7^\circ\,24'\,25''$. Polaris, the North star, has $\delta = 89^\circ\,15'\,51''$.

Declination is quoted in degrees (°), arcminutes (′), and arcseconds (″). There are 60 arcminutes in 1 degree and 60 arcseconds in 1 arcminute.

EXERCISE 1.1 — How far above our southern horizon will Betelgeuse be when it crosses our meridian? Our latitude is $42^\circ\ 43'\ 25''$N.

Declination measures how far north or south of the celestial equator a given object lies. To specify an east-west location, we need another reference point. Because of the Earth's rotation, we can't use a point on Earth, such as the Greenwich observatory (which is where the 0° of longitude is defined). We can, however, use the Earth's motion around the Sun: as the Earth moves around the Sun, the Sun appears to move eastwards relative to the fixed stars. This path the Sun takes around the celestial sphere is known as the *ecliptic*, and the constellations that lie along the ecliptic are the *zodiac*. Because the Earth's rotational axis is tilted at an angle of $23^\circ\ 16'$ with respect to its orbital axis, the Sun's declination varies over the course of a year.

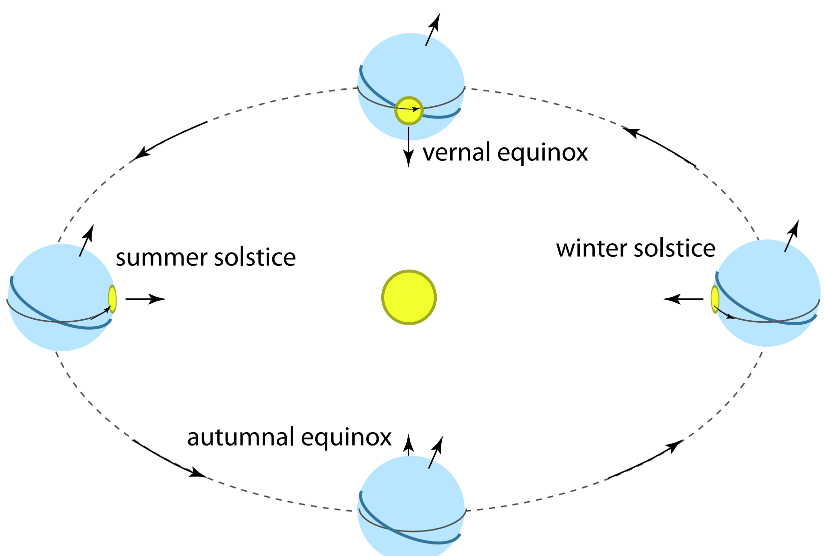


Figure 1.2: As Earth orbits the Sun, the Sun's declination traces out a path along the celestial sphere known as the *ecliptic*. Over the course of a year, the Sun appears to move eastward, relative to distant stars, along the ecliptic.

The Sun reaches its minimum declination, $-23^\circ\ 16'$, when it appears to lie in the direction of Sagittarius at the *winter solstice*. One quarter orbit later, the Sun crosses the celestial equator; at this point the Sun is in the direction of Pisces at the *vernal equinox*. Another quarter orbit brings the Sun in the direction of Gemini with declination $23^\circ\ 16'$; this is the *summer solstice*. A further quarter orbit, and the Sun crosses the celestial equator in the direction of Virgo at the *autumnal equinox*.

The ecliptic thus intersects the celestial equator at two points (Fig. 1.2), the *vernal and autumnal equinoxes*. We usually associate the equinoxes with a specific time of year, but they actually define unique directions on the sky. We can therefore define our second angular coordinate, *right ascension*, as the angle between an objects' meridian and the vernal equinox, measured eastwards along the celestial equator.

Rather than specify the right ascension by degrees, astronomers instead quote it in terms of hours (and minutes and seconds). The vernal equinox is therefore at $\mathrm{RA} = 00^{\mathrm{h}}\,00^{\mathrm{m}}\,00^{\mathrm{s}}$ and the autumnal equinox is at $\mathrm{RA} = 12^{\mathrm{h}}\,00^{\mathrm{m}}\,00^{\mathrm{s}}$.

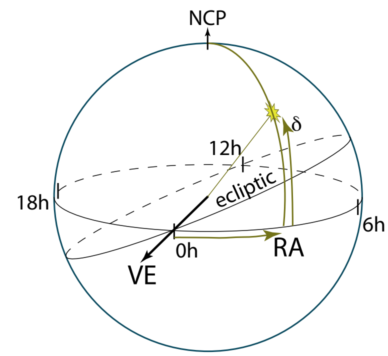


Figure 1.3: The right ascension (RA) and declination (δ) of a celestial object.

EXERCISE 1.2 — Estimate the Sun's current right ascension. Given that Betelgeuse is currently visible in the night sky, what is a a plausible value for its right ascension?

1.2 *Precession*

As we noted above, at the summer solstice, the Sun is in the direction of Gemini. On the solstice, the Sun will appear to be directly overhead at a latitude of $23°\,16'$N, which is known as the *Tropic of Cancer*. Why isn't it called the Tropic of Gemini?

The answer is that the Earth's rotation axis is not fixed; it precesses. The north and south celestial poles trace a circle on the sky relative to distant stars over a 26 000 yr period. The causes the direction of the equinoxes to move westward along the ecliptic on that timescale. There are 13 constellations, the *zodiac*, around the ecliptic; in the last two millennia the direction of the summer solstice has shifted one constellation over, from Cancer to Gemini. Likewise, the winter solstice used to be in the direction of Capricorn; now it is in the direction of Sagittarius.

As a practical matter, this means that the coordinates of right ascension and declination, which are based on the direction of the Earth's rotation axis, slowly change. To account for this, when giving the coordinates for an object astronomers specify an *epoch*—a reference time to which the right ascension and declination refer. The current epoch is J2000, which refers to roughly noon UTC on 1 January 2000.

EXERCISE 1.3 — Brainstorm some possible coordinate systems, and describe their advantages and disadvantages in comparison to right ascension and declination.

1.3 *Keeping time*

Our *local noon* is when the Sun crosses our meridian[1]. The time between two successive noons is one *solar day*, which we divide into 24 hours. This is slightly longer than the time for the earth to complete one rotation, however: because of the Earth's motion about the Sun, the position of the Sun shifts by about one degree over the course of a day, and the Earth

[1] The local noon is usually *not* at 12:00pm: our time zones are only to the nearest hour, and there is an adjustment for daylight savings time.

must rotate that amount in addition to one full rotation before the next noon (Fig. 1.4).

There are 365.24 solar days between successive solar crossings of the vernal equinox, which defines a *tropical year*. Over the course of this year, the extra rotation on each solar day adds up to one complete rotation of the Earth. The Earth rotates 366.24 times in one tropical year, and therefore the rotation period of the Earth is

$$\frac{365.24}{366.24} \times 24\,\mathrm{hr} = 23^{\mathrm{h}}\,56^{\mathrm{m}}\,04^{\mathrm{s}}.$$

In fact, the tropical year is slightly shorter, by about $20\,\mathrm{min} = 1\,\mathrm{yr}/26\,000$ because of the precession of the Earth's axis.

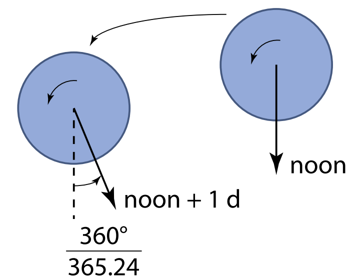


Figure 1.4: The movement of the Earth from noon to noon. The arrows indicate the direction towards the Sun.

Our time—hours and minutes—is tied to the position of the Sun, which is convenient for daily activity but not so convenient if we want to know when a particular star is observable. Instead of marking when the Sun crosses our meridian, we define our local *sidereal time* relative to our meridian crossing the vernal equinox. Because we also define right ascension relative to the vernal equinox, objects with a right ascension near that of the sidereal time will be high in the sky.

To compute our local sidereal time, first determine the right ascension of the Sun (Exercise 1.2); this will then fix the offset between the local sidereal time and the local noon in UTC. We can then compute our offset for local noon based on our longitude.

EXERCISE 1.4— Local noon at $0°$ longitude corresponds to 12:00 UTC. Given that our longitude is $84°\,28'\,33''$ W, what is our local noontime in UTC. What local time would this correspond to today? From this and your estimate of the Sun's current hour angle, what is the current sidereal time?

1.4 Parallax

The motion of the Earth around the Sun *does* cause a small shift in the apparent angular position of a star, a phenomena known as *parallax*. This effect is exploited to determine the distance to nearby stars.

The angular shift, ϖ, is related (see Fig. 1.5) to the radius of the Earth's orbit, 1 AU, and the distance to the star d via

$$\frac{1\,\mathrm{AU}}{d} = \tan\varpi \approx \varpi.$$

When ϖ is expressed in arcseconds, this gives

$$d = \frac{206\,265\,\mathrm{AU}}{\varpi/1''} = 1\,\mathrm{pc}\left(\frac{1''}{\varpi}\right), \tag{1.1}$$

which defines the *parsec*. In CGS units $1\,\mathrm{pc} = 3.086 \times 10^{18}\,\mathrm{cm}$, which is a bit over 3 light-years.

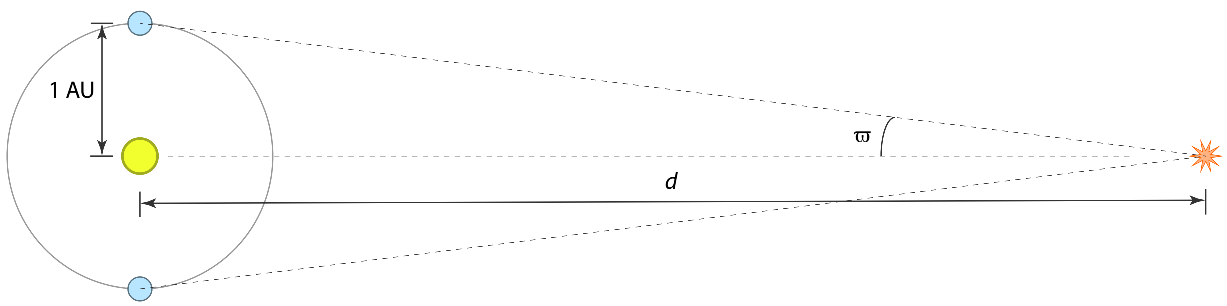


Figure 1.5: The parallax angle ϖ of a star induced by Earth's motion around the Sun.

1.5 *Angular distances between nearby objects*

To compute the angular distance between two points on the sky, we draw two vectors $\boldsymbol{a}$, $\boldsymbol{b}$ to these points and use

$$\cos\theta = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}.$$

Since both $\boldsymbol{a}$ and $\boldsymbol{b}$ lie on the unit sphere, $|\boldsymbol{a}| = |\boldsymbol{b}| = 1$; the (x, y, z) components of these vectors are

$$(\cos\delta_1\cos\eta_1, \cos\delta_1\sin\eta_1, \sin\delta_1)$$

and

$$(\cos\delta_2\cos\eta_2, \cos\delta_2\sin\eta_2, \sin\delta_2)\,,$$

respectively. Taking the dot product,

$$\begin{aligned}\cos\theta &= \cos\delta_1\cos\delta_2\left(\cos\eta_1\cos\eta_2 + \sin\eta_1\sin\eta_2\right) + \sin\delta_1\sin\delta_2 \\ &= \cos\delta_1\cos\delta_2\cos\left(\eta_1 - \eta_2\right) + \sin\delta_1\sin\delta_2. \qquad (1.2)\end{aligned}$$

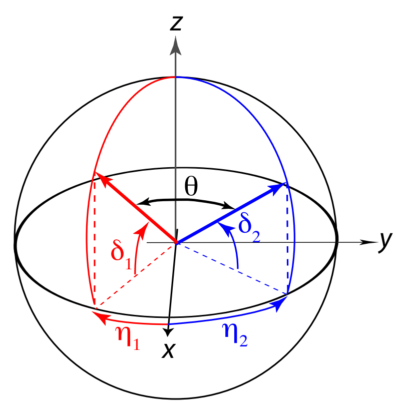


Figure 1.6: Two locations on the sphere separated by a distance θ.

We are usually interested in the angular distance between two nearby sources, with RAs $\eta_1 \approx \eta_2$ and declinations[2] $\delta_1 \approx \delta_2$. We can use the expansion rule,

$$\cos x \approx 1 - \frac{x^2}{2}, \qquad x \ll 1$$

on θ and $\eta_1 - \eta_2$ in equation (1.2):

$$\begin{aligned}1 - \frac{\theta^2}{2} &\approx \cos\delta_1\cos\delta_2\left[1 - \frac{(\eta_1-\eta_2)^2}{2}\right] + \sin\delta_1\sin\delta_2 \\ &= \cos(\delta_1 - \delta_2) - \cos\delta_1\cos\delta_2\frac{(\eta_1-\eta_2)^2}{2}.\end{aligned}$$

We can now expand $\cos(\delta_1 - \delta_2)$, cancel common factors and multiply by 2,

$$\theta^2 \approx (\delta_1 - \delta_2)^2 + \cos\delta_1\cos\delta_2(\eta_1 - \eta_2)^2.$$

[2] Notice our coordinates differ from the usual spherical polar coordinates: δ is measured from the x-y plane, not from the z-axis.

We make heavy use of the sine and cosine addition formula: $\cos(x + y) = \cos x \cos y - \sin x \sin y$, and $\sin(-y) = -\sin y$, $\cos(-x) = \cos(x)$.

Finally, we notice that to lowest order, $\cos\delta_1 \cos\delta_2 \approx \cos^2\delta$, where $\delta = (\delta_1 + \delta_2)/2$ is the average of the two declinations. This gives us a formula for the angular distance θ between two nearby points,

$$\theta \approx \sqrt{\cos^2\delta\,(\eta_1 - \eta_2)^2 + (\delta_1 - \delta_2)^2}. \tag{1.3}$$

This looks like the pythagorean formula; the factor of $\cos\delta$ accounts for the lines of constant RA converging as they approach the poles.

EXERCISE 1.5— Atlas A and Electra are two bright stars that lie on the east and west sides of the Pleiades star cluster. Atlas has right ascension $\mathrm{RA} = 03^{\mathrm{h}}\,49^{\mathrm{m}}\,09.7^{\mathrm{s}}$ and declination $\delta = 24^\circ\,03'\,12''$; Electra has $\mathrm{RA} = 03^{\mathrm{h}}\,44^{\mathrm{m}}\,52.5^{\mathrm{s}}$ and $\delta = 24^\circ\,06'\,48''$. Find the angular distance between these stars. If the distance to the Pleiades is 136 pc, what is the projected distance between these stars?

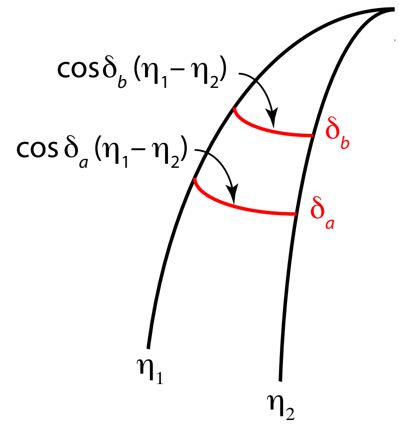


Figure 1.7: The distance between two RAs η_1 and η_2, measured along a circle of radius $\cos\delta$.

1.6 *Looking up*

Finally a note about directions when looking up at the sky. We've drawn our coordinates from the perspective of someone outside the celestial sphere; our perspective, however, is from the center. When we look up at the sky, if we face south, so that the direction northwards is at the top of our field of view, then the easterly direction is to our *left*. Objects of larger right ascension are therefore to our left as well.

2
Light and Telescopes

What do we actually measure when we observe a star? A star emits photons with a range of wavelengths over the electromagnetic spectrum. The total emitted energy per second over all wavelengths is the star's *luminosity*. For example, the solar luminosity is $L_\odot = 3.86 \times 10^{26}\,\mathrm{W}$. A telescope collects only a small fraction of this power: if a telescope has a collecting area $\mathcal{A}$ and is a distance d from the star, then it intercepts a fraction $\mathcal{A}/(4\pi d^2)$ of the star's light. We call $F = L/(4\pi d^2)$ the *flux*. The units of flux are $\mathrm{W\,m^{-2}}$.

More specifically, F is the *bolometric flux*, that is, the flux over all wavelengths. Of course, no telescope detects *all* wavelengths of light. Many wavebands, e.g., UV, X-ray, and infrared, do not even penetrate the Earth's atmosphere. Moreover, detectors (photographic plates or CCD's) are not uniformly efficient at converting photons into a signal.

In order to have a common standard, (optical) astronomers use *filters*, which transmit light only in certain wavelength bands. In this context, the flux refers to the power per area carried by light with wavelengths in that band. For historical reasons, astronomers define *magnitudes*, which are a relative logarithmic[1] scale for fluxes. The difference in magnitude between two stars is defined by

[1] In these notes, $\lg \equiv \log_{10}$ and $\ln \equiv \log_e$.

$$m_1 - m_2 = -2.5 \lg\left(\frac{F_1}{F_2}\right) \tag{2.1}$$

where the magnitudes m_1, m_2 and fluxes F_1, F_2 refer to light that has been passed through a particular filter.

Table 2.1: Selected common filters about the range of visible wavelengths [Binney and Merrifield, 1998]. Here "FWHM" means "Full width at half-maximum."

Filter	$\lambda_{\mathrm{eff}}/\mathrm{nm}$	FWHM/nm
U	365	66
B	445	94
V	551	88
R	658	138

Note that magnitudes are defined as the ratio of two fluxes. This is very useful when comparing the relative brightness of two stars; unfortunately it makes conversion to a physical unit ($\mathrm{W\,m^{-2}\,nm^{-1}}$) nontrivial. The magnitude scales are typically defined so that the star Vega has $U = B = V = \ldots = 0$.[2]

[2] But for historical reasons, $V(\mathrm{Vega}) = +0.04$.

EXERCISE 2.1—

1. Suppose we have two identical stars, A and B. Star A is twice as far away as star B. What is $m_A - m_B$?
2. Suppose a star's luminosity changes by a tiny amount δ. What is the corresponding change in that stars' magnitude?

IF WE TAKE A RATIO OF TWO MAGNITUDES USING DIFFERENT FILTERS FROM A SINGLE STAR, then we have a rough measure of the star's color. This ratio is called a *color index*. For example,

$$B - V \equiv m_B - m_V = -2.5 \lg \frac{F_B}{F_V}$$

gives a measure for how blue the star's spectrum appears.

EXERCISE 2.2— Which has the larger $B - V$ index: a red star, like Betelgeuse, or a blue-white star, like Rigel?

TWO STARS WITH THE SAME APPARENT BRIGHTNESS may have very different intrinsic brightnesses: one may be very dim and nearby, the other very luminous and faraway. To compare intrinsic brightness, we need to correct for the distance to the star[3]. We define the *distance modulus* as the difference in magnitude between a given star and the magnitude it would have if it were at a distance of 10 pc:

[3] This assumes we *know* the distance, which can be difficult!

$$\begin{aligned} \mathrm{DM} \equiv m - m(10\,\mathrm{pc}) &= -2.5 \lg \left[\frac{L}{4\pi d^2} \frac{4\pi (10\,\mathrm{pc})^2}{L} \right] \\ &= -2.5 \lg \left(\frac{10\,\mathrm{pc}}{d} \right)^2 \\ &= 5 \lg \left(\frac{d}{\mathrm{pc}} \right) - 5. \end{aligned}$$

The magnitude that the star would have if it were at 10 pc distance is called its *absolute magnitude*, $M \equiv m - \mathrm{DM}$.

2.1 Light is a wave

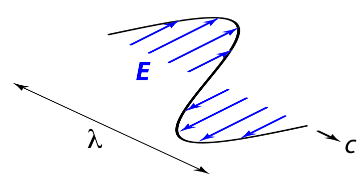


Figure 2.1: Schematic of the electric force (blue arrows) for a wave traveling towards us at speed c with wavelength λ.

Charges feel an electric force. When we detect light, what happens at the atomic level is that the charges in our detector (antenna, CCD, eye) feel an electric force that oscillates with frequency ν. If we could set up a grid of detectors and measure the electric force per unit charge, we would notice a sinusoidal pattern traveling at speed[4] $c = 299\,792\,458\,\mathrm{m/s}$ with a wavelength $\lambda = c/\nu$. We call this force per charge the electric field $\boldsymbol{E}(\boldsymbol{x}, t)$. The *intensity* of the light at our detector is proportional to $|\boldsymbol{E}|^2$.

[4] This velocity is exact; the meter is defined in terms of the speed of light.

In situations in which the wavelength is small (relative to the system in question), light propagates along *rays*. The rule for propagation is known as *Fermat's principle:* the path is that for which the propagation time is minimized. To illustrate this, we shall use it to derive the laws for reflection and refraction.

Consider light reflecting from a mirror as shown in the top panel of Figure 2.2. The time for light to propagate from source to observer is

$$\tau = \frac{1}{c}\left[\sqrt{h_s^2 + x^2} + \sqrt{h_o^2 + (w-x)^2}\right].$$

To minimize the path length, we compute $\mathrm{d}\tau/\mathrm{d}x$ and set it to zero,

$$0 = \frac{\mathrm{d}\tau}{\mathrm{d}x} = \frac{1}{c}\left[\frac{x}{\sqrt{h_s^2+x^2}} - \frac{w-x}{\sqrt{h_o^2+(w-x)^2}}\right] = \frac{1}{c}\left[\sin i - \sin r\right].$$

Hence the light travels such that $i = r$: the angles of incidence and reflection are equal.

For a second example, consider the passage of light from one medium to another, as depicted in the bottom panel of Figure 2.2. The interaction of matter with the oscillating electric field causes the light to travel at a speed c/n, where n is called the *index of refraction* and is a property of the material. For the situation in Fig. 2.2, the propagation time is

$$\tau = \frac{n_1}{c}\sqrt{h^2+x^2} + \frac{n_2}{c}\sqrt{d^2+(w-x)^2};$$

minimizing the propagation time with respect to x gives

$$0 = \frac{n_1}{c}\frac{x}{\sqrt{h^2+x^2}} - \frac{n_2}{c}\frac{w-x}{\sqrt{d^2+(w-x)^2}} = \frac{1}{c}\left[n_1 \sin i - n_2 \sin r\right].$$

This result, $n_1 \sin i = n_2 \sin r$, is also known as *Snell's law*.

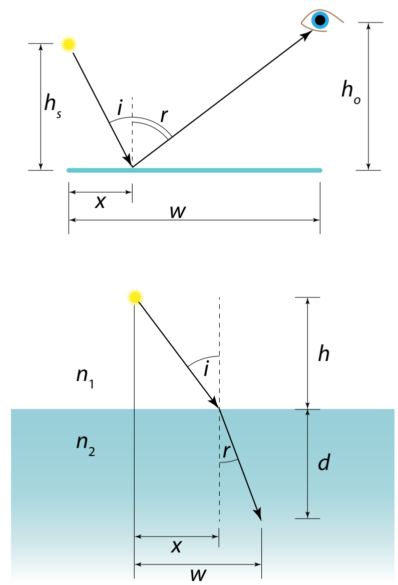


Figure 2.2: Top: reflection of light from a surface. Bottom: refraction of light as it passes from a medium with index n_1 into a medium with index n_2.

EXERCISE 2.3— A small stick of length ℓ is placed on the bottom of an empty swimming pool as shown in Fig. 2.3; when you look down on the stick from a height H above the bottom of the pool, the stick subtends an angle $\tan\theta = \ell/H$. The pool is then filled with water ($n = 4/3$) to a depth d. Because of refraction, the stick will appear to subtend a different angle θ'. Correct the right hand diagram of Fig. 2.3 to show how the light ray propagates from the ends of the stick to your eye. Is θ' larger or smaller than θ—is the image of the stick magnified or reduced? For the case $\ell \ll H$, so that $\theta \ll 1$, use the small angle expansions to derive an expression $\theta' = \theta\mathcal{M}$, where $\mathcal{M}$ depends on h, d, and n.

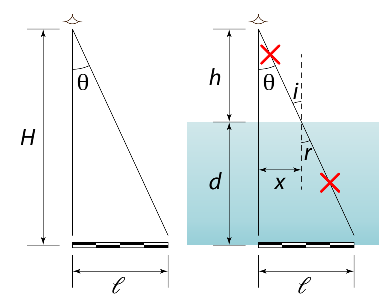


Figure 2.3: Change in angular size of an object in water.

2.2 Diffraction

A telescope makes an image by focusing the incoming rays of light onto a detector. Suppose we are at a fixed point and the wave is propagating past us. In general we would observe an electric field amplitude of the form

$$E(t) = A_0 \cos(2\pi\nu t) + B_0 \sin(2\pi\nu t)$$

where $\nu = c/\lambda$ is the frequency. Let's check this: in going from $t = 0$ to $t = T = 1/\nu$, the period of the wave, the argument of the cosine and sine goes from 0 to 2π, which is one oscillation. To find the net intensity I from a number of waves, we sum the amplitudes to get the net electric field $\boldsymbol{E}$ and then take the square $|\boldsymbol{E}|^2$.

Now imagine the electromagnetic wave incident on our telescope. The source is very distant, so the wavefront (a surface of constant phase) is a plane—think of sheets of paper moving downward onto the telescope. To make an image, the telescope focuses the incident radiation to a point on the detector. There is a limit, however, to how sharply the image can be focused. Let's look at a small angle θ away from the axis. Then the wavefront is incident on the telescope as shown in Figure 2.4. To keep the math tractable, we'll make our telescope opening one-dimensional and we'll break it into a $N + 1$ little detectors spaced a distance $d = D/N$ apart.

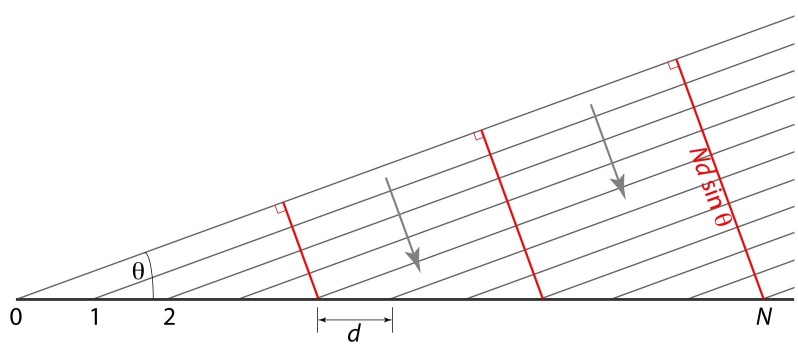


Figure 2.4: Schematic of a plane wave incident at angle θ on a detector.

Because of the angle, the light travels an extra distance $d \sin\theta$ to reach detector 1, $2d \sin\theta$ to reach detector 2, and so on. As a result, if the phase at the first detector (number 0) is χ, the phase at detector 1 is $\chi + 2\pi d \sin\theta/\lambda$, at detector 2, $\chi + 4\pi d \sin\theta/\lambda$, and so on. When we combine the signals from these detectors, the amplitude of the electric field will have the form

$$\begin{aligned} E &= A_0 \left[\cos\chi + \cos\left(\chi + 2\pi\frac{d\sin\theta}{\lambda}\right) + \cos\left(\chi + 2\pi\frac{2d\sin\theta}{\lambda}\right) + \right. \\ & \left. \quad + \cos\left(\chi + 2\pi\frac{3d\sin\theta}{\lambda}\right) + \ldots + \cos\left(\chi + 2\pi\frac{Nd\sin\theta}{\lambda}\right)\right] \\ & \quad + B_0\left[\sin\chi + \sin\left(\chi + 2\pi\frac{d\sin\theta}{\lambda}\right) + \ldots + \sin\left(\chi + 2\pi\frac{Nd\sin\theta}{\lambda}\right)\right]. \end{aligned}$$

When $\theta \to 0$, the amplitude goes to $E \to (N+1)\left[A_0 \cos\chi + B_0 \sin\chi\right]$, and so the brightness $I(\theta \to 0) = |E|^2$ is a very large number. That's good: the light from the star is focused to a point. Now, how large does θ have to be before E goes to zero?

To find this, let's first set $\chi = 0$ to keep things simple. There are a number of ways to find the sum; a particularly easy way is to recognize that this sum over cosines looks like adding up the x-component of vectors, and the sum over the sines looks like adding the y-component of vectors. We add the vectors by placing them nose-to-tail as shown in Fig. 2.5. The net amplitude is then A_0 times the x-component of the red vector, plus B_0 times the y-component of the red vector. Clearly if we want both the sum over sines and over cosines to vanish, we need the vectors to make a complete circle.

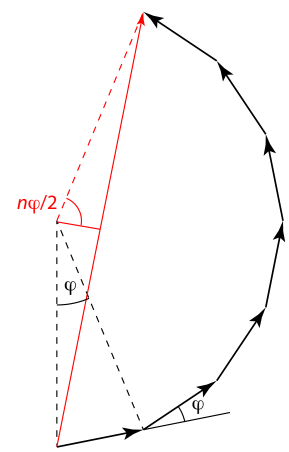


Figure 2.5: Addition of a series of vectors with a phase difference φ.

In this addition, each vector has length 1. If $N+1$ is large, then the circumference of the circle is approximately $(N+1) = 2\pi r$. For small $\varphi = (2\pi d/\lambda)\sin\theta$, the radius of the circle is $r \approx 1/\varphi$. Hence the condition for our vectors to sum to zero becomes

$$N + 1 = \frac{2\pi}{\varphi} = \frac{2\pi\lambda}{2\pi d \sin\theta}$$

Now, we assume that $N \gg 1$, so that $(N+1)d \approx Nd = D$, the diameter of our telescope's aperture. Then, the brightness falls to zero an angle

$$\sin\theta \approx \theta \approx \lambda/D$$

away from the center of the star's image.

The full form of the intensity as a function of angle from the beam axis is,

$$I = I_0 \left[\frac{\sin\left(\pi D/\lambda\ \sin\theta\right)}{\sin\left(\pi d/\lambda\ \sin\theta\right)}\right]^2. \tag{2.2}$$

EXERCISE 2.4— Write a `Python` function that computes eq. (2.2) for different values of N and D/λ. Plot $I/(I_0 n^2)$ against $\theta\lambda/D$. Describe your findings.

The wave nature of light places a limitation on the *resolving power* of a telescope, defined as the angular separation for which two point sources can be distinguished. Two point-like objects separated by an angular distance $\lesssim \lambda/D$ will have their images smeared into one.

EXERCISE 2.5— What is the resolving power of the *Hubble Space Telescope* ($D = 2.4\,\mathrm{m}$) and the Keck telescope ($D = 10\,\mathrm{m}$) at a wavelength $\lambda = 570\,\mathrm{nm}$? Estimate the angular resolution of the human eye at that wavelength. What is the resolving power of the Arecibo radio telescope ($D = 305\,\mathrm{m}$) at a frequency of 3 GHz?

For ground-based telescopes, an even more severe limitation is the refraction of light by the atmosphere. The atmosphere is turbulent, and the swirling eddies contain variations in density that change the refractive index and distort the wavefront. This distortion smears the image over an angular scale that is typically larger than $1''$.

EXERCISE 2.6 — What is the angular size of a solar-sized star ($R_\odot = 6.96 \times 10^5$ km) at a distance of 1 pc? What is the angular size of Mars ($R_♂ = 3\,390$ km) at a distance of 0.5 AU? How would the difference in angular size affect the appearance of these two objects?

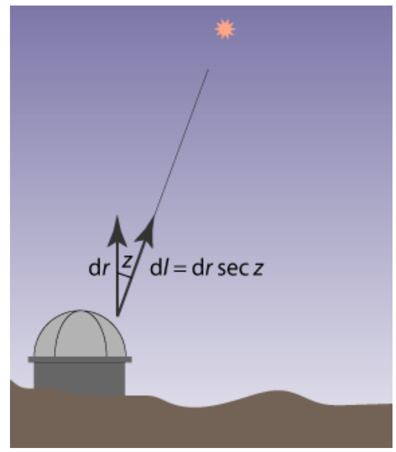


Figure 2.6: Illustration of the greater column of atmosphere (*airmass*) that the light from a star an angle z from the zenith must traverse.

IN ADDITION TO DISTORTING THE WAVEFRONT, THE AIR ALSO ATTENUATES THE BRIGHTNESS OF THE LIGHT. The amount of attenuation depends on the column, that is, the mass per unit area of air along the line of sight, which in turn depends on the viewing angle (Fig. 2.6).

Astronomers define the *airmass* m as a function of zenith angle z by

$$\text{air mass} = \frac{\int \rho(r)\,\mathrm{d}\ell}{\int \rho(r)\,\mathrm{d}r}$$

where ℓ is along the line of sight to the star. For a planar atmosphere, $\mathrm{d}\ell = \mathrm{d}r/\cos z = \sec z\,\mathrm{d}r$, and so the airmass is just $\sec z$. The dimming of the star is proportional to $\exp\left[-\int \rho(r)\,\mathrm{d}\ell\right]$, and therefore the magnitude of a star at zenith angle z varies as

$$m(z) = k \sec z + c,$$

where k and c are constants. By measuring the apparent brightness of the star at several different zenith angles, astronomers can empirically determine these constants.

3 Spectroscopy

3.1 Electromagnetic radiation is quantized

Electromagnetic radiation—light—is carried by massless particles known as *photons*. Being massless, they travel at a speed c in all frames. The energy of a photon depends on its frequency ν: $E_\nu = h\nu$. Since $\nu = \lambda/c$, we can also express the energy of a photon as $E_\lambda = hc/\lambda$. When matter absorbs or emits radiant energy, it does so by absorbing or emitting photons.

EXERCISE 3.1 — On a very dark night, the eye can make out stars down to visual magnitude $V \approx 6$. Given that the sun has $V = -26.71$ and that the flux from the sun in V-band is approximately $10^3\ \mathrm{W/m^2}$, estimate the radiant flux from this $V = 6$ star. If the V band photons have an average $\lambda = 550\ \mathrm{nm}$, how many photons from this barely visible star enter your pupil and strike your retina each second?

Suppose we shine a monochromatic (i.e., comprising a single wavelength) beam of light at a tinted piece of glass (sunglasses, for example). The light that emerges on the other side is the same color—meaning it has the same wavelength—but is dimmer. What are we to make of this? For the exiting light to be dimmer, some of the photons must have been absorbed. But if the photons are indistinguishable, why are only some absorbed? Once we have quantization, we are forced to adopt a probabilistic viewpoint: each photon has a certain probability of being absorbed.

3.2 The hydrogen atom

The electrons bound to an atom or molecule can only occupy states having a discrete set of energies. For example, the electron in a hydrogen atom only has energies

$$E_n = -13.6\ \mathrm{eV} \times \frac{1}{n^2}, \tag{3.1}$$

where $n > 0$ is an integer known as the principal quantum number. These energies are negative, relative to a free electron. For example, it takes 13.6 eV to remove an electron in its ground state from the atom.

Because the electrons in an atom can only have certain energies, the atom can only absorb or emit light at specific wavelengths, such that the energy of the photon matches the difference in energy between two levels. For example, a hydrogen atom can absorb a photon of energy

$$E_{1\to 2} = -13.6\,\mathrm{eV}\left(\frac{1}{2^2} - \frac{1}{1^2}\right) = 10.2\,\mathrm{eV}$$

corresponding to the energy required to move the electron from $n = 1$ to $n = 2$.

The wavelengths that can be emitted or absorbed by a hydrogen atom at rest can be found by substituting $E = hc/\lambda$ into equation (3.1):

$$\lambda_{m\to n} = \lambda_0 \left(\frac{1}{n^2} - \frac{1}{m^2}\right)^{-1}, \tag{3.2}$$

where $\lambda_0 = 91.2\,\mathrm{nm}$. The transitions to the lowest levels are named after their discoverers: Lyman for $m \to 1$, Balmer for $m \to 2$, Paschen for $m \to 3$. A greek letter is used to denote the higher state: for example Lyman α (abbr. Lyα) means $2 \to 1$, with $\lambda_{\mathrm{Ly}\alpha} = 121.6\,\mathrm{nm}$. The first line transition in the Balmer series is $3 \to 2$, and is designated Hα: $\lambda_{\mathrm{H}\alpha} = 656.3\,\mathrm{nm}$. The first 50 lines for the Lyman ($m \to 1$), Balmer ($m \to 2$), and Paschen ($m \to 3$) are shown in Fig. 3.1; note the $4 \to 3$ transition is outside the plot range.

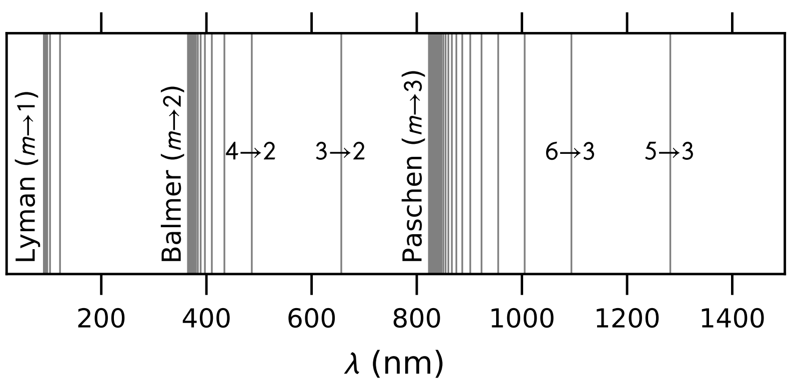


Figure 3.1: Spectral lines of neutral hydrogen.

3.3 *Diffraction Gratings*

To look at the different wavelengths in the light from a source, we use a diffraction grating, which is a series of fine, closely spaced lines etched on a surface. When light is projected onto the grating, it is reflected from the lines in all directions. Along a given direction, the light from two adjacent lines will travel a slightly different distance: if the spacing between lines is d, the extra distance traveled from a neighboring line is $d \sin \theta$, where θ is the angle between the incident and reflected rays. Because of this different path length, a distant detector will in general receive waves of many different phases. When the waves are added together, the peaks and troughs cancel, and the result is that the summed wave is greatly reduced in amplitude.

There are, however, certain directions along which the intensity is maximized. If the extra path length is a multiple of the wavelength then all the waves reach the distant detector so the intensity is bright. That is, at angles satisfying

$$d \sin \theta = m\lambda, \tag{3.3}$$

bright spots are produced.This situation is depicted in Fig. 3.2 for $m = 1$. For each line, the path length differs by one wavelength from its neighbors; as a result, the rays along a direction θ (at the right of the figure) are in phase. Since different wavelengths produce their bright spots at different angles, the light is dispersed in wavelength, producing a spectrum. A good home example of a grating is a compact disk: the tracks on the disk diffract light.

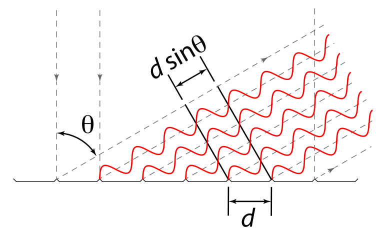
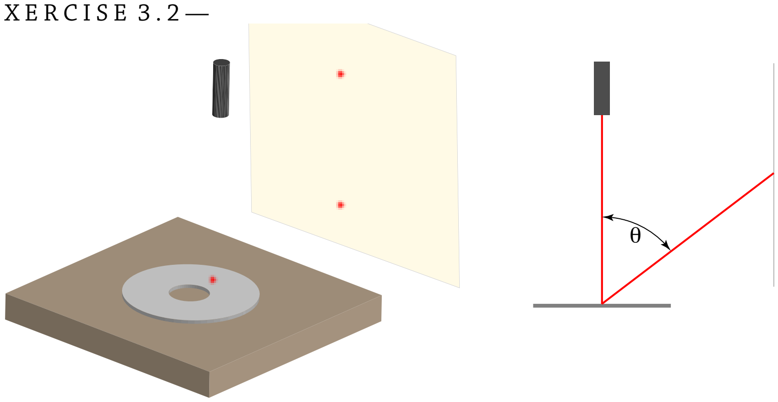


Figure 3.2: A diffraction grating.

EXERCISE 3.2—



You shine a red laser pointer ($\lambda = 650$ nm) onto a face-up CD, and observe that two dots appear on a blank screen, as shown above. The laser beam is vertical and the two dots that appear on the screen are at angles 23° and 52° from the vertical. There are no other dots appearing. From the information given, calculate the spacing between the tracks on the CD. Suppose we then shine a green laser pointer ($\lambda = 530$ nm) at the disk. At what angles would dots appear?

For a telescope, there is an additional complication: we don't have a single source, but rather an image of the entire field of view. To restrict our field of view, we overly our grating with a slit, as shown in Figure 3.3. The width of the slit is matched to the seeing so that if projects a line of light onto the diffraction grating. The dispersed light thus makes a two dimensional image, with position along the slit along one axis, and wavelength along the other axis.

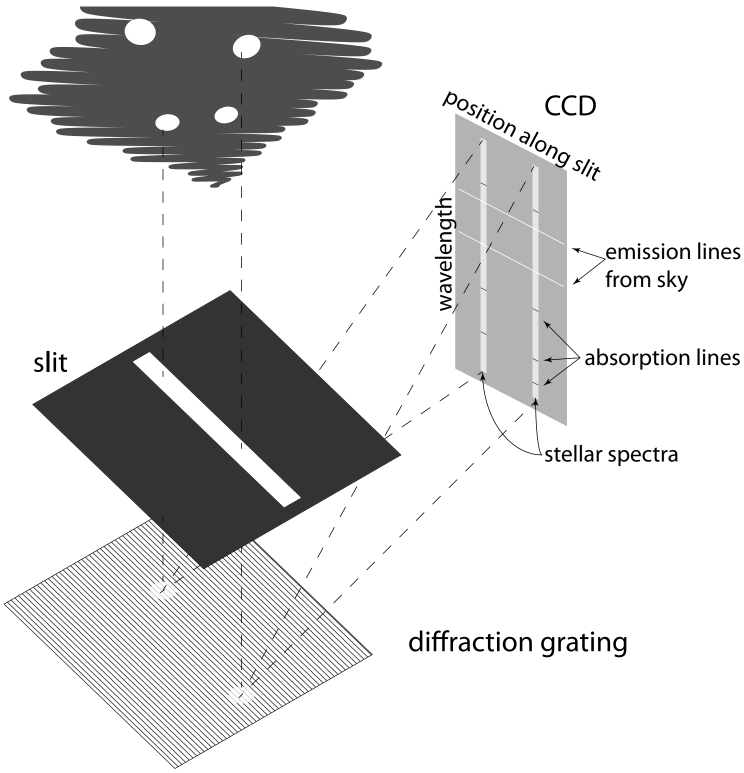


Figure 3.3: Taking a spectrum of an astronomical object.

EXERCISE 3.3 — The Goodman spectrograph on the SOAR telescope has a grating with 400 lines mm^{-1}. For the first order spectrum ($m = 1$), find the dispersion $\mathrm{d}\theta/\mathrm{d}\lambda$, in units of arcseconds per nanometer, at $\lambda = 500\,\mathrm{nm}$.

3.4 *Absorption and emission lines*

Now that we are taking spectra, what do we see? Suppose we look at a tenuous cloud of hot gas, and there is no light source behind this cloud. Because the gas is hot, collisions between atoms will excite electrons into excited states. When these electrons make a transition to the ground state, a photon is emitted. Thus, when we take a spectrum of the light from this cloud, we expect to see a series of discrete, bright lines at those frequencies. This is an *emission line spectrum*.

Emission lines are also produced in Earth's atmosphere from a variety of sources: for example collisions of molecues with cosmic rays and recombination of ions and electrons that had been photoionized by sunlight.

Conversely, suppose we have gas that is backlit by a strong source of photons—think of the atmosphere of a star. As the photons go through the gas, some are absorbed. Thus, the spectrum is a continuous blend of light, with darker lines corresponding to the absorption in the atmosphere. This is an *absorption line* spectrum.

When a gas becomes sufficiently dense that it is opaque, meaning that no light gets through it, then the surface emits a broad *continuous spectrum* of light, with the flux peaking at a wavelength that corresponds to the temperature of the gas. The hotter the gas, the shorter the peak wavelength.

3.5 *The Doppler Shift*

In addition to telling us about the intrinsic properties of the medium producing the spectrum—its temperature, density, and composition—the spectrum can also tell us about its velocity. Because light has wave-like behavior, it has properties in common with other waves you are familiar with, such as sound. One property that is very useful in astronomy is the *Doppler effect*: the wavelength changes depending on the motion of the source along your line of sight. To give a concrete example, suppose we have a source that is moving away from us with velocity v. We'll take v positive for motion away from us.[1]

[1] The convention here is not universal; in physics texts, v is usually taken as positive if the motion is towards the observer. In that case, replace v with $-v$ in eq. (3.4) below. As always, one must pay attention to the context before using a formula.

If the source is emitting light with wavelength λ, then the period (time between successive crests) is $T = \lambda/c$. In this time T, however, the source has moved away from us a distance vT. The tail of the wave is therefore not at a distance λ from the head, but rather at a distance $\lambda + vT$. As a result, the wavelength we receive is not λ, but rather

$$\lambda' = \lambda + vT = \lambda + \frac{v}{c}\lambda = \lambda\left(1 + \frac{v}{c}\right). \tag{3.4}$$

In deriving this equation, you may have noticed that the speed of the light wave c is unaffected by the motion of the source. Unlike other waves such as sound, a light wave always moves at a speed c regardless of the motion of either the emitter or the receiver. With light, only the relative speed of the source and observer matters in the expression for the doppler shift.

There is one further modification to equation (3.4). A consequence of c being a constant is that time passes at different rates for the emitter and receiver. The period of the wave T is what is measured at the source. The observer, however, measures that interval of time to be $T/\sqrt{1 - v^2/c^2}$. Since the wavelength is $\lambda = cT$, this means there is an additional redshift to the wavelength as well.

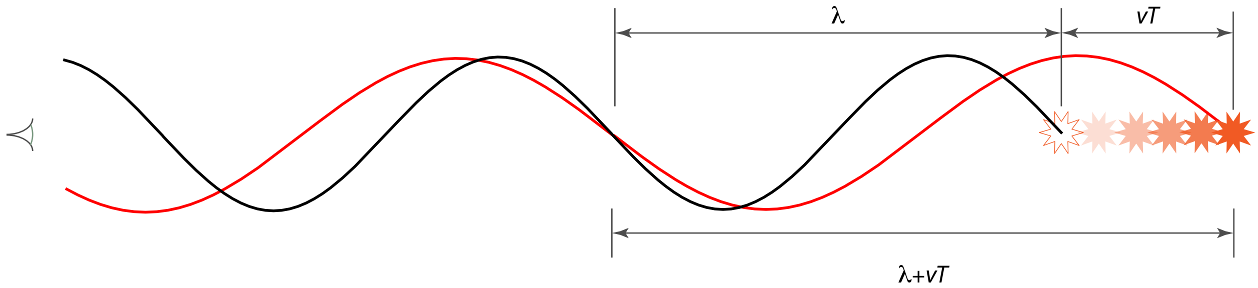


Figure 3.4: Schematic of the doppler effect for a source (red star) moving to the right at speed v.

When these changes are made, the formula for the wavelength observed from a source moving at radial velocity v is

$$\lambda_{\text{obs}} = \lambda_{\text{source}} \left[\frac{1 + v/c}{\sqrt{1 - v^2/c^2}} \right]. \tag{3.5}$$

In this equation, v is the velocity of the source along the line of sight. For a source moving towards us, the observed wavelength is shortened; as a result, a line in the middle of the visible spectrum (yellow-green) is shifted toward the blue. We call this a *blueshift*, irrespective of the actual wavelength of the light. For a source moving away from us, a line in the yellow-green is shifted toward the red; we term this a *redshift*.

EXERCISE 3.4 — A radar detector used by law enforcement measures speed by emitting a radar beam with frequency 22 GHz and measuring the frequency of the reflected signal.

1. What is the wavelength λ of the radar beam?
2. If a motorist is going 40 m/s (about 89 miles/hour) away from the officer, what is $\Delta\lambda = \lambda_{\text{motorist}} - \lambda_{\text{officer}}$? What is $\Delta\lambda/\lambda$?

4

Detection of Exoplanets

4.1 The Difficulty with Direct Detection

Suppose we want to observe exoplanets directly. Let's first estimate how far we have to look.

EXERCISE 4.1 — The density of stars in the solar neighborhood is $0.14\,\mathrm{pc}^{-3}$. Suppose 50% of the stars have planets, and we want a sample of about 20 planetary systems. What would be the radius (in parsec) of the volume containing this many systems? Given this radius, what is the average distance to a star in this sample?

Next let's estimate the difference in brightness between a planet and its host star. We shall use our solar system as an example.

EXERCISE 4.2 — The Sun, which is at a distance of 1 AU, has an apparent V-band magnitude $V_\odot = -26.74$. At its closest approach of approximately 4 AU, Jupiter has an apparent magnitude $V_{♃} = -2.94$. Compute the ratio of fluxes in V-band, i.e., $F_{♃}/F_\odot$, if both Jupiter and the Sun were at the same distance.

Finally, we know that there is a limit to the angular resolution of a telescope. This limit is imposed by both the atmospheric seeing and the telescope optics. Let's estimate how the angular separation of planet and star compares with a fiducial angular resolution.

EXERCISE 4.3 — Jupiter's mean distance from the Sun is 5.2 AU. Suppose we were to view the Sun-Jupiter system from the average distance derived in exercise 4.1; what would be the angular separation between Jupiter and the Sun? How does this compare with the atmospheric seeing under good conditions?

As these exercises illustrate, imaging a planet directly is a daunting task. Astronomers have therefore resorted to indirect means, in which the host star is observed to vary due to the influence of the planet's gravitational force. This motivates a review of Kepler's problem.

4.2 *Planetary Orbits: Kepler*

Suppose we have a exoplanet system with a planet p and a star s. The vector from the star to the planet is $\boldsymbol{r}_{sp} = \boldsymbol{r}_p - \boldsymbol{r}_s$, and the force that the star exerts on the planet is

$$\boldsymbol{F}_{sp} = -\frac{GM_pM_s}{|\boldsymbol{r}_{sp}|^3}\boldsymbol{r}_{sp}. \tag{4.1}$$

The planet exerts a force on the star $\boldsymbol{F}_{ps} = -\boldsymbol{F}_{sp}$.

To make this problem more tractable, we shall put the origin of our coordinate system at the center of mass, as shown in Fig. 4.1,

$$\boldsymbol{R} = \frac{M_s\boldsymbol{r}_s + M_p\boldsymbol{r}_p}{M_s + M_p};$$

in this frame the star and planet have positions

$$\boldsymbol{x}_s = \boldsymbol{r}_s - \boldsymbol{R} = -\frac{M_p}{M_p + M_s}\boldsymbol{r}_{sp} \tag{4.2}$$

$$\boldsymbol{x}_p = \boldsymbol{r}_p - \boldsymbol{R} = \frac{M_s}{M_p + M_s}\boldsymbol{r}_{sp} \tag{4.3}$$

and hence accelerations

$$\begin{aligned} \frac{\mathrm{d}^2\boldsymbol{x}_s}{\mathrm{d}t^2} &= -\frac{M_p}{M_p + M_s}\frac{\mathrm{d}^2\boldsymbol{r}_{sp}}{\mathrm{d}t^2} \\ \frac{\mathrm{d}^2\boldsymbol{x}_p}{\mathrm{d}t^2} &= \frac{M_s}{M_p + M_s}\frac{\mathrm{d}^2\boldsymbol{r}_{sp}}{\mathrm{d}t^2}. \end{aligned}$$

If we substitute this acceleration into the equation of motion for the planet,

$$M_p\frac{\mathrm{d}^2\boldsymbol{x}_p}{\mathrm{d}t^2} = \boldsymbol{F}_{sp},$$

and use eq. (4.1) for $\boldsymbol{F}_{sp}$, we get the reduced equation of motion

$$\frac{\mathrm{d}^2\boldsymbol{r}_{sp}}{\mathrm{d}t^2} = -G\frac{M_s + M_p}{|\boldsymbol{r}_{sp}|^3}\boldsymbol{r}_{sp}. \tag{4.4}$$

We recover this same equation if we substitute the accelerations into the equation of motion for the star. Hence for a two body problem, we only need to solve equation (4.4) for $\boldsymbol{r}_{sp}(t)$ and then use equations (4.2) and (4.3) to compute the positions $\boldsymbol{x}_s(t), \boldsymbol{x}_p(t)$ of the star and planet.

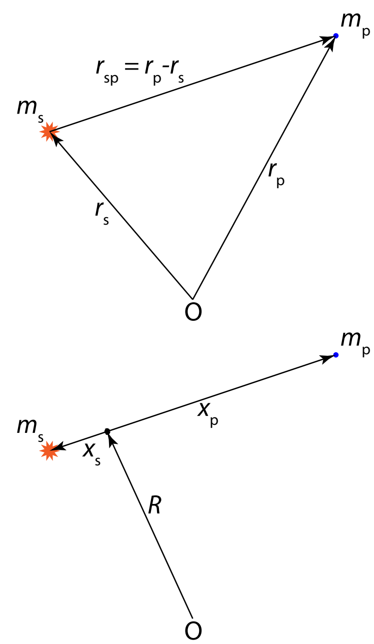


Figure 4.1: Center of mass in a star-planet system.

EXERCISE 4.4 — Locate the center of mass for the Sun-Jupiter system:

$$\frac{M_\odot}{M_♃} = 1047; \qquad \boldsymbol{r}_{\odot♃} = 5.2\,\mathrm{AU}.$$

The solution to equation (4.4) is an elliptical orbit (Fig. 4.2) with the center-of-force at one focus of the ellipse. The period T depends on the semi-major axis a of the ellipse,

$$T^2 = \frac{4\pi^2}{G(M_s + M_p)} a^3. \tag{4.5}$$

Suppose the orbit is circular, so that $|\boldsymbol{r}_{sp}| = a$ is constant. Then by combining equations (4.5) and (4.2) we can find the orbital speed of the star,

$$v_s = \frac{M_p}{M_s + M_p} \times \frac{2\pi a}{T} = \left[\frac{GM_p}{a} \frac{M_p}{M_s + M_p}\right]^{1/2}. \tag{4.6}$$

This speed is detectable via doppler shift of the stellar absorption lines.

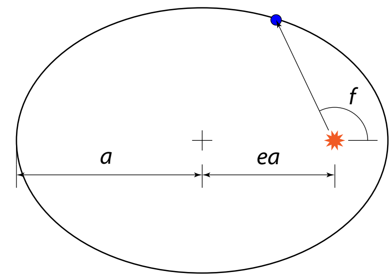


Figure 4.2: Orbital elements for a body moving in a gravitational potential about a fixed center of force, indicated by the yellow star.

EXERCISE 4.5— Compute the orbital speed of the Sun for the two-body Sun-Jupiter system;

$$\frac{M_\odot}{M_♃} = 1047; \qquad \boldsymbol{r}_{\odot ♃} = 5.2\,\mathrm{AU}.$$

EXERCISE 4.6— What is the wavelength shift induced by the motion of the Sun, computed in exercise 4.5, for an absorption line with rest wavelength 600 nm?

4.3 *Transits*

In § 3.5 we derived the doppler shift for motion along our line-of-sight. In general, however, the orbit is not edge-on, but rather inclined at an angle (Fig. 4.3). In this case the speed that is measured via doppler shift of stellar lines is $v_s \sin i$. Thus, our problem becomes, given a measurement of period T and projected speed $K = v_s \sin i$, what can we learn about the planet?

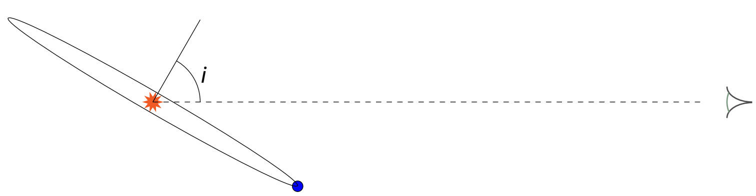


Figure 4.3: Schematic of the inclination of a planetary orbit to our line of sight.

We can combine equations (4.5) and (4.6) into the form

$$\frac{M_p^3 \sin^3 i}{(M_s + M_p)^2} = \frac{K^3 T}{2\pi G}. \tag{4.7}$$

The right-hand side is in terms of the observed quantities K and T, and is therefore determined from observations. We expect $M_s \gg M_p$, and can usually estimate M_s from spectroscopy of the star. Even with this information, we can only determine $M_p \sin i$.

For systems with sufficiently large inclination, we will observe the planet to *transit* the star, that is, to pass in front of the stellar disk. From Fig. 4.4, if

$$\cos i < \frac{R_s + R_p}{a},$$

then the light from the star will be partially blocked during some part of the orbit.[1]

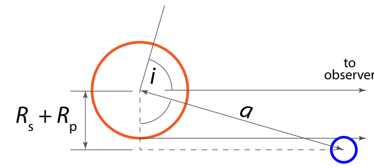


Figure 4.4: Schematic of a planetary transit.

[1] We are assuming that the star is sufficiently far away that we can ignore the angle subtended by the star.

EXERCISE 4.7 — For the Sun-Jupiter system ($R_\odot = 6.96 \times 10^5$ km, $R_{♃} = 71\,400$ km, $a = 5.2$ AU), what orbital inclination is required for an observer in a distant planetary system to witness a transit?

WHAT IS THE PROBABILITY DISTRIBUTION OF A STAR'S INCLINATION? To derive this, let's imagine each planet's orbital angular momentum as a vector having unit length. We don't care about whether, from out perspective, the planet orbits counterclockwise or clockwise, so we put all of the arrows with $0 \le i \le \pi/2$, as shown in figure 4.5.

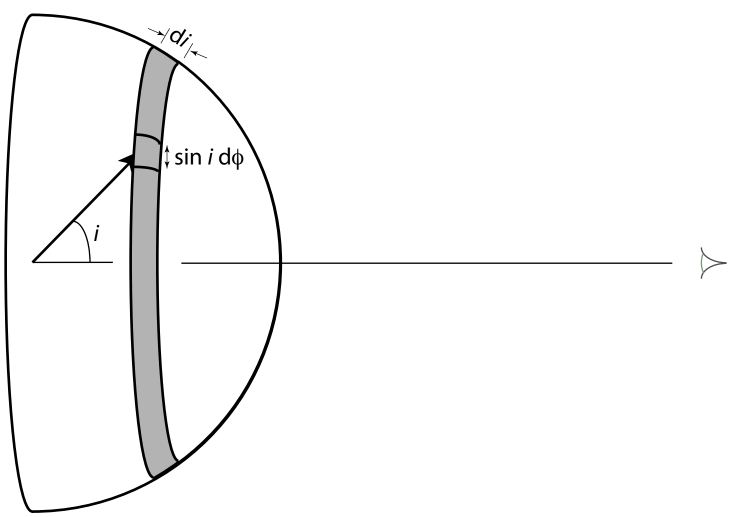


Figure 4.5: Schematic of the probability of the orbital inclination lying within $(i, i + \mathrm{d}i)$ and $(\varphi, \varphi + \mathrm{d}\varphi)$.

Now imagine a huge sample of planetary systems. If the orbits are randomly distributed, then we expect the arrows to be evenly distributed over our hemisphere; as a result, the probability of a planet having inclination in $(i, i + \mathrm{d}i)$ and azimuthal angle in $(\varphi, \varphi + \mathrm{d}\varphi)$ is the ratio of the area of that little coordinate patch to the area of the hemisphere,

$$p(i, \varphi)\, \mathrm{d}i\, \mathrm{d}\varphi = \frac{\sin i\, \mathrm{d}i\, \mathrm{d}\varphi}{2\pi}.$$

Since we aren't interested in the azimuthal angle, we can integrate over φ to find the probability distribution for a planet to have a given inclination, $p(i) = \sin i$.

EXERCISE 4.8 — From a solar-mass star you measure a periodic doppler shift with $T = 3\,\mathrm{yr}$ and $K = 18\,\mathrm{m\,s^{-1}}$. What is the probability that the planet has a mass $> 2\,M_{♃}$? What is the probability that the planet has a mass $> 10\,M_{♃}$?

EXERCISE 4.9 —

a) For an edge-on, circular orbit, show that the fraction of the orbit during which the planet is in transit is

$$f = \frac{T_{\mathrm{tr}}}{T} = \frac{R_s + R_p}{\pi a},$$

where a is the orbital separation.

b) Derive an expression for the transit duration T_{tr} in terms of a and the masses and radii of the star and planet.

c) For the Sun-Jupiter system, what is f and T_{tr}?

5

Beyond Kepler's Laws

When we studied the two-body problem, we treated the masses as simple points. In reality, they are complex extended objects. In this chapter, we'll explore some of the effects that arise when we go beyond the simple problem of two massive point particles orbiting one another.

5.1 *Tidal forces*

Because a planet is extended, the gravitational force exerted by another mass on it varies across its diameter. As a warm-up, let's imagine putting four test masses some distance from the Earth and letting them free-fall. We have a camera that is aligned with the center of mass of these four particles and that free-falls with them.

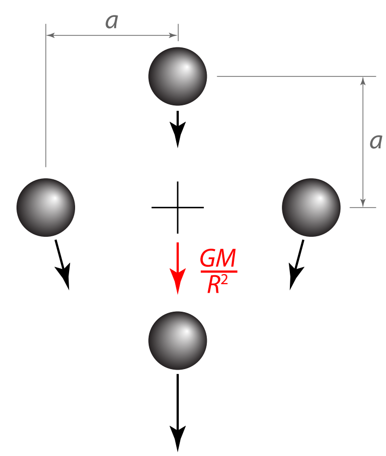


Figure 5.1: Four freely falling bodies. In a frame that falls with them, how does their motion appear?

Figure 5.1 depicts the setup: the particles are a distance a from the center of mass (indicated with a cross) and the center of mass is a distance R from the Earth's center. When we release the particles and camera, the camera and center of mass both move downward with acceleration $-GM/R^2\ \hat{\boldsymbol{z}}$. Because each particle feels a slightly different gravitational force, however, none of the particles falls with that exact acceleration: the top particle has a lower acceleration and the bottom, higher; while the left and right particles have some horizontal acceleration toward the center of mass.

EXERCISE 5.1— Compute the difference between the acceleration of each test mass and that of the center of mass. Expand this difference to lowest order in a/R. This difference is the **tidal force**. Sketch the tidal force on each particle from the point of view of the free-falling camera.

For the Earth-moon system (Fig. 5.2), we can decompose the tidal force exerted by the moon into radial and tangential components. The Earth-Moon separation is $a = 60.3R_\oplus$, so expanding our expression for the tidal force to lowest order in $R_\oplus/a$ is a good approximation.

Upon expanding the tidal acceleration components to lowest order in

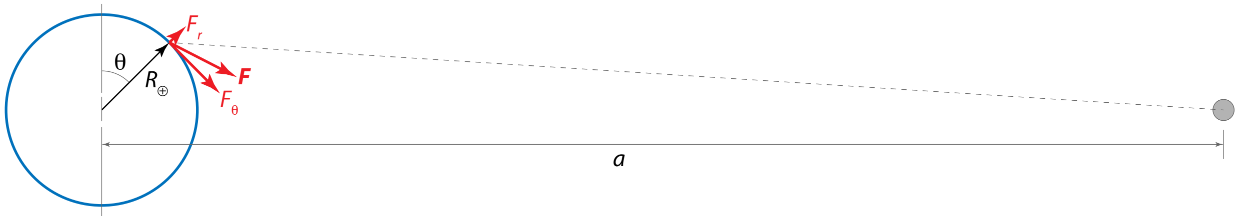


Figure 5.2: Schematic of the tidal force on the Earth raised by the Moon.

$R_\oplus/a$, the components[1] are found to be

$$\hat{\boldsymbol{r}}: \quad \frac{GM_{☾} R_\oplus}{a^3}\left(3\sin^2\theta - 1\right) \tag{5.1}$$

$$\hat{\boldsymbol{\theta}}: \quad \frac{3GM_{☾} R_\oplus}{2a^3}\sin 2\theta. \tag{5.2}$$

[1] The geometry can be worked out by consulting Fig. 5.2; it is straightforward, but tedious, and I won't go through the algebra here.

EXERCISE 5.2— Sanity check: does the radial component of the tidal force, eq. (5.1), agree with the calculation in Exercise 5.1?

The ratio of the radial component of the tidal acceleration, neglecting the angular dependence, to the Earth's surface gravity is

$$\frac{M_{☾}}{M_\oplus}\left(\frac{R_\oplus}{a}\right)^3 = 5.6\times 10^{-8}.$$

This is quite small, and you might wonder how the tidal force can produce such large daily flows of water in the ocean. But consider the tangential component, eq. (5.2): it has a maximum at $\theta = 45°$, $135°$ and, although it is also small, there is nothing to oppose it.

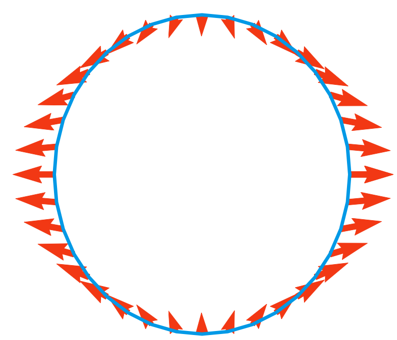


Figure 5.3: Tidal force field exerted by the Moon on the Earth.

THE EARTH'S ROTATIONAL PERIOD IS SHORTER THAN THE MOON'S ORBITAL PERIOD. Because of viscosity (resistance to flow) the tidal bulge is carried ahead of the line joining the centers of the Earth and Moon (Figure 5.4). As a result, the Moon's pull exerts a torque on the Earth and gradually slows its rotation; the oblate Earth in turn exerts a torque on the Moon and gradually forces it to greater orbital separation.

5.2 *Motion in a rotating frame*

To work out the equations of motion in a rotating frame, we start from an inertial frame in polar coordinates. In this system, the particle is located at (r,θ); the position vector of the particle is $\boldsymbol{r} = r\hat{\boldsymbol{r}}$. After an internal Δt, the particle's position is $(r+\Delta r, \theta+\Delta\theta)$, as shown in Fig. 5.5.

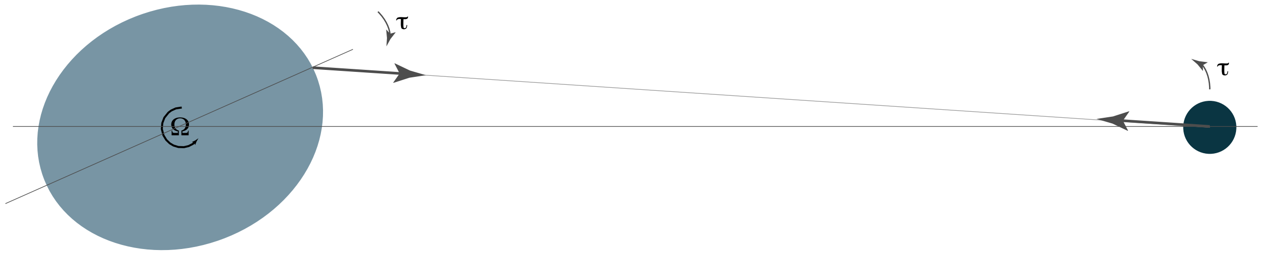


Figure 5.4: The torque resulting from the misalignment of Earth's tidal bulge.

As the particle moves, both $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$ change as well. Since both $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$ are unit vectors, only their direction changes with their magnitude remaining constant. Neither vector changes under purely radial motion, $\Delta\theta = 0$. Under a change in angle $\Delta\theta$, however, both $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$ rotate by an angle $\Delta\theta$, as shown in Fig. 5.6. In the limit $\Delta\theta \to 0$,

$$\Delta\hat{\boldsymbol{r}} \to \Delta\theta\hat{\boldsymbol{\theta}}; \quad \Delta\hat{\boldsymbol{\theta}} \to -\Delta\theta\hat{\boldsymbol{r}}.$$

Dividing by Δt and calling $\omega = \mathrm{d}\theta/\mathrm{d}t$ the angular velocity, we have $\mathrm{d}\hat{\boldsymbol{r}}/\mathrm{d}t = \omega\hat{\boldsymbol{\theta}}$ and $\mathrm{d}\hat{\boldsymbol{\theta}}/\mathrm{d}t = -\omega\hat{\boldsymbol{r}}$.

We can then differentiate the particle's position with respect to time to get its velocity in polar coordinates, and then differentiate again to get the acceleration.

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}t}\hat{\boldsymbol{r}} + r\omega\hat{\boldsymbol{\theta}}; \tag{5.3}$$

$$\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = \frac{\mathrm{d}^2r}{\mathrm{d}t^2}\hat{\boldsymbol{r}} + 2\frac{\mathrm{d}r}{\mathrm{d}t}\omega\hat{\boldsymbol{\theta}} + r\frac{\mathrm{d}\omega}{\mathrm{d}t}\hat{\boldsymbol{\theta}} - r\omega^2\hat{\boldsymbol{r}}. \tag{5.4}$$

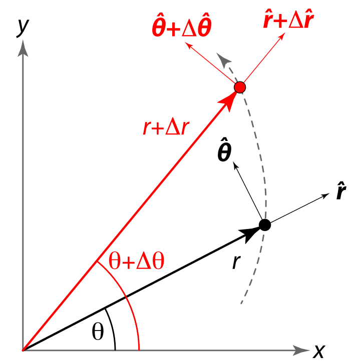


Figure 5.5: Polar coordinates for a particle.

Now suppose further that the angular velocity has two parts: $\omega = \Omega + \omega'$, a uniform rotation at velocity Ω plus a remaining portion ω'. Further, since Ω represents uniform rotation, $\mathrm{d}\Omega/\mathrm{d}t = 0$ and the acceleration is

$$\begin{aligned}\frac{1}{m}\boldsymbol{F} = \frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} &= \left(\frac{\mathrm{d}^2r}{\mathrm{d}t^2} - r\omega'^2\right)\hat{\boldsymbol{r}} + \left(2\frac{\mathrm{d}r}{\mathrm{d}t}\omega' + r\frac{\mathrm{d}\omega'}{\mathrm{d}t}\right)\hat{\boldsymbol{\theta}} \\ &\quad -r\Omega^2\hat{\boldsymbol{r}} + 2\Omega\left(\frac{\mathrm{d}r}{\mathrm{d}t}\hat{\boldsymbol{\theta}} - r\omega'\hat{\boldsymbol{r}}\right).\end{aligned} \tag{5.5}$$

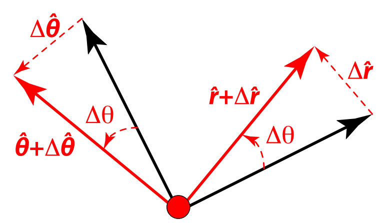


Figure 5.6: Change in the unit vectors $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$ under a change in the angular coordinate $\Delta\theta$.

Here $\boldsymbol{F}$ is the force in an inertial frame.

Now the first two terms on the right-hand side are just the acceleration $\mathrm{d}^2\boldsymbol{r}'/\mathrm{d}t^2$ that an observer rotating with velocity Ω would write down (cf. eq. [5.4]). Hence, if we move the last two terms of equation (5.5) to the left, we are left with the equations of motion in a rotating frame,

$$\frac{\mathrm{d}^2\boldsymbol{r}'}{\mathrm{d}t^2} = \frac{1}{m}\boldsymbol{F}_{\mathrm{rot}} = \frac{1}{m}\boldsymbol{F} + \underbrace{r\Omega^2\hat{\boldsymbol{r}}}_{\text{centrifugal}} + \underbrace{2\Omega\left(v_\theta\hat{\boldsymbol{r}} - v_r\hat{\boldsymbol{\theta}}\right)}_{\text{Coriolis}}. \tag{5.6}$$

We also make the identification

$$v_r = \mathrm{d}r/\mathrm{d}t, \quad v_\theta = r\omega'.$$

The **centrifugal** force is outwards (along $\hat{\boldsymbol{r}}$); the **Coriolis** force depends on velocity and deflects the motion of a particle at right angles to its velocity[2]. If you've ever tried to walk in a straight line on a spinning merry-go-round, then you've met the Coriolis force.

[2] That is, if you are moving in the $\hat{\boldsymbol{r}}$ direction, the Coriolis force is in the $\hat{\boldsymbol{\theta}}$ direction, and vice versa.

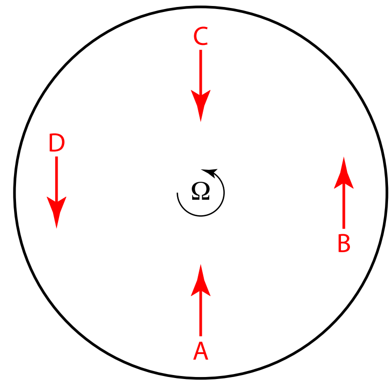


Figure 5.7: Schematic for Exercise 5.3.

EXERCISE 5.3— Figure 5.7 depicts a merry-go-round rotating counter-clockwise with velocity $\Omega > 0$. Four points, A–D are moving as shown. Draw the deflections of their trajectories due to the Coriolis force.

5.3 *Lagrange and Roche*

For analyzing the motion of a test particle in the vicinity of two massive orbiting bodies, we transform to a frame with an origin at the center of mass mass and with an angular velocity Ω. The bodies have masses M_1 and M_2, and we take M_1 to be the more massive of the two bodies. The two bodies are located at coordinates

$$M_1: \qquad x_1 = -a\frac{M_2}{M},\ y_1 = 0; \tag{5.7}$$

$$M_2: \qquad x_2 = a\frac{M_1}{M},\ y_2 = 0, \tag{5.8}$$

Here $M = M_1 + M_2$ is the total mass of the two bodies and a their separation.

Let's check that our rotating coordinate system is consistent: since M_2 is at rest, the net force on it vanishes, so from equation (5.6),

$$-\frac{GM_1}{a^2} + a\frac{M_1}{M_1+M_2}\Omega^2 = 0,$$

or

$$P_{\mathrm{orb}}^2 = \left(\frac{2\pi}{\Omega}\right)^2 = \frac{4\pi^2}{GM}a^3.$$

This is just what we would expect from Kepler's law.

Remember, "at rest in this frame" means the particle is co-rotating with our two masses.

Now we are in a position to ask, are there any points where a particle could sit at rest in this frame? Between the two masses, for example, we expect that the net force must vanish at some point. The acceleration of a test mass is

$$\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = -\frac{GM_1}{|\boldsymbol{r}-\boldsymbol{r}_1|^3}(\boldsymbol{r}-\boldsymbol{r}_1) - \frac{GM_2}{|\boldsymbol{r}-\boldsymbol{r}_2|^3}(\boldsymbol{r}-\boldsymbol{r}_2) + \frac{G(M_1+M_2)}{a^3}\boldsymbol{r}. \tag{5.9}$$

Along the x-axis, points where a particle would feel no acceleration are given by the roots of the equations

$$x < x_1: \qquad \frac{GM_1}{(x_1-x)^2} + \frac{GM_2}{(x_2-x)^2} + \frac{G(M_1+M_2)}{a^3}x = 0;$$

$$x_1 < x < x_2: \qquad -\frac{GM_1}{(x-x_1)^2} + \frac{GM_2}{(x_2-x)^2} + \frac{G(M_1+M_2)}{a^3}x = 0;$$

$$x_2 < x: \qquad \frac{GM_1}{(x-x_1)^2} + \frac{GM_2}{(x-x_2)^2} + \frac{G(M_1+M_2)}{a^3}x = 0.$$

This is a nasty quintic equation; if, however, we take the limit $M_2 \ll M_1$ then after some inspired algebra we find that there are three roots, which are the first three **Lagrange points**:

$$
\begin{aligned}
&L_1 \qquad && x_{L1} \approx a\left\{\frac{M_1}{M_1+M_2} - \left[\frac{M_2}{3(M_1+M_2)}\right]^{1/3}\right\}; \\
&L_2 \qquad && x_{L2} \approx a\left\{\frac{M_1}{M_1+M_2} + \left[\frac{M_2}{3(M_1+M_2)}\right]^{1/3}\right\}; \\
&L_3 \qquad && x_{L3} \approx a\left\{-\frac{M_1+2M_2}{M_1+M_2} + \frac{7M_2}{12M_1}\right\}.
\end{aligned}
$$

These points are depicted in Fig. 5.8 for a system with $M_2 = 0.05\,M_1$. The remaining two Lagrange points L_4 and L_5 form equilateral triangles with M_1 and M_2.

We can draw an equipotential surface (in the rotating frame) that crosses through L_1: the surface is dumbbell-shaped and forms two **Roche lobes** (Fig. 5.8) that touch at L_1. Within each lobe the gradient of the potential is inward toward the center of the lobe.

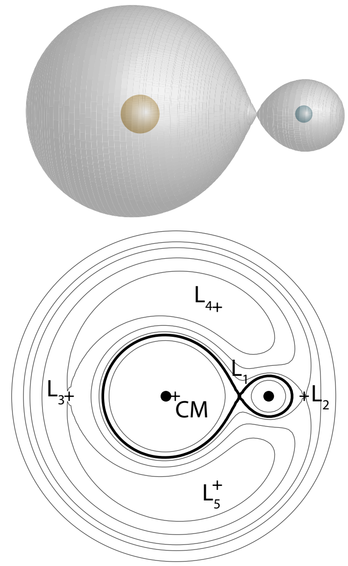


Figure 5.8: Lagrange points for a system with $M_2 = 0.1\,M_1$.

EXERCISE 5.4 — Show that the acceleration vanishes at L_4:

a) Find the coordinates of L_4;

b) Compute the net gravitational acceleration, due to both M_1 and M_2, on a particle at point L_4 and show that it points toward the center of mass; then

c) Show that the gravitational acceleration cancels the centrifugal, so that the net acceleration vanishes.

FROM THE EXPRESSIONS FOR L_1 AND L_2, WE NOTICE THAT they can be written as [3]

$$x_{L1} \approx x_2 - R_{\mathrm{H}}; \quad x_{L2} \approx x_2 + R_{\mathrm{H}},$$

with

$$R_{\mathrm{H}} \approx a\left[\frac{M_2}{3(M_1+M_2)}\right]^{1/3}.$$

Particles within a sphere of radius R_{H} are dominated by the gravitational attraction of M_2; R_{H} is called the **Hill radius**.

[3] Recall that

$$a\frac{M_1}{M_1+M_2} = x_2,$$

the location of body 2.

EXERCISE 5.5 — Compute the Hill radius for the Sun-Jupiter system.

EXERCISE 5.6 — Speculate on what would happen if M_2 had an atmosphere that extended outside its Roche lobe.

6
Planetary Atmospheres

> It's more important to know whether there will be weather than what the weather will be. —Norton Juster, *The Phantom Tollbooth*

6.1 *Hydrostatic equilibrium*

Let's consider a fluid at rest in a gravitational field. By *at rest*, we simply mean that the fluid velocity is sufficiently small that we can neglect the inertia of the moving fluid in our equation for force balance. By a *fluid*, we mean that the pressure is isotropic[1] and directed perpendicular to a surface. Let's now imagine a small fluid element, with thickness Δr and cross-sectional area ΔA, as depicted in Fig. 6.1.

[1] Meaning the pressure is the same in all directions.

The weight of the fluid element is $\Delta m\, g$, where g is the gravitational acceleration and $\Delta m = \Delta A \times \Delta r \times \rho$ is the mass of the fluid element with ρ being the mass density. The force on the upper face is $\Delta A \times P(r + \Delta r)$; on the lower face, $\Delta A \times P(r)$. Here $P(r)$ is the pressure. For the element to be in hydrostatic equilibrium the forces must balance,

$$\Delta A\left[-P(r+\Delta r) + P(r) - \Delta r \rho g\right] = 0;$$

dividing by Δr and taking the limit $\Delta r \to 0$ gives us the equation of hydrostatic equilibrium:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\rho g. \tag{6.1}$$

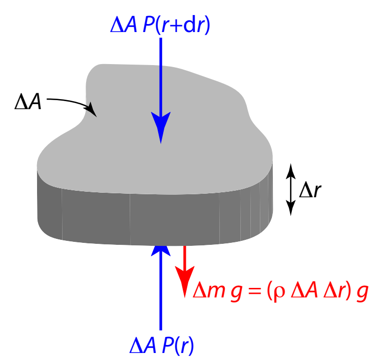


Figure 6.1: A fluid element in hydrostatic equilibrium.

For an incompressible fluid in constant gravity, the pressure increases linearly with depth. This is a good approximation to the pressure in Earth's oceans: the density of sea water changes by less than 5% between surface and floor. In general, however, the density ρ depends on the pressure P, and we need more information to solve for the atmospheric structure.

The SI unit of pressure is the **Pascal**: $1\,\mathrm{Pa} = 1\,\mathrm{N\,m^{-2}}$. The mean pressure at terrestrial sea level is $1\,\mathrm{atm} = 1.013 \times 10^5\,\mathrm{Pa}$. Other common units of pressure are the **bar** ($1\,\mathrm{bar} = 10^5\,\mathrm{Pa}$) and the **Torr** ($760\,\mathrm{Torr} = 1\,\mathrm{atm}$).

EXERCISE 6.1 — Water is nearly incompressible and has a density of $10^3\,\mathrm{kg\,m^{-3}}$. How deep would you need to dive for the pressure to increase by $1\,\mathrm{atm} = 1.013 \times 10^5\,\mathrm{Pa}$? The gravitational acceleration at Earth's surface is $9.8\,\mathrm{m\,s^{-2}}$.

Let's look at this in a bit more detail. Suppose we take our fluid layer to be thin, so that g is approximately constant. Then we can write equation (6.1) as

$$\int_{P_0}^{P(z)} \mathrm{d}P = -g \int_0^z \rho\,\mathrm{d}z.$$

Now consider a cylinder of cross-section ΔA that extends from 0 to z. The mass of that cylinder is

$$m(z) = \Delta A \times \int_0^z \rho\,\mathrm{d}z.$$

and its weight is $m(z)g$.

The difference in pressure between the bottom and top of the cylinder is just

$$P_0 - P(z) = gm(z)/\Delta A,$$

that is, the weight per unit area of our column. Let's apply this to our atmosphere: if we take the top of our column to infinity and the pressure at the top to zero, then the pressure at the bottom (sea level) is just the weight of a column of atmosphere with a cross-sectional area of $1\,\mathrm{m}^2$.

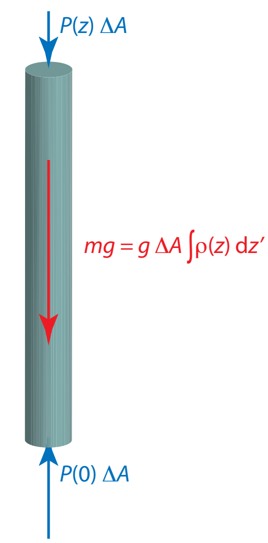


Figure 6.2: The mass of a column of fluid.

6.2 *The ideal gas*

To solve equation (6.1) we need at a minimum a relation between pressure and density. A relation between pressure, density, and temperature is called an **equation of state**. For an ideal gas[2] of N particles in a volume V at pressure and temperature P and T, the equation of state is

$$PV = NkT \tag{6.2}$$

where $k = 1.381 \times 10^{-23}\,\mathrm{J\,K^{-1}}$ is **Boltzmann's constant**.

In chemistry, it is convenient to count the number of particles by **moles**. One mole of a gas has $N_A = 6.022 \times 10^{23}$ particles[3], and the number of moles in a sample is $n = N/N_A$. If we divide and multiply equation (6.2) by N_A, then our ideal gas equation becomes

$$PV = n\,[N_A k]\,T \equiv nRT,$$

where $R = N_A k = 8.314\,\mathrm{J\,K^{-1}\,mol^{-1}}$ is the gas constant. This is perhaps the most familiar form of the ideal gas law—but it is not in a form useful to astronomers.

We astronomers don't care about little beakers of fluid—we have whole worlds to model! Let's take our ideal gas law and introduce the molar weight m as the mass of one mole of our gas. Then the ideal gas law can be written

$$P = \left(\frac{mN/N_A}{V}\right)\frac{kN_A}{m}T \equiv \rho\frac{kN_A}{m}T. \tag{6.3}$$

[2] By *ideal* gas, we mean that the particles are non-interacting; as a result, the energy of the gas only depends on the kinetic energy of the particles and in particular is independent of the volume.

[3] The constant N_A is known as **Avogadro's number**.

The quantity in parenthesis is the mass per volume, or density ρ, of our fluid. This is the same mass density that appears in equation (6.1). Equation (6.3) is the form most convenient for fluid dynamics, because it is in terms of intrinsic fluid properties rather than in terms of a laboratory quantity like volume.

6.3 The scale height

Let's take a first stab at modeling Earth's atmosphere with equation (6.1). We'll take Earth's atmosphere to be an ideal gas and for simplicity we'll assume the temperature doesn't change with altitude[4]. The molar weight of dry[5] air is $0.02897\,\mathrm{kg\,mol^{-1}}$. Using equation (6.3) to eliminate ρ in equation (6.1), we obtain

$$\frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}z} = -\frac{mg}{N_A kT}, \quad \text{or} \quad \frac{\mathrm{d}P}{P} = -\frac{mg}{N_A kT}\,\mathrm{d}z.$$

Integrating from $z = 0$, where $P(z = 0) = P_0$, to a height z gives us an equation for the pressure as a function of height,

$$P(z) = P_0 \exp\left[-\frac{mgz}{N_A kT}\right]. \tag{6.4}$$

Since the argument of the exponential is dimensionless, we see that we can write $P(z) = P_0 e^{-z/H_P}$, where

$$H_P = \frac{N_A kT}{mg}$$

is the **pressure scale height**—the height over which the pressure decreases by a factor $1/e$.

EXERCISE 6.2— Evaluate H_P for dry air at a temperature of 288 K (15 °C). Check that your answer is reasonable based on your experience. In fact, this value of H_P is overly large because the temperature in the troposphere does, in fact, decrease with height at an average **lapse rate** of

$$\frac{\mathrm{d}T}{\mathrm{d}z} = -6.5\,^\circ\mathrm{C\,km^{-1}}.$$

6.4 The adiabatic thermal gradient

Hot air rises. This simple phenomenon sets the lapse rate in the troposphere. Warm surface air rises quickly enough that there is little exchange of heat with colder, downward moving air. As a result, the fluid motions are **adiabatic**. To understand what this means, recall the first law of thermodynamics[6], which relates the change in internal energy $\mathrm{d}U$

[4] This isn't true, of course, but let's keep things simple and see how we do.

[5] The water vapor content of air varies considerably depending on ambient conditions.

[6] Enrico Fermi. *Thermodynamics*. Dover, 1956

and in volume $\mathrm{d}V$ to the heat transferred $\mathrm{d}Q$:

$$\mathrm{d}Q = \mathrm{d}U + P\mathrm{d}V, \tag{6.5}$$

where P is the pressure. Now, we aren't using volume to describe our fluid[7] so let's apply this equation to 1 mol of our fluid, and divide both sides by the molar mass m. Then Q refers to the heat transferred *per kilogram*, and U refers to the internal energy *per kilogram*. Instead of $\mathrm{d}V$, we then have $\mathrm{d}V/(1\,\mathrm{mol} \times m) = \mathrm{d}(1/\rho) = -\rho^{-2}\mathrm{d}\rho$. Our first law, rewritten in terms of mass-specific quantities, is thus

[7] cf. eq. (6.3)

$$\mathrm{d}Q = \mathrm{d}U - \frac{P}{\rho^2}\mathrm{d}\rho. \tag{6.6}$$

Suppose we wish to express quantities in terms of temperature T and density ρ: then

$$\mathrm{d}U = \left(\frac{\partial U}{\partial T}\right)_\rho \mathrm{d}T + \left(\frac{\partial U}{\partial \rho}\right)_T \mathrm{d}\rho,$$

and

$$\mathrm{d}Q = \left(\frac{\partial U}{\partial T}\right)_\rho \mathrm{d}T + \left[\left(\frac{\partial U}{\partial \rho}\right)_T - \frac{P}{\rho^2}\right] \mathrm{d}\rho.$$

Hence the heat needed to raise the temperature of one kilogram of fluid when holding density fixed is

$$C_\rho \equiv \left(\frac{\partial Q}{\partial T}\right)_\rho = \left(\frac{\partial U}{\partial T}\right)_\rho. \tag{6.7}$$

For an ideal gas, $U = U(T)$ and C_ρ is approximately constant; hence we may integrate equation (6.7) to obtain $U = C_\rho T + \text{const.}$.

In Eq. (6.6), the last term is $-(P/\rho)\,\mathrm{d}\rho/\rho = -(P/\rho)\,\mathrm{d}\ln\rho$. This illustrates a useful trick: take the logarithm of the equation of state, $\ln(P) = \ln(\rho) + \ln(T) + \ln(kN_\mathrm{A}/m)$, and then take the differential to obtain

$$\frac{\mathrm{d}P}{P} = \frac{\mathrm{d}\rho}{\rho} + \frac{\mathrm{d}T}{T}.$$

Now eliminate $\mathrm{d}\rho/\rho$ in the equation

$$\mathrm{d}Q = C_\rho \mathrm{d}T - \frac{P}{\rho}\frac{\mathrm{d}\rho}{\rho}$$

to obtain an expression for the heat transferred as a function of temperature and pressure,

$$\mathrm{d}Q = \left[C_\rho + \frac{P}{\rho T}\right]\mathrm{d}T - \frac{1}{\rho}\mathrm{d}P = \left[C_\rho + \frac{kN_\mathrm{A}}{m}\right]\mathrm{d}T - \frac{1}{\rho}\mathrm{d}P.$$

From this we see that the heat needed to raise the temperature of one mole when holding pressure fixed is

$$C_P \equiv \left(\frac{\partial Q}{\partial T}\right)_P = C_V + \frac{kN_\mathrm{A}}{m}. \tag{6.8}$$

The specific heat of one mole of various ideal gases is given in Table 6.1. It is important to remember that these relations for the specific heats are for an ideal gas and are not universally true.

gas	C_ρ	$C_P = C_\rho + kN_A/m$	$\gamma = C_P/C_\rho$
monatomic	$(3/2)kN_A/m$	$(5/2)kN_A/m$	$5/3$
diatomic	$(5/2)kN_A/m$	$(7/2)kN_A/m$	$7/5$

Table 6.1: Specific heats for ideal gases.

DURING CONVECTION, HOT AIR RISES AND COOL AIR DESCENDS, AND BOTH MOVE ADIABATICALLY. By adiabatically, we mean that there is no heat exchange:

$$0 = \mathrm{d}Q = C_P \mathrm{d}T - \frac{1}{\rho}\mathrm{d}P.$$

Using the ideal gas equation of state we can eliminate $\frac{1}{\rho} = (kN_A/m)T/P$ and write

$$\frac{\mathrm{d}T}{T} = \frac{kN_A}{mC_P}\frac{\mathrm{d}P}{P} = \frac{C_P - C_V}{C_P}\frac{\mathrm{d}P}{P} = \frac{\gamma-1}{\gamma}\frac{\mathrm{d}P}{P}.$$

Integrating both sides of the equation gives

$$\ln T = \frac{\gamma-1}{\gamma}\ln P + \text{const.},$$

or

$$T = T_0\left(\frac{P}{P_0}\right)^{(\gamma-1)/\gamma}, \tag{6.9}$$

where T_0 and P_0 are the temperature and pressure at the beginning of the adiabatic process. Equation (6.9) tells us how the temperature changes with pressure along an adiabat for an ideal gas.

EXERCISE 6.3— Use equations (6.9) and (6.1) to compute the lapse rate $\mathrm{d}T/\mathrm{d}z$ at sea level. Dry air is composed of mostly diatomic gases with a molar weight $0.02897\,\mathrm{kg\,mol^{-1}}$. You should find an answer around $-10\,^\circ\mathrm{C/km}$, which is almost twice as large as the value quoted earlier. Can you guess why the value you calculated might be off? (*Hint: there is a process we haven't yet accounted for. If you want a hint, go outside and look up.*)

6.5 *Atmospheric circulation on a rotating Earth*

The Sun heats the Earth unevenly; this in turn creates pressure gradients that drive a circulation of the atmosphere and a transfer for heat from the equator polewards. The Coriolis force deflects the horizontal motion of the air, and this sets up large-scale features in the atmosphere.

Because of the Earth's rotation, in the frame of a particular location on Earth there is both a Coriolis and a centrifugal acceleration:

$$\text{Coriolis} \qquad \boldsymbol{a}_{\mathrm{Cor}} = -2\boldsymbol{\Omega}\times\boldsymbol{v} \tag{6.10}$$

$$\text{centrifugal} \qquad \boldsymbol{a}_{\mathrm{cen}} = -\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{R}) \tag{6.11}$$

where $\boldsymbol{R}$ is the location of our particle and $\boldsymbol{\Omega}$ is the rotation vector of the Earth.

The centrifugal component just depends on the latitude λ and causes the Earth to bulge at the equator to compensate. It doesn't, however, change the motion of air currents. The vertical component of the Coriolis acceleration will be quite small compared to $\boldsymbol{g}$, so we can neglect it as well. For the horizontal component, if we are at latitude λ,

$$a_{\mathrm{Cor}} = 2\Omega v \sin\lambda.$$

This acceleration is to the right in the northern hemisphere and to the left in the southern. At the equator it vanishes.

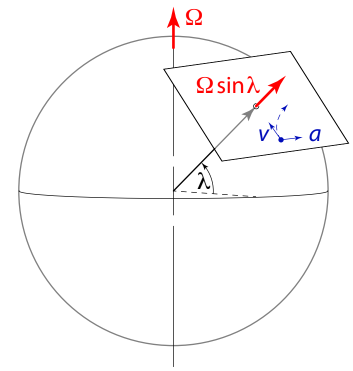


Figure 6.3: Motion in a horizontal layer in a small region at latitude λ.

EXERCISE 6.4— Suppose we have a river flowing at 3 km/hr. At our latitude, how does the Coriolis acceleration compare to the centripetal acceleration if the river has a bend with radius of curvature r? How large would r need to be for the Coriolis force to dominate?

In addition to the Coriolis acceleration from the Earth rotation, horizontal pressure gradients will also produce an acceleration

$$-\frac{1}{\rho}\boldsymbol{\nabla} P.$$

A typical horizontal gradient for a weather system is about 0.03 mbar/km. Consider a **cyclone** in which the winds swirl counterclockwise about a low. Let's look at a small parcel of fluid a distance r from the center of the cyclone, which has a height H. The mass of our fluid parcel is $\Delta S\,\Delta r\,H\,\rho$, and the acceleration of the fluid is $-v^2/r\,\hat{r}$. The equation for force and acceleration along $\hat{r}$ is therefore

Recall that 1 bar $= 1.013 \times 10^5$ Pa. The density of air at sea level is $1.3\,\mathrm{kg\,m^{-3}}$.

$$[P(r) - P(r+\Delta r)]\,\Delta S\,H + 2\Delta S\,\Delta r\,H\,\rho\,\Omega v \sin\lambda = -\Delta S\,\Delta r\,H\,\rho\frac{v^2}{r}.$$

or

$$\underbrace{\frac{v^2}{r}}_{\text{centripetal}} + \underbrace{2v\Omega\sin\lambda}_{\text{Coriolis}} - \underbrace{\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}}_{\text{pressure}} = 0. \qquad (6.12)$$

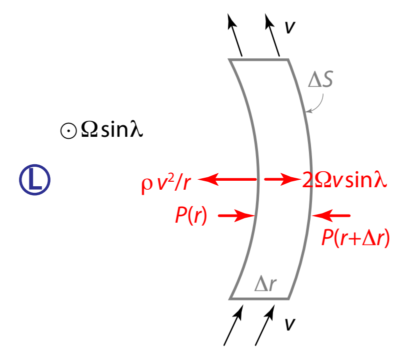


Figure 6.4: Forces on a parcel of air circulating about a low.

EXERCISE 6.5—

a) When r is sufficiently large, we can neglect the centripetal term in equation (6.12). In that case, for a pressure gradient of 3 mbar/100 km, what is a velocity satisfying this equation. Does this seem realistic?

b) Using the velocity you found in part a, determine the size r of the weather system at which the centripetal term becomes comparable to the Coriolis term.

In tropical regions the latent heat released from condensing water vapor in rising updrafts can produce a strong pressure gradient around a low—a tropical depression. If the pressure gradient is strong enough, a hurricane forms. In this case the pressure gradient can be as strong as 0.3 mbar/km, and the centripetal term cannot be neglected. If we consider a hurricane located at latitude $\lambda = 20°$ and take the eye region to have $r = 100$ km, then solving equation (6.12) gives

$$v = -r\Omega \sin\lambda + \sqrt{(r\Omega\sin\lambda)^2 + \frac{r}{\rho}\frac{dP}{dr}} \approx 46\,\mathrm{m/s},$$

which is typical of hurricane-strength winds.

EXERCISE 6.6 — You may have wondered why the strongest storms are associated with low-pressure systems. Repeat the analysis leading to equation (6.12) for air circulating in an anti-cyclone around a pressure high. There is one crucial difference in the equation which leads to a limitation on the pressure gradient and velocities in this case; explain this difference.

A

Constants and Units

A.1 Selected constants

constant	symbol	value in MKS	
speed of light	c	2.998×10^{8}	$\mathrm{m\,s^{-1}}$
Newton constant	G	6.674×10^{-11}	$\mathrm{m^3\,kg^{-1}\,s^{-2}}$
Planck constant	h	6.626×10^{-34}	$\mathrm{J\,s}$
Planck constant, reduced	$\hbar$	1.055×10^{-34}	$\mathrm{J\,s}$
Boltzmann constant	k	1.381×10^{-23}	$\mathrm{J\,K^{-1}}$
Stefan-Boltzmann constant	σ	5.670×10^{-8}	$\mathrm{W\,m^{-2}\,K^{-4}}$
	$a = 4\sigma/c$	7.566×10^{-16}	$\mathrm{J\,m^{-3}\,K^{-4}}$
mass, hydrogen atom	m_{H}	1.673×10^{-27}	kg
atomic mass unit	m_{u}	1.661×10^{-27}	kg
electron mass	m_e	9.109×10^{-31}	kg
electron volt	eV	1.602×10^{-19}	J
	Astronomical		
solar mass	$M_\odot$	1.989×10^{30}	kg
solar radius	$R_\odot$	6.960×10^{8}	m
solar luminosity	$L_\odot$	3.842×10^{26}	W
solar effective temperature	$T_{\mathrm{eff},\odot}$	5780	K
astronomical unit	AU	1.496×10^{11}	m
parsec	pc	3.086×10^{16}	m
year	yr	3.154×10^{7}	s

A.2 Properties of selected stellar types

Spectral Type	T_{eff} (K)	$M/M_\odot$	$L/L_\odot$	$R/R_\odot$	V mag.
B5	15 400	5.9	830	3.9	-1.2
G0	5 940	1.05	1.4	1.1	4.4
M5	3 170	0.21	0.0066	0.27	12.3

A.3 Planets of the solar system

Planet	symbol	a (AU)	M (10^{24} kg)	R (km)	$I/(MR^2)$
Mercury	☿	0.387	0.330	2 440	0.353
Venus	♀	0.723	4.869	6 052	0.33
Earth	♁	1.000	5.974	6 371	0.331
Mars	♂	1.524	0.642	3 390	0.365
Jupiter	♃	5.203	1900	69 900	0.254
Saturn	♄	9.543	569	58 200	0.210
Uranus	⛢	19.192	86.8	35 400	0.23
Neptune	♆	30.069	102	24 600	0.23

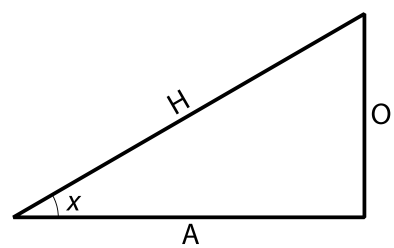
B

Mathematics Review

B.1 A Brief Refresher on Trigonometry

Definitions in terms of the unit circle

You may remember that in high school you memorized the definitions of the sine, cosine, and tangent of an angle in a right triangle. The $\sin x$ is the ratio of the side of the triangle opposite the angle x to the hypotenuse; the $\cos x$ is the ratio of the side adjacent the angle x to the hypotenuse; and the $\tan x$ is the ratio of the side opposite the angle x to the side adjacent the angle x. A useful mnemonic is S O H - C A H - T O A: Sine-Opposite-Hypotenuse — Cosine-Adjacent-Hypotenuse — Tangent-Opposite-Adjacent.



You may have wondered why the tangent, for instance, is called by that name. Now that you are a collegiate sophisticate, we can delve more deeply into how the sine, cosine, tangent, cotangent, secant, and cosecant are constructed. Draw a circle with a radius of unit length. Now draw a line from the origin O to intersect the circle at a point A, as shown in Fig. B.1. Denote by x the length along the arc from the horizontal to point A.

From the point A, we draw a vertical line to the horizontal. The length of this line AB is $\sin x$. Likewise, we draw a horizontal line from point A to the vertical; the length of this line, which is equal to OB, we call $\cos x$.

Next, we construct a line tangent to the arc at point A and extend this *tangent* to where it intersects the horizontal axis, at point C, and to where it intersects the vertical axis, at point D. We call the length of the line AC $\tan x$; the length of the line AD we call $\cot x$.

Finally, we draw from the origin O lines along the horizontal to intersect the tangent at point C and along the vertical to interest the cotangent at point D. The line from the origin O to point C is the *secant* and we call the length OC $\sec x$. The line OD is the *cosecant* and its length is $\csc x$.

The relationships between these quantities can be deduced by studying

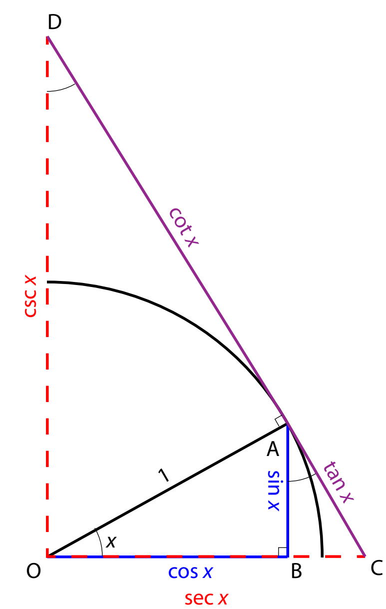


Figure B.1: Construction of the sine, tangent, secant, cosine, cotangent, and cosecant from the unit circle.

Fig. B.1. First, the triangle ABC is similar to the triangle OBA. The ratio of AC to AB is therefore equal to the ratio of OA to OB,

$$\frac{AC}{AB} = \frac{\tan x}{\sin x} = \frac{OA}{OB} = \frac{1}{\cos x}, \quad \text{so } \tan x = \frac{\sin x}{\cos x}. \tag{B.1}$$

Likewise, the triangle OAC is similar to OBA; therefore

$$\frac{OC}{OA} = \frac{\sec x}{1} = \frac{OA}{OB} = \frac{1}{\cos x}, \quad \text{so } \sec x = \frac{1}{\cos x}. \tag{B.2}$$

Finally, the triangle DAO is similar to OBA, giving

$$\frac{DO}{AO} = \frac{\csc x}{1} = \frac{OA}{BA} = \frac{1}{\sin x}, \qquad \text{so } \csc x = \frac{1}{\sin x}, \tag{B.3}$$

$$\frac{DA}{OA} = \frac{\cot x}{1} = \frac{OB}{BA} = \frac{\cos x}{\sin x}, \qquad \text{so} \cot x = \frac{\cos x}{\sin x} = \frac{1}{\tan x}. \tag{B.4}$$

Some further relations follow immediately from the Pythagorean theorem:

$$\begin{aligned} \sin^2 x + \cos^2 x &= 1 \\ \tan^2 x + 1 &= \sec^2 x \\ \cot^x x + 1 &= \csc^2 x. \end{aligned}$$

At small angles x, we see from the diagram that $\sin x < x < \tan x$, and $\cos x < 1$. Further, one can show[1] that

$$\lim_{h\to 0} \frac{\sin h}{h} = 1 \quad \text{and} \tag{B.5}$$

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = 0. \tag{B.6}$$

[1] Richard Courant, Herbert Robbins, and Ian Stewart. *What is Mathematics?* Oxford University Press, 2d edition, 1996

Trigonometric addition formulae

To develop the formulae for $\sin(x+y)$ and $\cos(x+y)$, consider the schematic in Figure B.2. The grey arc is again a segment of a unit circle. The length of OF is $\cos(x+y)$ and the length of CF is $\sin(x+y)$. By construction, $CE = \sin y$ and $OE = \cos y$.

The length $CF = \sin(x+y)$ is the sum of $DF = EG$ and CD. Now triangle CDE is similar to OBA; hence

$$\frac{CD}{CE} = \frac{CD}{\sin y} = \frac{OB}{OA} = \frac{\cos x}{1}, \quad \text{so } CD = \sin y \cos x.$$

By a similar argument, $EG = \cos y \sin x$. Hence

$$CF = \sin(x+y) = \sin x \cos y + \cos x \sin y. \tag{B.7}$$

To construct $\cos(x+y) = OF = OG - FG$, we use a similar approach to find that $FG = DE = \sin y \sin x$ and $OG = \cos y \cos x$; therefore

$$\cos(x+y) = \cos x \cos y - \sin x \sin y. \tag{B.8}$$

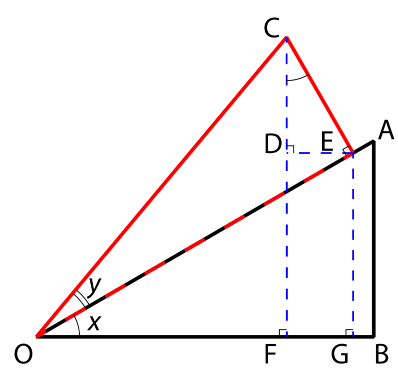


Figure B.2: Schematic of the addition of two angles x and y.

Application to calculus

We can now use equations (B.5), (B.6), (B.7), and (B.8) to establish formulae for the derivatives of the sine and cosine. For the sine, using $\lim_{h\to 0} \sin h/h = 1$ and $\lim_{h\to 0}(\cos h - 1)/h = 0$,

$$\begin{aligned}\frac{\mathrm{d}\sin x}{\mathrm{d}x} &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}\\ &= \lim_{h\to 0}\frac{\sin x\cos h + \cos x \sin h - \sin x}{h}\\ &= \cos x.\end{aligned} \tag{B.9}$$

Likewise,

$$\begin{aligned}\frac{\mathrm{d}\cos x}{\mathrm{d}x} &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}\\ &= \lim_{h\to 0}\frac{\cos x\cos h - \sin x \sin h - \cos x}{h}\\ &= -\sin x.\end{aligned} \tag{B.10}$$

The formulae for the derivatives of the tangent, cotangent, secant, and cosecant can be derived by using the chain rule on equations (B.1)–(B.4).

FOR A CONTINUOUSLY DIFFERENTIABLE FUNCTION $f(x)$, Taylor's theorem allows us to expand the function at a point $x_0 + h$ as a series in powers of h,

$$f(x_0+h) = f(x_0) + \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x_0} h + \frac{1}{2}\left.\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right|_{x_0} h^2 + \frac{1}{3\cdot 2}\left.\frac{\mathrm{d}^3 f}{\mathrm{d}x^3}\right|_{x_0} h^3 + \dots$$

See § B.2 for details.

Applying this expansion to $\sin x$ and $\cos x$ about the point $x_0 = 0$, we have to order h^2,

$$\sin h = \sin(0) + \cos(0)h - \frac{1}{2}\sin(0)h^2 + \dots \approx h, \tag{B.11}$$

$$\cos h = \cos(0) - \sin(0)h - \frac{1}{2}\cos(0)h^2 + \dots \approx 1 - \frac{h^2}{2} \tag{B.12}$$

since $\sin(0) = 0$, $\cos(0) = 1$.

B.2 The Taylor Expansion

Suppose we wish to approximate a function $f(x)$ in the neighborhood of some point x_0 by a power series. That it, we wish to write for some $h \ll 1$,

$$f(x = x_0 + h) = c_0 + c_1 h + c_2 h^2 + c_3 h^3 + c_4 h^4 + \dots$$

We assume that $f(x)$ is differentiable, and all those derivatives exist—no discontinuities or places where the derivative blows up. To find the constants $c_0, c_1, c_2, c_3, \dots$, we first set $h = 0$ and obtain

$$f(x_0) = c_0,$$

which fixes the first constant. Next, we take the derivative and set $h = 0$, $x = x_0$

$$\left.\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right|_{x=x_0} = \left[c_1 + 2c_2h + 3c_3h^2 + 4c_4h^3 \ldots\right]_{h=0} = c_1.$$

For the next term, we take another derivative,

$$\left.\frac{\mathrm{d}^2f(x)}{\mathrm{d}x^2}\right|_{x=x_0} = \left[2c_2 + 3\cdot 2c_3h + 4\cdot 3c_4h^2 \ldots\right]_{h=0} = 2c_2.$$

Thus our expansion out to the term in h^2 is

$$f(x_0 + h) = f(x_0) + \left.\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right|_{x=x_0} h + \frac{1}{2}\left.\frac{\mathrm{d}^2f(x)}{\mathrm{d}x^2}\right|_{x=x_0} h^2 + \mathcal{O}(h^3).$$

Here the expressions $\mathcal{O}(h^3)$ means that the remaining terms are of the same size as h^3.

C
Probability and Statistics

> The true logic of this world is in the calculus of probabilities. —James Clerk Maxwell

ASTRONOMICAL OBSERVATIONS PRODUCE *DATA*—sets of numbers from measurements. To advance our understanding of astronomy, we must compare this data to an underlying hypothesis or model. That is, we compute some *statistic* s from the data $\{\mathcal{D}\}$ and assess the likelihood of the value of s.

As a naive example, we might sum over the differences between the predictions $\mathcal{P} = \{p_i\}$ of a model and the observations $\{D_i\}$:

$$s = \sum_i (p_i - D_i)^2 .$$

In this case, $s = 0$ would signify perfect agreement. Nothing is ever perfect, however; what would we make of s being small but non-zero? We need a figure-of-merit[1]: given some small value of s, is it *likely* that the model is consistent with the data? If we judge the value of s to be implausible, we say that the model, or hypothesis, is not supported by the data.

[1] And of course, we want to find the best choice of statistic s for assessing how well the model fits the data.

The converse case of s having a value with a high probability does *not*, however, "rule in" the hypothesis—at best, the hypothesis is consistent with observations, but other hypotheses may also be consistent. The goal is to amass an ever larger body of evidence supporting the hypothesis, but one can never prove it conclusively.

C.1 Basic Rules of Probability and Combinatorics

Having motivated the problem, we now step back and ask, what is meant by probability? We are familiar with many examples from our everyday experience. What is the probability of drawing an ace from a deck of cards? What is the probability of rain tomorrow? What is the probability that our candidate will win the election?

EXERCISE C.1— Think of some different situations in which you might use the word *probability*. How does the definition of probability differ among these situations?

It is not immediately obvious that different usages of the term are consistent. To give two examples:

1. A "fair" die is cast; we say that the probability of rolling a $\bullet$ is $\mathcal{P}(\bullet) = 1/6$. What does this mean? We may mean that if we were to roll the die a very large number of times N, or roll a large number N of dice, then the number of those tries yielding a $\bullet$ tends toward[2]

$$\mathcal{P}(\bullet) = \lim_{N\to\infty} \frac{N(\bullet)}{N} = \frac{1}{6}.$$

   Note that this is an *assertion*: if we did this experiment and found that $\mathcal{P}_{\mathrm{exp}}(\bullet) \neq 1/6$, we would claim the die is *loaded*!

2. The Newtonian constant of gravitation is

$$G = (6.67384 \pm 80) \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}.$$

   What does the "± 80" mean? It signifies that the value of G has some specified probability of lying in the interval $6.67304 \leq G \times 10^{11} \leq 6.67464$. This is a different sense of probability than that in the first example: the value of G has a single, definite value, and here the probability reflects the *degree of certainty* we attach to its measured value.

[2] This definition carries the prior assumption that all sides are equally likely and that $0 \leq \mathcal{P} \leq 1$.

TO START MAKING THIS MORE PRECISE, let's introduce some terms[3]: For an experiment or observation there is a set of all possible outcomes, called the *sample space*. A subset of possible outcomes is an *event*. We describe our events as subsets of a sample space Ω, as shown in Figure C.1. We write, e.g., $A \subset \Omega$. An impossible event is $\emptyset$, the empty set. When we say "not A" we mean A^c, the complement of A (shaded region in Fig. C.2). When we say "A or B" we mean "A or B or both" and denote this by $A \cup B$ (Fig. C.3). Finally, when we say "A and B" we write $A \cap B$ (Fig. C.4). If "A and B are mutually exclusive" then we write $A \cap B = \emptyset$ and we say that the sets are *disjoint*, like A and B in Fig. C.1.

[3] Richard Durrett. *The Essentials of Probability*. Duxbury Press, Belmont, CA, 1994

For example, if we roll two dice, there are $6 \times 6 = 36$ possible outcomes. This is our sample space. How many events are there for which the sum of the two dice is a nine? Answer: there are four such possibilities, $\{(3,6),(4,5),(5,4),(6,3)\}$. Let's call this event A. If event B denotes those rolls in which at least one die is a 3, then $A \cap B = \{(3,6),(6,3)\}$.

EXERCISE C.2 — We have a deck of cards consisting of 4 suits (♣, ♢, ♡, ♠) with 13 cards per suit (A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K). Suppose we draw one card. There are $13 \times 4 = 52$ possible outcomes.

1. How many events draw a ♠?
2. How many events draw a 4?
3. How many events draw a 4♠?
4. How many events draw a 4 or a ♠?

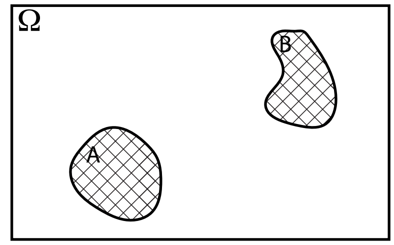


Figure C.1: Sets in Ω.

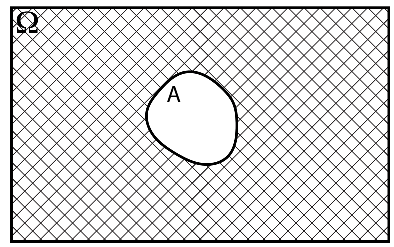


Figure C.2: The complement of $A \subset \Omega$.

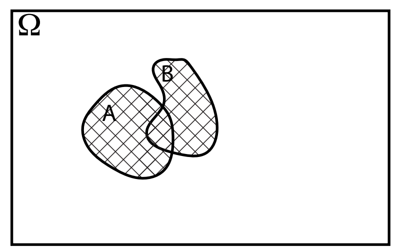


Figure C.3: $A \cup B$.

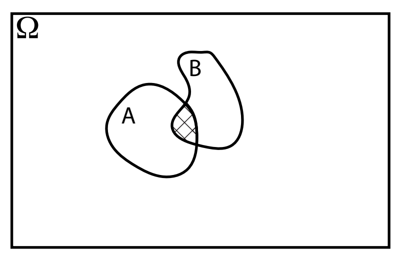


Figure C.4: $A \cap B$.

A PROBABILITY IS A RULE that assigns a number $\mathcal{P}(A)$ to an event A and obeys the following conditions:

1. $0 \le \mathcal{P}(A) \le 1$
2. $\mathcal{P}(\Omega) = 1$
3. For a set of disjoint (mutually exclusive) events[4] $\{A_i\}$,

$$\begin{aligned} \mathcal{P}(\cup_i A_i) &= \sum_i \mathcal{P}(A_i) : \\ \mathcal{P}(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_N) &= \mathcal{P}(A_1) + \mathcal{P}(A_2) + \mathcal{P}(A_3) + \ldots + \mathcal{P}(A_N). \end{aligned}$$

4. If A and B are independent—meaning that the outcome of A has no influence on the outcome of B, and vice versa—then the probability of both events occurring is $\mathcal{P}(A \cap B) = \mathcal{P}(A)\mathcal{P}(B)$.

   For example, suppose we roll a die. Each of the possible outcomes are mutually exclusive, so by rules 2 and 3,

$$1 = \mathcal{P}(\{1,2,3,4,5,6\}) = \mathcal{P}(1) + \mathcal{P}(2) + \ldots + \mathcal{P}(6).$$

If we assert that all outcomes are equally likely, $\mathcal{P}(1) = \mathcal{P}(2) = \ldots = \mathcal{P}(6) = p$, then $6p = 1$, so $p = 1/6$.

There are a few other properties of sets that are useful to know.

$$\begin{aligned} A \cup B = B \cup A, &\qquad A \cap B = B \cap A; \\ A \cap (B \cap C) = (A \cap B) \cap C, &\qquad A \cup (B \cup C) = (A \cup B) \cup C; \\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C), &\qquad A \cup (B \cap C) = (A \cup B) \cap (A \cup C); \\ A \cup A^c = \Omega, &\qquad A \cap A^c = \emptyset; \\ \Omega \cap A = A, &\qquad \emptyset \cap A = \emptyset. \end{aligned}$$

Using these properties and our rules for assigning probabilities, we can deduce a few more formulae. For example, $\mathcal{P}(A^c \cup A) = \mathcal{P}(A^c) + \mathcal{P}(A) = \mathcal{P}(\Omega) = 1$; therefore, $\mathcal{P}(A^c) = 1 - \mathcal{P}(A)$. Likewise, we can show that $\mathcal{P}(\emptyset) = 0$. Finally, we can show that if A and B are not mutually exclusive, but have some overlap, then

$$\mathcal{P}(A \cup B) = \mathcal{P}(A) + \mathcal{P}(B) - \mathcal{P}(A \cap B).$$

[4] We'll need to be careful when we deal with continuous, rather than discrete, sets.

We encountered an instance of this last rule in exercise C.2.

EXERCISE C.3 — Suppose we draw 1 card from each of 2 decks. Find the probability that at least one card is an ace, using the following two formulae:

$$\begin{aligned}\mathcal{P}(\text{“from deck 1 or from deck 2”}) &= \mathcal{P}(\text{“from deck 1”}) + \mathcal{P}(\text{“from deck 2”}) \\ &\quad -\mathcal{P}(\text{“from both deck 1 and deck 2”}). \\ \mathcal{P}(\text{“at least one ace”}) &= 1 - \mathcal{P}(\text{“drawing no ace”}).\end{aligned}$$

For an example of independent events,

$$\mathcal{P}(\text{“drawing a 4 and a } \spadesuit\text{”}) = \mathcal{P}(4)\mathcal{P}(\spadesuit) = \frac{1}{13}\frac{1}{4} = \frac{1}{52}.$$

BEFORE CONTINUING, WE NEED TO DISCUSS TECHNIQUES FOR HANDLING LARGE NUMBERS OF POSSIBLE OUTCOMES. For example, suppose we want to put 5 people in a line. How many ways are there to do this? For the first spot there are 5 choices. After assigning this first spot, we move to the second for which there are 4 choices. Proceeding along in this fashion, the number of possible arrangements is $P_5 = 5{\cdot}4{\cdot}3{\cdot}2{\cdot}1 \equiv 5!$.

The factorial fcn. is recursively defined by $m! = m \cdot (m-1)!$, with $0! = 1, 1! = 1$.

Suppose we are not picking everything; for example, in our class of 32 we may wish to pick a president, vice-president, secretary, and treasurer. The number of possibilities is

$$P_{32} = 32 \cdot 31 \cdot 30 \cdot 29 = \frac{32 \cdot 31 \cdot 30 \cdot 29 \cdot 28 \cdot \ldots \cdot 1}{28 \cdot \ldots 1} = \frac{32!}{(32-4)!}. \tag{C.1}$$

Now suppose we aren't picking individuals for distinct offices, but just 4 individuals. The order of how we pick is irrelevant—Autumn, Brook, Collin, and Dustin is the same as Brook, Dustin, Collin, and Autumn. To avoid over-counting different arrangements, we divide P_{32} from equation (C.1) by 4!, giving

$$\text{“32 choose 4”} \equiv \binom{32}{4} \equiv C_4^{32} = \frac{32!}{(32-4)!4!}. \tag{C.2}$$

More formally, $\binom{n}{m}$ is the number of ways of choosing m objects from a set of n, without regard to order.

One example you may have seen often is the expansion of a binomial. For example, suppose we wish to expand $(a+b)^5$. There are a total of $2^5 = 32$ terms of the form a^5b^0, a^4b, a^3b^2, and so on:

$$(a+b)^5 = S_5a^5 + S_4a^4b + S_3a^3b^2 + S_2a^2b^3 + S_1ab^4 + S_0b^5.$$

For S_5, there is only one way to get a^5: we must take one a from each of the terms. For S_4, we pick an a from four of the terms, and a b from the fifth. There are five ways to do this. To get S_3, we must pick an a from 3 of terms. We don't care about order, so there are $\binom{5}{3} = 5!/(2!3!) = 5 \cdot 4 \cdot 3/(3 \cdot 2) = 10$ ways to do this. Our coefficients are therefore

m	term	S_m
5	a^5	$\binom{5}{5} = 1$
4	a^4b	$\binom{5}{4} = 5$
3	a^3b^2	$\binom{5}{3} = 10$
2	a^2b^3	$\binom{5}{2} = 10$
1	a^1b^4	$\binom{5}{1} = 5$
0	b^5	$\binom{5}{0} = 1$

and

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

These coefficients $\binom{n}{m}$ obey several neat recurrence relations. When we are picking our m objects, we have a choice: to pick or not to pick the last item, item number n. If we pick the last item, then we must pick the remaining $m-1$ objects from the set $n-1$. There are $\binom{n-1}{m-1}$ ways to do so. If we do not pick the last item, then we must pick all m objects from the set $n-1$. There are $\binom{n-1}{m}$ ways to do so. The number of ways for both of these choices must add up to the total number of ways of picking m from n:

$$\binom{n}{m} = \binom{n-1}{m-1} + \binom{n-1}{m}. \tag{C.3}$$

We can make a nice table by putting the coefficients for each n on a row, with n increasing as we go down the table. If $m < 0$ or $m > n$ in one of the terms in equation (C.3), we take that term to be 0. Also, we stagger the entries, so that the terms on the RHS of equation (C.3) are diagonally to the left and right above $\binom{n}{m}$, like so:

$$\begin{array}{ccc} \binom{n-1}{m-1} & & \binom{n-1}{m} \\ & \binom{n}{m} & \end{array}.$$

This gives us the following arrangement, known as *Pascal's triangle*:

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
 & & & & & & \dots & & & & & &
\end{array}$$

EXERCISE C.4 — Show that

$$\binom{n}{m+1} = \binom{n}{m}\frac{n-m}{m+1}.$$

Then use this recurrence relation to derive $\binom{6}{m}$, $m = 1, \dots 6$, starting from $\binom{6}{0} = 1$.

We now have enough machinery to compute the probability of drawing certain hands in poker. To make this concrete, we insist on no wild cards. If we draw 5 cards from a deck, there are

$$\binom{52}{5} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2\,598\,960$$

different possible hands. Suppose we want the probability of getting a full house (3 of a kind plus one pair; e.g., 3 eights and 2 kings)? First, there are 13 possibilities for the 3 of a kind. For each of those kinds, we pick 3 out of 4 cards. We then have 12 choices for the pair, and once we have that choice we'll pick 2 of 4 cards. The number of such full house combinations is therefore

$$13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{2} = 13 \cdot 4 \cdot 12 \cdot 6 = 3744$$

and the probability of a full house is therefore $3744/2\,598\,960 = 0.0014$.

EXERCISE C.5 — What is the probability of drawing a flush (5 cards of the same suit)?

C.2 A Probability Distribution: The Random Walk

We are now ready to tackle a problem that occurs when modeling molecular motion: the random walk. Imagine a person who flips a coin before each step—heads to go right, tails to go left. On average, this person doesn't go anywhere, but from experience you know that sometimes you will get several heads or tails in a row; you wouldn't want to try this random walk if you were a few steps from the edge of a cliff!

To formulate this problem, call the probability to go right p; the probability to go left is then $(1-p)$. We wish to find the probability $\mathcal{P}_n(m;p)$ that after n steps, m will have been to the right and $(n-m)$ to the left, for a net displacement $m-(n-m) = 2m-n$ steps. Clearly $\mathcal{P}_n(m;p) = 0$ for $m > n$ or $m < 0$. Since each step is independent of the others, the probability for a specific sequence, e.g., RRLRLLRRRL, is

A positive distance means to the right; negative, to the left.

$$\mathcal{P}_n(RRLRLLRRRL) = p^6(1-p)^4 \tag{C.4}$$

since there were 6 steps to the right and 4 to the left. Of course, any sequence of 6 steps to the right and 4 steps to the left has the same probability, so to get the total probability $\mathcal{P}_{10}(6)$ of having 6 steps out of ten be to the right, we must multiply $p^6(1-p)^4$ by the number of ways of picking 6 steps out of 10 total, which is just $\binom{10}{6}$. More generally, the probability of taking m steps out of n to the right, with each step having a probability p to be to the right, is

$$\mathcal{P}_n(m;p) = \binom{n}{m} p^m (1-p)^{n-m}. \tag{C.5}$$

This function $\mathcal{P}_n(m;p)$ is called the *binomial distribution*.

For example, suppose you flip a coin 20 times. What is the probability of getting exactly 10 heads?

$$\text{Answer:} \quad \mathcal{P}_{20}(10) = \binom{20}{10}\left(\frac{1}{2}\right)^{20} = 0.176.$$

EXERCISE C.6 — Compute $\mathcal{P}_{20}(m)$ for $m = 0 \ldots 20$ and $p = 1/2$. What is the probability of getting 9, 10, or 11 heads? What is the probability of getting between 7 and 13 heads?

Of course, as the next exercise illustrates, this probability distribution occurs in many contexts, not just in the context of flipping coins or staggering home.

EXERCISE C.7 — A student takes an exam with 10 multiple choice questions, each with 4 possible responses. Suppose the student guesses randomly for each question. What is the probability the student gets 5 or more correct?

C.3 Describing the distribution

The mean

You are probably familiar with taking a simple average of a set of numbers: do a sum over the set and divide by the number of items in the set. A related quantity for a probability distribution is the *mean*,

$$\langle m \rangle \equiv \sum_{m=0}^{n} m\mathcal{P}_n(m;p). \tag{C.6}$$

To show that this behaves as expected, there are some mathematical preliminaries we need to address. First, let's demonstrate that $\sum \mathcal{P}_n(m;p) = 1$. The easiest way to do this is to take a concrete example, say $n = 5$. Then

$$\begin{aligned}\sum_{m=0}^{5} \mathcal{P}_5(m;p) &= \sum_{m=0}^{5} \binom{5}{m} p^m (1-p)^{n-m} \\ &= p^5 + 5p^4(1-p) + 10p^3(1-p)^2 + 10p^2(1-p)^3 \\ &\quad +5p(1-p)^4 + (1-p)^5.\end{aligned}$$

More formally, we can define taking the moment of a distribution with respect to a function $f(x)$ as

$$\langle f(x) \rangle = \sum f(x)\mathcal{P}(x).$$

In general, $\langle f(x) \rangle \neq f(\langle x \rangle)$.

Look familiar? You should convince yourself that this is the binomial

$$\sum_{m=0}^{5} \mathcal{P}_5(m;p) = (p+q)^5\Big|_{q=1-p} = [p + (1-p)]^5 = 1.$$

We can use this identity, $\sum \mathcal{P}_n(m;p) = \sum \binom{n}{m} p^m q^{n-m} = (p+q)^n$ with $q = 1-p$, to help us evaluate various sums over the distribution.

To evaluate the mean, we first notice that each term in the sum of equation (C.6) can be written $m\mathcal{P}_n(m;p) = \binom{n}{m} m p^m q^{n-m}$. Now $q = 1-p$; but if we temporarily let q and p vary independently, we can write

$$m\mathcal{P}_n(m;p) = \binom{n}{m} p \left(\frac{\partial}{\partial p}\right)_q p^m q^{n-m} = p \left(\frac{\partial}{\partial p}\right)_q \left[\binom{n}{m} p^m q^{n-m}\right].$$

By $\partial f/\partial x$, we mean, "the derivative of f with respect to x, holding other variables fixed." For example, if $f = f(x,y) = x^2 y e^y$, then $\partial f/\partial x = 2xye^y$ and $\partial f/\partial y = x^2(1+y)e^y$. Sometimes we write $(\partial f/\partial x)_y$ just to make it clear we are holding y fixed.

Hence, the mean is

$$\begin{aligned}
\langle m \rangle &= \sum_{m=0}^{n} m\mathcal{P}_n(m;p) \\
&= \left[p\left(\frac{\partial}{\partial p}\right)_q \sum_{m=0}^{n} \binom{n}{m} p^m q^{n-m}\right]_{q=1-p} \\
&= \left[p\left(\frac{\partial}{\partial p}\right)_q (p+q)^n\right]_{q=1-p} \\
&= \left[pn\,(p+q)^{n-1}\right]_{q=1-p} = np. \qquad \text{(C.7)}
\end{aligned}$$

This makes sense: if we flip a fair coin n times, we expect the average number of heads to be $n/2$. Notice also that for this distribution the mean is the value for which the probability is highest.

The standard deviation

Although the mean $\langle m \rangle = np$ gives the most likely value, you know from exercise C.6 that there is a substantial probability of getting values other than $\langle m \rangle$. What we want to develop is a measure for how tightly the distribution is clustered about the mean. We might want something like $\langle m - \langle m \rangle \rangle$—that is, what is the average difference from the mean—but this won't do: from the definition of the mean,

$$\langle m - \langle m \rangle \rangle = \langle m \rangle - \langle m \rangle = 0.$$

We could choose something like $\langle |m - \langle m \rangle| \rangle$, which gives a positive-definite measure of the width; a more convenient measurement, however, is the root-mean-square (rms) width $\sqrt{\langle (m - \langle m \rangle)^2 \rangle}$. To calculate this, let's first expand the square:

From the definition of the mean,

$$\begin{aligned}
\langle a + bx \rangle &= \sum \left[(a+bx)\,\mathcal{P}(x)\right] \\
&= a\sum \mathcal{P}(x) + b\sum x\mathcal{P}(x) \\
&= a + b\,\langle x \rangle\,,
\end{aligned}$$

since $\sum \mathcal{P}(x) = 1$.

$$\left\langle (m - \langle m \rangle)^2 \right\rangle = \left\langle m^2 - 2m\,\langle m \rangle + \langle m \rangle^2 \right\rangle = \left\langle m^2 \right\rangle - \langle m \rangle^2.$$

To calculate $\left\langle m^2 \right\rangle$ for the binomial distribution, we use a similar trick from the calculation of $\langle m \rangle$:

$$m^2\mathcal{P}_n(m;p) = p\left(\frac{\partial}{\partial p}\right)_q \left\{p\left(\frac{\partial}{\partial p}\right)_q \left[\binom{n}{m} p^m q^{n-m}\right]\right\}.$$

Therefore

$$\begin{aligned}
\langle m^2 \rangle &= \sum_{m=0}^{n} m^2 \mathcal{P}_n(m;p) \\
&= \left\{ p\left(\frac{\partial}{\partial p}\right)_q \left[p\left(\frac{\partial}{\partial p}\right)_q \sum_{m=0}^{n} \binom{n}{m} p^m q^{n-m} \right] \right\}_{q=1-p} \\
&= \left\{ p\left(\frac{\partial}{\partial p}\right)_q \left[p\left(\frac{\partial}{\partial p}\right)_p (p+q)^n \right] \right\}_{q=1-p} \\
&= p\left(\frac{\partial}{\partial p}\right)_q \left[pn\,(p+q)^{n-1} \right]_{q=1-p} \\
&= np + n(n-1)p^2.
\end{aligned}$$

Our expression for the rms width is therefore

$$\begin{aligned}
\left[\left\langle (m - \langle m \rangle)^2 \right\rangle\right]^{1/2} &= \left[np + (np)^2 - np^2 - (np)^2\right]^{1/2} \\
&= \left[np(1-p)\right]^{1/2}. \qquad \text{(C.8)}
\end{aligned}$$

Notice that although the width of the distribution increases with n, the ratio of the width to the average value decreases, width/mean $\propto 1/\sqrt{n}$. Thus the relative size of fluctuates about the mean decreases as n becomes larger.

C.4 *The Poisson distribution*

A limiting case that comes up often is when p is very small, but n is large. For example, suppose we are receiving X-rays from a dim source with a photon countrate $36\,\mathrm{hr}^{-1}$; that is, in one hour we receive on average just 36 photons. In any given second the probability of receiving a photon is $36\,\mathrm{hr}^{-1} \times 1\,\mathrm{hr}/3600\,\mathrm{s} = 0.01\,\mathrm{s}^{-1}$. If we point our detector at the source for 500 s, however, we can expect to receive on average $\lambda = 500\,\mathrm{s} \times 0.01\,\mathrm{s}^{-1} = 5$ photons.

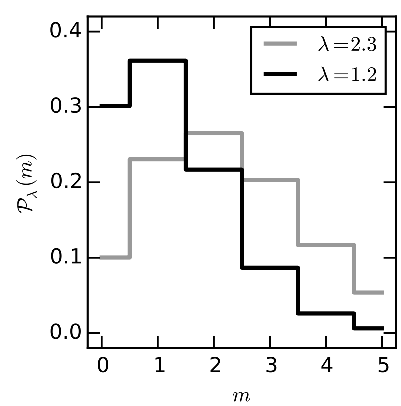


Figure C.5: The Poisson distribution for $\lambda = 2.3$ (thick gray lines) and $\lambda = 1.2$ (thin black lines).

If we take the binomial distribution, eq. (C.5), in the limit of $p \ll 1$ while holding $Np = \lambda =$ const., we obtain the *Poisson distribution*:

$$\mathcal{P}(m,\lambda) = \frac{\lambda^m}{m!} e^{-\lambda}. \qquad \text{(C.9)}$$

Thus in our example, if we have an average photon countrate of $36\,\mathrm{hr}^{-1}$ and we stare at our source for 500 s, then $\lambda = 5$ is our expected number of photons, and $\mathcal{P}(3,\lambda)$ would be the probability of receiving 3 photons in that time.

Not surprisingly, the mean number of events $\langle m \rangle$ is

$$\langle m \rangle = \sum_{m=0}^{\infty} m \frac{\lambda^m}{m!} e^{-\lambda}$$

Recall the expansion

$$e^\lambda = \sum_{m=0}^{\infty} \frac{\lambda^m}{m!}.$$

$$
\begin{aligned}
&= e^{-\lambda}\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \\
&= \lambda.
\end{aligned}
$$

The standard deviation is

$$
\begin{aligned}
\sigma = \sqrt{\left\langle (m - \langle m \rangle)^2 \right\rangle} = \sqrt{\langle m^2 \rangle - \langle m \rangle^2} &= \left[e^{-\lambda} \left(\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \right)^2 \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} - \lambda^2 \right]^{1/2} \\
&= \left[e^{-\lambda} \lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\lambda e^{\lambda} \right) - \lambda^2 \right]^{1/2} \\
&= \sqrt{\lambda}
\end{aligned}
$$

As with the normal distribution, the ratio of width to mean, $\sigma / \langle m \rangle = 1/\sqrt{\lambda}$, decreases as the mean number of events increases.

EXERCISE C.8 — A high school graduating class has 400 students. What is the probability that 2 people in that class have a birthday on January 1? What about 3 students?

A historical example of a Poisson distribution is from WWII, when London was targeted by V-1 flying bombs. A total of 537 V-1 bombs hit London. To look at where the bombs hit, Clarke[5] divided London into 576 districts, each of $0.25\,\mathrm{km}^2$ area. The distribution of bomb strikes went as follows.

[5] R. D. Clarke. An application of the poisson distribution. *Journal of the Institute of Actuaries*, 72(3):481, 001 1946. DOI: 10.1017/S0020268100035435

no. bombs	0	1	2	3	4	> 5
no. districts	229	211	93	35	7	1

That is, 229 districts were unscathed, 211 districts were hit once, and so on, with one unfortunate district being hit by 7 bombs. Question: were certain districts of London deliberately targeted? Suppose instead that the bombs were just launched in the general direction of London and fell randomly over the area. In that case, the probability of a district being hit by any one bomb is small—1/576 to be exact. The average number of bombs per district is $\lambda = 537/576 = 0.9323$, so we would expect a Poisson distribution if the bombs were distributed randomly, with the expected number of districts hit by m bombs as follows.

$$
\mathcal{N}(m) = 576 \times \mathcal{P}_\lambda(m) = 576 \times \frac{\lambda^m}{m!} e^{-\lambda},
$$

m	0	1	2	3	4	5
$\mathcal{P}_\lambda(m)$	0.3937	0.3670	0.1711	0.0532	0.0124	0.0023
$\mathcal{N}(m)$	227	211	99	31	7	1

This matches the observed distribution well, and the pattern of hits is therefore consistent with the bombs being scattered randomly over the London area.[6]

[6] This was probably small consolation to the residents of the districts hit multiple times.

C.5 An alternate derivation of the Poisson distribution

This section is not essential and can be omitted

Another probability distribution occurs in situations where the probability of an individual event p is very small but there are a large number of trials. In astronomy, this comes up frequently in looking at some sources in X- or γ-rays: the probability of receiving a single photon in any given second is small, but over a long period of time (several thousands of seconds, e.g.) there will be a sizable number of photons collected. What is the probability of receiving N photons in a given time interval?

To derive this, assume that our time intervals $\mathrm{d}t$ are sufficiently short that the chance of receiving more than one photon in $\mathrm{d}t$ is negligible. Then in $\mathrm{d}t$ we either receive one photon with probability $\mu\mathrm{d}t \ll 1$ or we receive no photons with probability $(1-\mu)\mathrm{d}t$. Let us take μ to be a constant, and assume that non-overlapping intervals of time are statistically independent—that is, the chance of receiving a photon in a given interval doesn't depend on what happened in the previous interval. Then there are two ways to receive N photons in a time $t+\mathrm{d}t$: we can receive N photons in a time t and no photons in the interval $(t, t+\mathrm{d}t)$; or we can receive $N-1$ photons in a time t and one photon in the interval $(t, t+\mathrm{d}t)$. The probability of receiving N photons in a time $t+\mathrm{d}t$ is the sum of the probabilities of these two scenarios,

$$\mathcal{P}_\mu(N; t+\mathrm{d}t) = (1-\mu\mathrm{d}t)\mathcal{P}_\mu(N;t) + \mu\mathrm{d}t\mathcal{P}_\mu(N-1;t).$$

Rearranging terms and taking the limit $\mathrm{d}t \to 0$,

$$\frac{\mathrm{d}\mathcal{P}_\mu(N;t)}{\mathrm{d}t} = \lim_{\mathrm{d}t\to 0}\frac{\mathcal{P}_\mu(N;t+\mathrm{d}t) - \mathcal{P}_\mu(N;t)}{\mathrm{d}t} = \mu\left[\mathcal{P}_\mu(N-1;t) - \mathcal{P}_\mu(N;t)\right]. \tag{C.10}$$

The probability of getting no events in $t+\mathrm{d}t$ is easier:

$$\mathcal{P}_\mu(N=0; t+\mathrm{d}t) = (1-\mu\mathrm{d}t)\mathcal{P}_\mu(N=0;t),$$

with solution $\mathcal{P}_\mu(N=0;t) = e^{-\mu t}$. We fix the constant of integration by setting $\mathcal{P}_\mu(N=0;t=0) = 1$.

Once we have $\mathcal{P}_\mu(N=0;t)$, we can solve the differential equation (C.10) for $\mathcal{P}_\mu(N=1;t)$,

$$\frac{\mathrm{d}\mathcal{P}_\mu(N=1;t)}{\mathrm{d}t} = \mu\left[e^{-\mu t} - \mathcal{P}_\mu(N=1;t)\right],$$

with solution $\mathcal{P}_\mu(N=1;t) = (\mu t)e^{-\mu t}$, as you can verify. We can then find a solution for $\mathcal{P}_\mu(N=2;t)$ and continue in that fashion to find the *Poisson distribution*:

$$\mathcal{P}_\mu(N;t) = \frac{\lambda^N}{N!}e^{-\lambda} \tag{C.11}$$

with $\lambda = \mu t$.

EXERCISE C.9— Show that $\mathcal{P}_\mu(N;t)$ (eq. [C.11]) satisfies the recursion relation, equation (C.10). Since $\mathcal{P}_\mu(N=1;t)$ also satisfies the equation, this proves via induction that equation (C.11) is the correct distribution.

C.6 The normal, or Gaussian, distribution

To motivate our discussion of the probability distribution for a continuous variable, suppose we wish to find the average location of a person doing the random walk. If all the steps have the same unit length, then for m steps to the right the position is $x = m - (n-m) = 2m - n$ with mean value $\langle x \rangle = 2\langle m \rangle - n$. Now suppose that the steps are not all of the same length, but instead have some random distribution with mean length $\langle \ell \rangle$. We'd like to know the probability, $\mathcal{P}_n(x)$, of our walker being at position x.

THE FIRST CONCEPTUAL HURDLE WE REACH IS THAT IT MAKES NO SENSE TO ASK THIS QUESTION. We cannot ask, "What is the probability $\mathcal{P}_n(x)$ of the walker being at *exactly* $x = 0.2914578329$?": there are an innumerable infinity of real numbers, so the probability of hitting any particular real number exactly is vanishingly small. What we instead must ask is, "What is the probability $\mathcal{P}_n(x;\Delta x) = p(x)\Delta x$ of being in some interval Δx about x"? We call $p(x)$ the *probability density* or *probability distribution* and assume that $\mathcal{P}(x) \propto \Delta x$ for sufficiently small Δx.

Although the formal proof is beyond the scope of the course, it can be shown that in the limit of large N,

$$p(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]; \qquad \text{(C.12)}$$

for this random walk with steps of average length $\langle s \rangle$, $\mu = (2\langle m \rangle - n)\langle s \rangle$ and $\sigma \propto \sqrt{n}\langle s \rangle$. This distribution is known as the normal, or gaussian, probability distribution. The factor $1/(\sqrt{2\pi}\sigma)$ ensures that the probability has the correct normalization, The peak, or most probable value, of this distribution is at $x = \mu$.

Recall that
$$\int_{-\infty}^{\infty} e^{-\beta x^2}\,\mathrm{d}x = \sqrt{\frac{\pi}{\beta}}.$$

We can verify that the mean of the normal distribution is μ: letting $z = x - \mu$,

$$\int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]\mathrm{d}x = \int_{-\infty}^{\infty} \frac{z}{\sqrt{2\pi}\sigma} \exp\left[-\frac{z^2}{2\sigma^2}\right]\mathrm{d}z + \int_{-\infty}^{\infty} \frac{\mu}{\sqrt{2\pi}\sigma} \exp\left[-\frac{z^2}{2\sigma^2}\right]\mathrm{d}z$$

The first term on the RHS vanishes because it is odd in z; that is,

$$\int_{-\infty}^{0} \frac{z}{\sqrt{2\pi}\sigma} \exp\left[-\frac{z^2}{2\sigma^2}\right]\mathrm{d}z = -\int_{0}^{\infty} \frac{z}{\sqrt{2\pi}\sigma} \exp\left[-\frac{z^2}{2\sigma^2}\right]\mathrm{d}z.$$

The second term is just $\mu \int_{-\infty}^{\infty} p(z)\,\mathrm{d}z = \mu$.

$$\int_{-\infty}^{\infty} p(x;\mu,\sigma)\,\mathrm{d}x = 1.$$

To compute the standard deviation, $\sqrt{\langle(x-\langle x\rangle)^2\rangle} = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$, we use the following trick:

$$\int_{-\infty}^{\infty} x^2 e^{-\beta x^2}\,\mathrm{d}x = -\frac{\partial}{\partial\beta}\int_{-\infty}^{\infty} e^{-\beta x^2}\,\mathrm{d}x = -\frac{\partial}{\partial\beta}\sqrt{\frac{\pi}{\beta}} = \frac{1}{2}\frac{\sqrt{\pi}}{\beta^{3/2}}.$$

To use this, we first change variables to $z = x - \mu$; then

$$\begin{aligned}
\left\langle x^2\right\rangle - \langle x\rangle^2 &= \int_{-\infty}^{\infty} \frac{z^2 + 2z\mu + \mu^2}{\sqrt{2\pi}\sigma} \exp\left[-\frac{z^2}{2\sigma^2}\right] \mathrm{d}z - \mu^2 \\
&= \int_{-\infty}^{\infty} \frac{z^2}{\sqrt{2\pi}\sigma} \exp\left[-\frac{z^2}{2\sigma^2}\right] \mathrm{d}z + 2\mu \int_{-\infty}^{\infty} \frac{z}{\sqrt{2\pi}\sigma} \exp\left[-\frac{z}{2\sigma^2}\right] \mathrm{d}z \\
&\quad + \mu^2 \left\{ \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{z}{2\sigma^2}\right] \mathrm{d}z - 1 \right\}.
\end{aligned}$$

You should see that the last term (in { }) is zero; also, the middle term vanishes because it is odd in z. We then use our trick to evaluate the first term and obtain,

$$\begin{aligned}
\left\langle x^2\right\rangle - \langle x\rangle^2 &= \int_{-\infty}^{\infty} \frac{z^2}{\sqrt{2\pi}\sigma} \exp\left[-\frac{z^2}{2\sigma^2}\right] \mathrm{d}z \\
&= \frac{\sqrt{\pi}}{2}(2\sigma^2)^{3/2}\frac{1}{\sqrt{2\pi}\sigma} = \sigma^2.
\end{aligned}$$

Plots of the normal distribution for different values of σ and $\mu = 0$ are shown in Figure C.7.

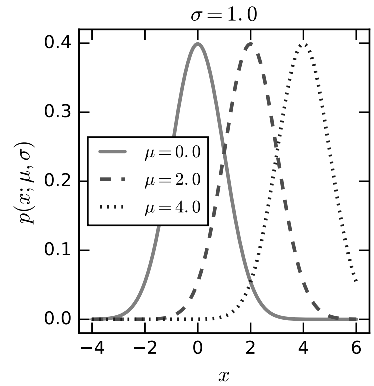


Figure C.6: Normal, or Gaussian, probability distribution for different values of μ with $\sigma = 1$.

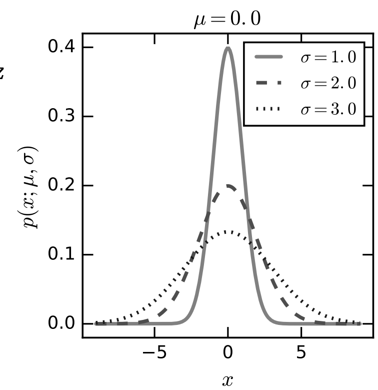


Figure C.7: Normal distribution for $\mu = 0$ and varying values of σ.

C.7 *The cumulative probability distribution*

Now that we have our probability distribution, we can ask questions such as, "For a mean value μ and standard deviation σ, what is the probability that $a < x < b$?" or "What is the probability that $x < c$?"

To answer questions like these, we integrate p over the range of interest:

$$\begin{aligned}
\mathcal{P}(a < x < b) &= \int_a^b \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] \mathrm{d}x, \\
\mathcal{P}(x < c) &= \int_{-\infty}^{c} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] \mathrm{d}x.
\end{aligned}$$

A common application is to assess the probability that a measurement will lie within some range about μ: for example, "What is the probability of measuring x in a range $(\mu - \sigma, \mu + \sigma)$?"

As shown in Figure C.8, the 1σ region (light gray) contains 68% of the probability; that is, for a normal distribution you would expect about 2/3 of your measurements to lie within $\mu \pm \sigma$. For the range $\mu \pm 2\sigma$, the probability is 95%; that is, you would expect only 1 in 20 measurements to lie outside this range.

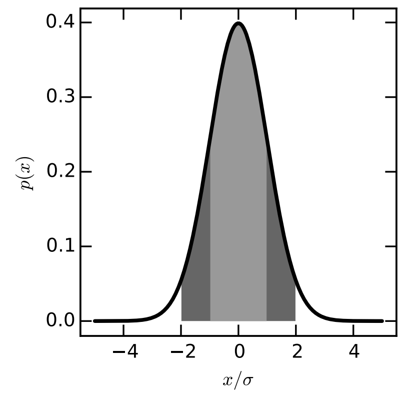


Figure C.8: The 2σ (dark gray) and 1σ (light gray) probability regions, comprising 95% and 68% probability, respectively.

C.8 *Measurements with random fluctuations*

Taking a measurement is not a simple affair. Suppose, for example, we want to measure the brightness of a star. First, we do not directly measure the brightness; what happens instead is that the light from the star is focused on a charge-coupled device (CCD)—a semiconductor chip—that converts the photons into electric charge. The charge is read out from a chip, and we relate that charge to the flux of light incident on the chip. Now, the power supply to the CCD is not perfect but has some fluctuations in voltage. There is turbulence in the atmosphere that refracts the starlight. The telescope is a big mechanical device that vibrates. And so on. If we take a set of measurements of some quantity, we will have a distribution of values; our task is to estimate the most likely value for that quantity given the measured values.

It is plausible that each of these fluctuations produces an upward or downward error in our measurement, and the size of each fluctuation varies randomly. We can therefore think of our measurement as being like a random walk: each source of variation contributes "one step", and the end result is that the value x that we measure has a probability to lie in $(x, x + \mathrm{d}x)$ of

$$p(x)\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] \mathrm{d}x.$$

In this case μ is the "true" value of the signal, which we don't know *a priori*. We also don't know beforehand the value of σ, which indicates the precision of our measurement.

SUPPOSE WE MAKE N MEASUREMENTS WITH VALUES $x_i, i = 1, \ldots, N$. Question: what is the best estimate of μ and σ? Intuitively we expect that these should be

We use $\mathcal{E}(\mu)$ to mean, "The expected value of μ", and likewise for $\mathcal{E}(\sigma)$.

$$\mathcal{E}(\mu) = \frac{1}{N}\sum x_i \quad \text{and} \quad \mathcal{E}(\sigma) = \sqrt{\frac{1}{N}\sum (x_i - \bar{x})^2},$$

respectively. To put our expectations on a firmer footing, we note that since the probability to measure a value x is $\mathcal{P}(x)$ and the N measurements $x_1, x_2, \ldots, x_N$ are independent, the probability that we measured the set $\{x_i\}_{i=1,\ldots,N}$ is

The symbol $\prod$ indicates a product,

$$\prod_{i=1}^{N} a_i \equiv a_1 \times a_2 \times \ldots \times a_N.$$

$$\mathcal{P}\left(\{x_i\}_{i=1,\ldots,N}\right) = \mathcal{P}(x_1)\mathcal{P}(x_2) \times \ldots \times \mathcal{P}(x_N)$$

$$= \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x_i-\mu)^2}{2\sigma^2}\right] \mathrm{d}x_i$$
$$= (2\pi)^{-N/2}\,\sigma^{-N} \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{N}(x_i-\mu)^2\right] \mathrm{d}x_1 \ldots \mathrm{d}x_N.$$

In the absence of additional information, our best guess for μ and σ is to pick values that maximize the probability of our measurements. To find μ, for example, we take the derivative of $\mathcal{P}\left(\{x_i\}_{i=1,\ldots,N}\right)$ with respect to μ and set it to zero to find the maximum,

Recall that
$$\frac{\mathrm{d}\exp[f(x)]}{\mathrm{d}x} = \exp[f(x)] \times \frac{\mathrm{d}f}{\mathrm{d}x}.$$

$$0 = \frac{\partial}{\partial\mu}\mathcal{P}\left(\{x_i\}_{i=1,\ldots,N}\right) = \mathcal{P}\left(\{x_i\}_{i=1,\ldots,N}\right) \times \left[\frac{1}{\sigma^2}\sum_{i=1}^{N}(x_i-\mu)\right]$$
$$= \mathcal{P}\left(\{x_i\}_{i=1,\ldots,N}\right)\frac{1}{\sigma^2} \times \left[\left(\sum_{i=1}^{N}x_i\right) - N\mu\right].$$

For this derivative to vanish, the quantity in [ ] must vanish. We therefore find that the value of μ which maximizes the probability that we made this set of measurements to be

$$\mathcal{E}(\mu) = \frac{1}{N}\sum_{i=1}^{N}x_i \equiv \bar{x}. \tag{C.13}$$

EXERCISE C.10— Show that

$$\mathcal{E}(\sigma) = \left[\frac{1}{N}\sum_{i=1}^{N}(x_i-\mu)^2\right]^{1/2}.$$

You may notice that $\mathcal{E}(\sigma)$ depends on μ, which we don't know, but must rather estimate as $\bar{x}$. If one uses $\bar{x}$ as an *estimate* for μ, then one has to make the uncertainty a bit larger,

$$\mathcal{E}(\sigma) = \left[\frac{1}{N-1}\sum_{i=1}^{N}(x_i-\bar{x})^2\right]^{1/2}.$$

To quote Press et al. [2007], "if the difference between N and $N-1$ [in the formula for $\mathcal{E}(\sigma)$] ever matters to you, then you are probably up to no good."

WE OFTEN WANT TO KNOW THE PROBABILITY DISTRIBUTION FOR A FUNCTION OF SEVERAL RANDOM VARIABLES. For example, we might measure the volume of a block as $V = L \times W \times H$, where L, W, and H are the measured length, width, and height, respectively, and each has an associated uncertainty σ_L, σ_W, and σ_H. What is the resulting uncertainty in V?

To make this general, suppose $f(\{x_i\})$ is a function of the N independent random variables $x_1, x_2, \ldots, x_N$. If the relative values of the uncertainties are small, that is $\sigma_i/x_i \ll 1$, then we can expand f about the values $x_i = \mu_i$:

$$f(\{x_i\}) \approx f(\{\mu_i\}) + \sum_{i=1}^{N}\left.\frac{\partial f}{\partial x_i}\right|_{x_i=\mu_i}(x_i-\mu_i). \tag{C.14}$$

To find the width of our distribution, we will compute

$$\sigma_f^2 = \langle (f - \langle f\rangle)^2\rangle$$

In the limit of a large number of measurements, we assume that $\langle f\rangle \approx f(\{\mu_i\})$; then

$$f - \langle f\rangle \approx \sum_{i=1}^{N} \left.\frac{\partial f}{\partial x_i}\right|_{x=\mu_i} (x_i - \mu_i)$$

and the width σ_f^2 is

$$\left\langle (f - \langle f\rangle)^2 \right\rangle = \left\langle \left[\sum_{i=1}^{N} \left.\frac{\partial f}{\partial x_i}\right|_{x=\mu_i} (x_i - \mu_i) \right]^2 \right\rangle .$$

We then expand the square of the term in [ ] to obtain

$$\left\langle \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i}\right)^2 (x_i - \mu_i)^2 + \sum_{i\neq j} \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} (x_i - \mu_i)\left(x_j - \mu_j\right) \right\rangle =$$
$$\sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i}\right)^2 \left\langle (x_i - \mu_i)^2 \right\rangle + \sum_{i\neq j} \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} \left\langle (x_i - \mu_i)\left(x_j - \mu_j\right) \right\rangle . \tag{C.15}$$

Now, if the variables x_i are completely independent, then

$$\left\langle (x_i - \mu_i)\left(x_j - \mu_j\right) \right\rangle = \langle (x_i - \mu_i)\rangle \left\langle \left(x_j - \mu_j\right) \right\rangle = 0$$

and therefore

$$\sigma_f^2 = \langle (f - \langle f\rangle)^2 \rangle = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i}\right)^2 \sigma_i^2 \tag{C.16}$$

This is an important result: it tells us how to propagate uncertainties.

To go back to our example, suppose we measure the length L, width W, and height H of a block with associated uncertainties σ_L, σ_W, and σ_H. The uncertainty in our volume is then

We use the relation $\partial V/\partial L = W \times H = V/L$, and so on for the derivatives w.r.t. W and H.

$$\sigma_V^2 = \left(\frac{V}{L}\right)^2 \sigma_L^2 + \left(\frac{V}{W}\right)^2 \sigma_W^2 + \left(\frac{V}{H}\right)^2 \sigma_H^2 \tag{C.17}$$

or

$$\frac{\sigma_V}{V} = \sqrt{\left(\frac{\sigma_L}{L}\right)^2 + \left(\frac{\sigma_W}{W}\right)^2 + \left(\frac{\sigma_H}{H}\right)^2} .$$

Using equation (C.16) we can derive general rules for propagating uncertainties.

EXERCISE C.11 — Demonstrate the following relations. In these equations, x and y are independent random variables, and a and b are constants.

1. For $f = ax + by$, show that

$$\sigma_f = \sqrt{a^2\sigma_x^2 + b^2\sigma_y^2} .$$

2. For $f = x^a y^b$, show that

$$\frac{\sigma_f}{f} = \sqrt{a^2\left(\frac{\sigma_x}{x}\right)^2 + b^2\left(\frac{\sigma_y}{y}\right)^2} .$$

AN INTERESTING SPECIAL CASE OF EQUATION (C.16) IS WHEN WE MAKE A NUMBER OF REPEATED MEASUREMENTS $x_i, i = 1, \ldots, N$ AND

$$f = \frac{1}{N} \sum_{i=1}^{N} x_i = \langle x \rangle,$$

that is, f is the average of our measurements. Since each measurement has the same uncertainty σ, the uncertainty in our average—the error in the mean—is

$$\sigma_{\langle x \rangle} = \frac{1}{N} \sqrt{\sum_{i=1}^{N} \sigma^2} = \frac{1}{N} \sqrt{N\sigma^2} = \frac{\sigma}{\sqrt{N}}. \tag{C.18}$$

By making many repeated measurements, the uncertainty in the mean due to random fluctuations can be made much less than the uncertainty of any single measurement.

EXERCISE C.12— Suppose during an election ten independent polls each show that candidate A is leading with 50.3% of the vote. Since the margin of error—the uncertainty—in each poll is $\pm 1\%$, the news anchor reports that the election is a dead heat. Is this correct? What is the probability of candidate A receiving more than 50% of the vote, assuming that the errors in the polls follow a normal (Gaussian) distribution with $\sigma = 1\%$?

NOW WE CAN GENERALIZE OUR DISCUSSION to the case of making N measurements, but with each measurement x_i having a different uncertainty σ_i. What is our estimate for μ, and what is our estimate of σ_μ, the uncertainty in μ? To answer, we go back to finding the value of μ that maximizes the probability of us obtaining a sequence of measurements $\{x_i\}_{i=1,\ldots,N}$,

$$\begin{aligned} 0 = \frac{\partial}{\partial \mu} \mathcal{P}\left(\{x_i\}_{i=1,\ldots,N}\right) &= \frac{\partial}{\partial \mu} \mathcal{P}(x_1)\mathcal{P}(x_2)\ldots\mathcal{P}(x_N) \\ &= \mathcal{P}\left(\{x_i\}_{i=1,\ldots,N}\right) \times \sum_{i=1}^{N} \frac{x_i - \mu}{\sigma_i^2} \end{aligned}$$

Hence the expected (most likely) value of μ is

$$\mathcal{E}(\mu) = \frac{\sum_{i=1}^{N} x_i/\sigma_i^2}{\sum_{i=1}^{N} 1/\sigma_i^2}. \tag{C.19}$$

Note that in the limit $\sigma_i \to \sigma$, then $\mathcal{E}(\mu) \to \sum x_i/N$, as it should.

We can compute the uncertainty in $\mathcal{E}(\mu)$ from equation (C.16),

$$\sigma_\mu^2 = \sum_{i=1}^{N} \left(\frac{\partial \mu}{\partial x_i}\right)^2 \sigma_i^2 = \frac{\sum_{i=1}^{N} 1/\sigma_i^2}{\left(\sum_{i=1}^{N} 1/\sigma_i^2\right)^2} = \frac{1}{\sum_{i=1}^{N} 1/\sigma_i^2}. \tag{C.20}$$

As before, in the limit $\sigma_i \to \sigma$, $\sigma_\mu \to \sigma/\sqrt{N}$.

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