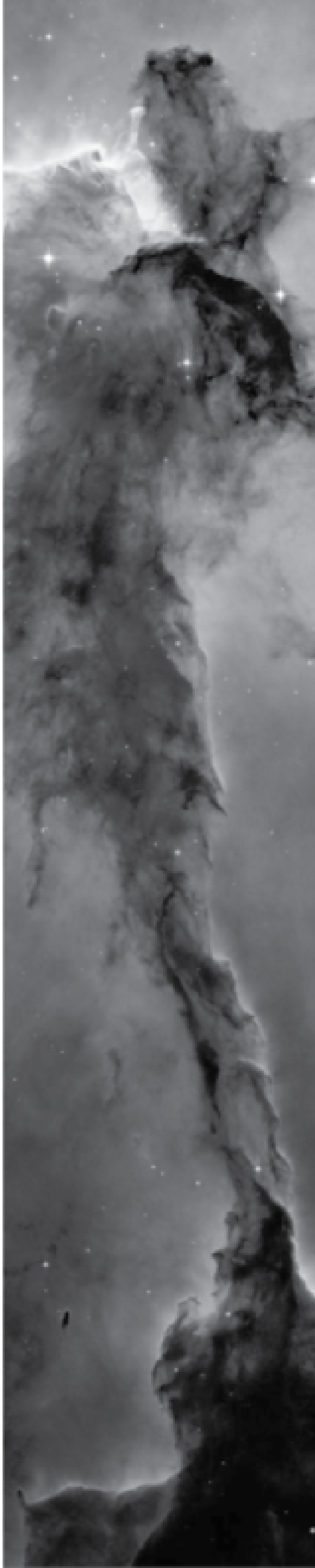
EDWARD BROWN

RADIATION IN ASTROPHYSICS



About the cover: The image is of a dust pillar in the Eagle Nebula.
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Preface

These notes are from a graduate-level course on radiative processes in astrophysics at Michigan State University. Because the course is taught in fall semesters of alternating years, the only preparation assumed is that the students have completed an undergraduate degree in physics or astronomy.

The notes are meant as a supplement to the main text, Rybicki and Lightman[1], and the secondary text, Shu[2]. The coverage therefore expands upon topics covered in those texts, rather than aiming to be a standalone monograph. The first two chapters are meant to fill in a gap between this course and undergraduate coursework on quantum mechanics and electromagnetism, since astronomy students at Michigan State do not typically take graduate-level quantum or a second semester of electromagnetism prior to taking this course.

[1] George B. Rybicki and Alan P. Lightman. *Radiative Processes in Astrophysics*. Wiley, 1979

[2] Frank H. Shu. *Radiation*, volume I of *The Physics of Astrophysics*. University Science Books, 1991

Some of the topics and the style of presentation were inspired by three courses taught at UC-Berkeley in the mid-90's: Fluid Mechanics, taught by Professor J. Graham; Radiation Astrophysics, taught by the late Professor D. Backer; and Physics of the Interstellar Medium, taught by Professor C. McKee. I also am grateful for extensive notes on these topics from Professor J. Arons. Finally, I am indebted to the students who are taking the MSU course for their questions, feedback, and encouragement.

The text layout uses the `tufte-book`[3] LaTeX class: the main feature is a large right margin in which the students can take notes; this margin also holds small figures and sidenotes. Exercises are embedded throughout the text. These range from "reading exercises" to longer, more challenging problems.

[3] https://tufte-latex.github.io/tufte-latex/

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From Coulomb to Ampère to Faraday

1.1 Maxwell's Equations for Electromagnetism

Radiation is an electromagnetic phenomenon. It is useful, therefore, to give a brief review of the governing equations of electromagnetism. As we do so, we will also indicate how the units for the actors in electromagnetism—charges and fields—are defined. Unlike much of physics and engineering, astronomy does not use the Système International (SI) units, but rather the Gaussian system of units. Hopefully this brief introduction, based on the discussion in Jackson [1975], will ease the transition from undergraduate coursework[1].

[1] For further information on different systems of units, see Appendix A.

The equations of electromagnetism are based on a few experimental relations. The first experimental relation is Coulomb's law,

$$\boldsymbol{F}_{\mathrm{C}} = k_{\mathrm{C}} \frac{q_1 q_2}{d^2} \boldsymbol{e}_r, \tag{1.1}$$

which establishes that the force F_{C} on charge q_2 due to charge q_1 is inversely proportional to the square of the distance d between them. Here k_{C} is a constant of proportionality. The unit vector $\boldsymbol{e}_r$ points along the line connecting q_1 and q_2.

In general, describing a system of charges in terms of the forces between pairs of particles is cumbersome. It is more useful to define the electric field of a charge q,

$$\boldsymbol{E} = k_{\mathrm{C}} \frac{q}{d^2} \boldsymbol{e}_r,$$

as the force on a test charge at a given position in the limit of an infinitesimally small test charge. It is found experimentally that the fields obey superposition: the electric field at a given point is the linear sum of the electric field produced by individual charges. To be completely general, we could have defined the electric field as being proportional to the force, so that $\boldsymbol{E} = k_{\mathrm{E}} k_{\mathrm{C}} q/d^2 \ \boldsymbol{e}_r$. In all commonly used systems of units, however, the electric field is defined so that $k_{\mathrm{E}} \equiv 1$; we shall not bother with this distinction any further.

If we have a system of many small, numerous charges, such that Δq is the charge in an infinitesimal volume ΔV located at position $\boldsymbol{x}$, then we can define a charge density $\rho(\boldsymbol{x}) = \Delta q/\Delta V$. Integrating the electric field over a surface enclosing a volume dV and converting to a differential relation gives the first Maxwell equation,

$$\nabla \cdot \boldsymbol{E} = 4\pi k_C \rho. \tag{1.2}$$

The second experimental relation is Ampère's law for the force per unit length between two infinitely long, parallel wires a distance d apart and carrying currents I_1 and I_2:

$$\frac{\mathrm{d}\boldsymbol{F}_\mathrm{A}}{\mathrm{d}l} = -2k_\mathrm{A}\frac{I_1 I_2}{d}\boldsymbol{e}_r. \tag{1.3}$$

Here k_A is another proportionality constant and the factor 2 is foresight. The force points along a vector $\boldsymbol{e}_r$ from wire 1 to wire 2 and is perpendicular to the wires. The force is attractive if the currents in the wires flow in the same direction.

Most systems of units define current as charge per time[2] $I = \mathrm{d}q/\mathrm{d}t$, so charge has dimension [current × time]. In analogy with the charge density, we define a current density $\boldsymbol{J}(\boldsymbol{x}, t)$ as the current per unit area. Conservation of charge means that the change in charge density at a point must be accounted for by a net divergence of the current density:

[2] In modified Gaussian units, $I = c^{-1}\mathrm{d}q/\mathrm{d}t$, so that current and charge manifestly form a 4-vector.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} = 0. \tag{1.4}$$

With this definition of current, we compute the dimensionless ratio $F_\mathrm{C}/F_\mathrm{A}$ and find that $k_\mathrm{C}/k_\mathrm{A}$ must have dimension [length/time]^2. Experimentally this ratio is found to be $k_\mathrm{C}/k_\mathrm{A} = c^2$; we can therefore choose either k_C or k_A and then the other constant is fixed.

The magnetic field is defined as the force per unit length per unit of current,

$$B = k_\mathrm{B}\frac{\mathrm{d}F_\mathrm{A}}{\mathrm{d}l}\frac{1}{I} = -2k_\mathrm{A}k_\mathrm{B}\frac{I}{d}.$$

The ratio of the electric and magnetic fields therefore has dimension

$$\left[\frac{E}{B}\right] \sim \left[\frac{\text{length}}{\text{time}}\right]\frac{1}{k_\mathrm{B}}.$$

We need the constant of proportionality k_B to allow for $\boldsymbol{B}$ having different dimensions from $\boldsymbol{E}$.

The lack of magnetic monopoles—our third experimental relation—implies that

$$\nabla \cdot \boldsymbol{B} = 0, \tag{1.5}$$

which is the second Maxwell equation. The third Maxwell equation is Faraday's law that the electromotive force—the integral of the electric field around a circuit—is proportional to the rate of change of the magnetic flux threading that circuit. In vector form,

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -k_\mathrm{F}\frac{\partial \boldsymbol{B}}{\partial t} \tag{1.6}$$

From this equation, the dimension of k_F is

$$[k_F] \sim \left[\frac{\text{time}}{\text{length}}\right] \cdot \left[\frac{E}{B}\right] \sim \left[\frac{1}{k_B}\right].$$

From the general relation between $\boldsymbol{B}$ and a system of currents we obtain an equation for magnetostatics,

$$\boldsymbol{\nabla} \times \boldsymbol{B} = 4\pi k_A k_B \boldsymbol{J}. \qquad (1.7)$$

When dealing with time-dependent phenomena, such as charging a capacitor, the four equations (1.2), (1.5), (1.6), and (1.7) do not give consistent results. Maxwell realized that the fix was to enforce charge conservation by replacing $\boldsymbol{J}$ in equation (1.7) with

$$\boldsymbol{J} \to \boldsymbol{J} + \frac{1}{4\pi k_C}\frac{\partial \boldsymbol{E}}{\partial t}.$$

This completes Maxwell's equations:

$$\nabla \cdot \boldsymbol{E} = 4\pi k_C \rho \qquad \nabla \cdot \boldsymbol{B} = 0$$
$$\boldsymbol{\nabla} \times \boldsymbol{E} = -k_F \frac{\partial \boldsymbol{B}}{\partial t} \qquad \boldsymbol{\nabla} \times \boldsymbol{B} = 4\pi k_A k_B \boldsymbol{J} + \frac{k_A k_B}{k_C}\frac{\partial \boldsymbol{E}}{\partial t}$$

If the charge density ρ and current density $\boldsymbol{J}$ are zero, then the two equations for $\boldsymbol{\nabla} \times \boldsymbol{E}$ and $\boldsymbol{\nabla} \times \boldsymbol{B}$ can be combined to give a wave equation for $\boldsymbol{B}$ and $\boldsymbol{E}$,

$$\nabla^2 \left\{ \begin{array}{c} \boldsymbol{B} \\ \boldsymbol{E} \end{array} \right\} = \frac{k_A k_B k_F}{k_C}\frac{\partial^2}{\partial t^2} \left\{ \begin{array}{c} \boldsymbol{B} \\ \boldsymbol{E} \end{array} \right\}. \qquad (1.8)$$

The wave propagation speed is

$$\sqrt{\frac{k_C}{k_A k_B k_F}} = \frac{c}{\sqrt{k_B k_F}}.$$

Since electromagnetic waves do indeed propagate with velocity c, we must have $k_F \equiv k_B^{-1}$. The vectors $\boldsymbol{E}$, $\boldsymbol{B}$, and direction of propagation $\boldsymbol{k}$ form a right-handed triad (Fig. 1.1).

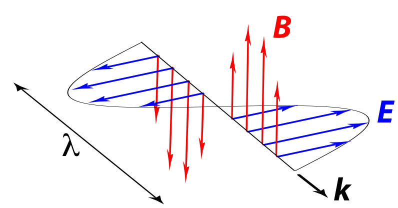

Figure 1.1: Propagation of an electromagnetic wave in free space.

Finally, from the two homogeneous equations $\nabla \cdot \boldsymbol{B} = 0$ and $\boldsymbol{\nabla} \times \boldsymbol{E} + k_F \partial \boldsymbol{B}/\partial t = 0$, we can define potentials $(\Phi, \boldsymbol{A})$ such that $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ and $\boldsymbol{E} = -\boldsymbol{\nabla}\Phi - k_F \partial \boldsymbol{A}/\partial t$. These potentials will be used in Ch. 2 when we quantize the electromagnetic field.

WE HAVE TWO INDEPENDENT CONSTANTS TO SPECIFY OUR SYSTEM OF ELECTROMAGNETIC UNITS: k_F and either k_C or k_A. For the SI system of units, the original definition of current was based on the mass of silver deposited per unit time by electrolysis in a standard silver voltameter. Because this is an *independent* definition of current, the constant k_A *must* be defined so that Ampère's law is consistent. The unit of current, known as the Ampère (A), is now defined as the amount of current that when flowing through two infinitely long wires 1 m apart produces a force per unit length of exactly $2 \times 10^{-7}\,\mathrm{N\,m^{-1}}$. With this definition, $k_A = 10^{-7}\,\mathrm{N\,A^{-2}}$ and $k_C = c^2 k_A$. For convenience, SI introduces the vacuum permeability

$$\mu_0 = 4\pi k_A = 4\pi \times 10^{-7}\,\mathrm{N\,A^{-2}}$$

and the vacuum permittivity

$$\epsilon_0 = 1/(4\pi k_C) = (4\pi k_A c^2)^{-1} = (c^2 \mu_0)^{-1}.$$

Finally, in SI $k_F = 1$; this implies that the electric and magnetic fields have different dimensions.

With these choices for k_F and k_A, Maxwell's equations are written as

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \cdot \boldsymbol{B} = 0$$
$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$$

The force on a charged particle traveling with velocity $\boldsymbol{v}$ is

$$\boldsymbol{F} = q\,(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})\,.$$

This system of units is convenient for dealing with laboratory and engineering applications. The unit of charge is given by $1\,\mathrm{A}\cdot\mathrm{s}$ and is called a *Coulomb* (C). The charge of a single electron is 1.602×10^{-19} C.

For problems involving the interaction of individual particles and photons, it is more convenient to adopt the Gaussian system of units. In this system, the speed of light c appears explicitly. We set $k_F = c^{-1}$, so that in Maxwell's equations, time derivatives are multiplied by c^{-1} and $\boldsymbol{E}$ and $\boldsymbol{B}$ have the same dimensions. Second, we choose $k_C = 1$, so that $k_A = c^{-2}$. With these choices, Maxwell's equations are written

$$\nabla \cdot \boldsymbol{E} = 4\pi\rho \qquad \nabla \cdot \boldsymbol{B} = 0$$
$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} \qquad \boldsymbol{\nabla} \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{J} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}$$

and the force on a charged particle traveling with velocity $\boldsymbol{v}$ is

$$\boldsymbol{F} = q\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}\right).$$

For historical reasons, the units of mass, length, and time in this system are the gram, the centimeter, and the second. Because $k_C = 1$, the unit of charge is therefore $(\mathrm{erg}\cdot\mathrm{cm})^{1/2}$ and is termed a *statcouloub*.

1.2 Propagation in matter: elementary treatment

When an electromagnetic wave passes through some medium, the oscillating electric field perturbs the charges in the medium; those oscillating charges in turn emit electromagnetic radiation. Some of this radiation may be sent back along the path of the original, incident wave, forming a *reflected* wave; some of this radiation adds to the forward-propagating wave and modifies it, thereby forming a *refracted* wave.

We shall develop a more thorough picture of the interaction of radiation and matter in this course; for now, however, we will just review the simplest case. Suppose the effect of the electric field is to induce an average dipole moment $\langle \boldsymbol{p} \rangle$ on each atom, so the net polarization per unit volume is $\boldsymbol{P} = n\langle \boldsymbol{p} \rangle$, where n is the density of atoms. In a macroscopically small volume (but still large enough to contain many microscopic diploes) centered at $\boldsymbol{x}'$, the potential due to the dipoles is

For the remainder of these notes, we work with Gaussian units.

$$\Phi(\boldsymbol{x}) = \int \frac{\boldsymbol{P}(\boldsymbol{x}') \cdot (\boldsymbol{x} - \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|^3} \mathrm{d}V = \int \boldsymbol{P}(\boldsymbol{x}') \cdot \boldsymbol{\nabla}' \left(\frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} \right) \mathrm{d}V.$$

Integrating by parts gives

$$\Phi(\boldsymbol{x}) = -\int \frac{\boldsymbol{\nabla}' \cdot \boldsymbol{P}(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} \mathrm{d}V.$$

This expression is just the standard formula for the potential, if we identify the induced charge density as $\rho = -\nabla \cdot \boldsymbol{P}$. We can obtain an even simpler formula, if we assume the polarization is proportional to the electric field, $\boldsymbol{P} = \chi \boldsymbol{E}$, with χ a scalar constant. Then Coulomb's law becomes[3] $\nabla \cdot \boldsymbol{E} = -4\pi \nabla \cdot \boldsymbol{P}$ or

[3] We are assuming here that there are no "free" charges present.

$$(1 + 4\pi\chi)\nabla \cdot \boldsymbol{E} \equiv \varepsilon \nabla \cdot \boldsymbol{E} = 0.$$

There is an analogous relation for the induced magnetic moment per unit volume; if the response is again linear and isotropic, Maxwell's equations become

$$\begin{aligned} \nabla \cdot \boldsymbol{E} &= 0 & \nabla \cdot \boldsymbol{B} &= 0 \\ \boldsymbol{\nabla} \times \boldsymbol{E} &= -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} & \boldsymbol{\nabla} \times \boldsymbol{B} &= \frac{\mu\varepsilon}{c}\frac{\partial \boldsymbol{E}}{\partial t}. \end{aligned}$$

The solution to this system of equations is again a traveling wave, but with propagation speed $c/\sqrt{\mu\varepsilon}$. Hence the index of refraction (the ratio of the speed of light in vacuum to the speed in a medium) is $n = \sqrt{\mu\varepsilon}$. For most non-ferromagnetic materials, $|\mu - 1| \ll 1$ and $n \approx \sqrt{\varepsilon} = \sqrt{1 + 4\pi\chi}$.

In physical terms, the electric field generated by the induced dipoles, when added to the "external" electric field, shifts the phase of the wave such that the effective propagation speed is modified.

1.3 Geometrical optics: propagation along rays

We have an intuitive feel for the propagation of light along straight paths, or rays. Our experience is based on optical wavelengths ($\sim 500\,\mathrm{nm}$)

being much smaller than ourselves. The propagation along rays is clearly not an accurate description of light when our system is small, e.g., an atom or molecule. Let's therefore examine the propagation of light when the scales over which external conditions change are much longer than the wavelength of the light itself.

In the absence of interactions with matter, we know that the light propagates as a free wave: if f is some quantity that characterizes our electromagnetic disturbance, then we can write

$$f(\boldsymbol{x}, t) = \xi \exp\left[i\left(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t\right)\right].$$

Now, in the presence of matter the propagation is not so simple; more generally,

$$f(\boldsymbol{x}, t) = a(\boldsymbol{x}, t)\exp\left[i\psi(\boldsymbol{x}, t)\right]. \tag{1.9}$$

Here ψ is the phase.

We are in the limit that the wavelength λ is much smaller than some macroscopic length scale. Then we can expand ψ about $\boldsymbol{x} = \boldsymbol{0}$, $t = 0$,

$$\psi(\boldsymbol{x}, t) \approx \psi_0 + \boldsymbol{x}\cdot\boldsymbol{\nabla}\psi + t\partial_t\psi. \tag{1.10}$$

Note that since ψ changes by 2π over a distance λ, we need $\psi_0 \gg 2\pi$. Inserting equation (1.10) into equation (1.9), we obtain

$$f \approx \left[a e^{i\psi_0}\right]\exp\left[i\boldsymbol{x}\cdot\boldsymbol{\nabla}\psi + it\partial_t\psi\right].$$

Thus, if a is also slowly varying, our variable f looks like a wave with wavenumber and frequency

$$\boldsymbol{k} = \boldsymbol{\nabla}\psi \tag{1.11}$$

$$\omega = -\partial_t\psi, \tag{1.12}$$

respectively. For this approximation to be valid, we need $\boldsymbol{k}\cdot\boldsymbol{k} = \omega^2/c^2$, or

$$\left(\boldsymbol{\nabla}\psi\right)^2 - \left(\frac{1}{c}\frac{\partial\psi}{\partial t}\right)^2 = 0. \tag{1.13}$$

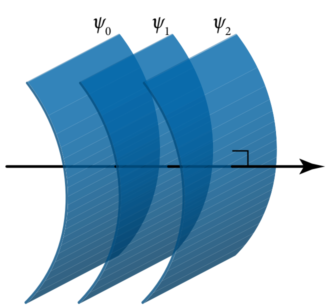

Figure 1.2: A ray (arrow) is tangent to the $\perp$ of each surface of constant phase ψ (labeled here ψ_0, ψ_1, ψ_2).

This is the *eikonal equation*, which determines the path of the ray. To define a ray, we construct at a given time surfaces of constant phase (Fig. 1.2). A given ray is tangent to the perpendicular of each surface.

DO EQUATIONS (1.11) AND (1.12) LOOK FAMILIAR? As a hint, multiply their right-hand sides by $\hbar/i$; then you might be reminded of quantum mechanics with ψ the wavefunction, $\boldsymbol{p} = (\hbar/i)\boldsymbol{\nabla}\psi$ the momentum and $\mathcal{H} = -(\hbar/i)\partial_t\psi$ the Hamiltonian. The formulation of mechanics in terms of a Hamiltonian implies that we can bring in the machinery of advanced classical mechanics to derive a better description of the path followed by a ray of light.

In classical mechanics, the analogous equations to (1.11) and (1.12) are

$$\begin{aligned} \boldsymbol{p} &= \frac{\partial S}{\partial \boldsymbol{q}} \\ \mathcal{H} &= -\frac{\partial S}{\partial t}. \end{aligned}$$

Here $\boldsymbol{p}$ and $\boldsymbol{q}$ are the generalized momenta and coordinates, and

$$S = \int_1^2 L \, \mathrm{d}t$$

is the *action*, with $L = \boldsymbol{p} \cdot \dot{\boldsymbol{q}} - \mathcal{H}$ being the Lagrangian. The path a particle takes between points 1 and 2 (Fig. 1.3) is the one that minimizes S.

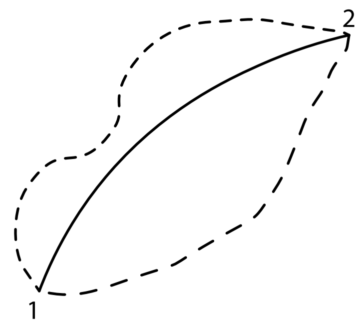

Figure 1.3: Possbile paths between two points at times t_1 and t_2.

Suppose we write the action as a function of the coordinates: $S = S(\boldsymbol{q}, t)$ where $\boldsymbol{q} = \boldsymbol{q}(t_2)$; then when we vary S we obtain

$$\delta S = \frac{\partial S}{\partial t}\delta t + \frac{\partial S}{\partial \boldsymbol{q}}\delta \boldsymbol{q} = -\mathcal{H}\delta t + \frac{\partial S}{\partial \boldsymbol{q}}\delta \boldsymbol{q}.$$

Since we are fixing $\boldsymbol{q}$, the second term vanishes and $\delta S = -\mathcal{H}\delta t$.

For all paths, let the time when the particle leaves point 1 be t_1 and the time when the particle arrives at point 2 be t_2. Further, let $\mathcal{H}$ be constant[4]. Then,

$$S = \int_1^2 L \, \mathrm{d}t = \int_1^2 (\boldsymbol{p} \cdot \dot{\boldsymbol{q}} - \mathcal{H}) \, \mathrm{d}t = \int_1^2 \boldsymbol{p} \cdot \mathrm{d}\boldsymbol{q} - \mathcal{H}(t_2 - t_1) \,. \qquad (1.14)$$

[4] This requires that there be no explicit dependence on time.

But, $t = t_2 - t_1$, so when we vary S,

$$\delta S = \delta \int_1^2 \boldsymbol{p} \cdot \mathrm{d}\boldsymbol{q} - \mathcal{H}\delta t.$$

Since $\delta S = -\mathcal{H}\delta t$, we must have

$$\delta \int_1^2 \boldsymbol{p} \cdot \mathrm{d}\boldsymbol{q} = 0 \qquad (1.15)$$

for the path taken by a particle. Translating equation (1.15) to our optics language ($\boldsymbol{p} \rightarrow \boldsymbol{\nabla}\psi$), the path a ray takes between points 1 and 2 is determined by

$$\delta \int_1^2 \boldsymbol{\nabla}\psi \cdot \mathrm{d}\boldsymbol{x} = 0. \qquad (1.16)$$

Equation (1.16) is a generalization of Fermat's principle, which you learned about in introductory optics.

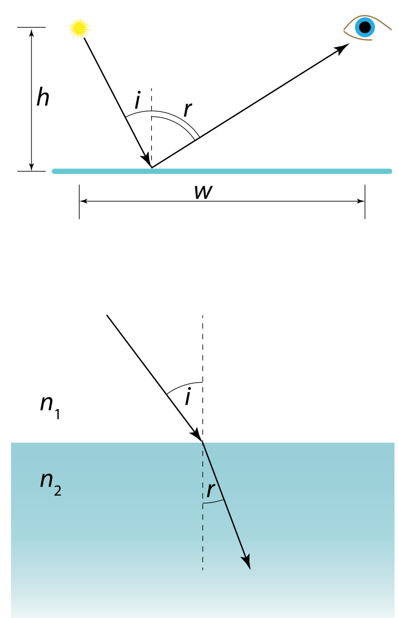

Figure 1.4: Top: reflection of light from a surface. Bottom: refraction of light as it passes from a medium with index n_1 into a medium with index n_2.

EXERCISE 1.1—

1. Suppose we reflect light as shown in the top panel of Fig. 1.4. Show that in the geometrical optics limits, $i = r$: the angles of incidence and reflection are equal.
2. Consider a ray of light incident on a pool of water as shown in the bottom panel of Fig. 1.4. Show that equation (1.16) implies Snell's law.

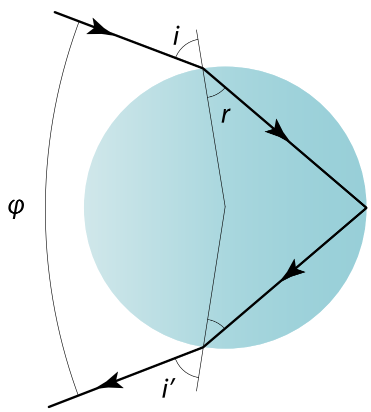

Figure 1.5: Scattering of light by water droplet with one internal reflection.

EXERCISE 1.2— Many atmospheric optical effects are caused by small droplets of water. Suppose we have a ray of light that enters a droplet of water, reflects from the back surface, and re-emerges as depicted in Figure 1.5. The ray enters with angle of incidence i and exits with angle of incidence i'; the angle between the entering and exiting rays is ϕ. We shall assume the droplet of water is sufficiently large that we may work in the geometrical optics limit.

1. Show that $i' = i$, and derive a formula for ϕ in terms of i and r.
2. Use Snell's law to relate r in terms of the angle of incidence i and index of refraction n. For water, $n \approx 4/3$; use this to plot $\phi(i)$. Argue that the backscattered light is most intense at the maximum value of ϕ.
3. Now redo part (2) for red light ($n = 1.330$ at $\lambda = 700$ nm) and violet light ($n = 1.342$ at $\lambda = 400$ nm). What is the difference $\phi_{\text{red}} - \phi_{\text{violet}}$?
4. Verify that your calculations are correct against observations.

1.4 Phase and Group Velocity

A pure wave propagates at a *phase velocity* $\boldsymbol{v}_{\text{phase}} = \mathrm{d}\boldsymbol{x}/\mathrm{d}t|_{\text{phase}}$. In reality, sources emit a spread of wavevectors Δk about some central k_0. As shown in the previous section, the presence of a medium may alter the propagation speed of the electromagnetic wave. Typically, the speed becomes wavelength dependent, so that after traversing the medium waves of different frequencies will be at a different phase.

Such a wave packet will have spatial extent $\sim 1/\Delta_k$. The center of the wave will move at the *group velocity* (see exercise 1.3)

$$v_{\text{group}} = \frac{\mathrm{d}\omega}{\mathrm{d}k}. \tag{1.17}$$

Outside of the packet, the waves will tend to cancel out, so that the amplitude decays away. Since the energy density is $\propto |A|^2$, the energy carried by the packet also travels at the group velocity.

EXERCISE 1.3 — Let's construct a one-dimensional wave packet centered on $x = 0$ at time $t = 0$. The amplitude of each component wave k follows a Gaussian distribution,

$$A_0 \exp\left[-\frac{(k-k_0)^2}{2\Delta^2}\right].$$

The (complex) wave amplitude at a position x and time t is then

$$A(x,t) = A_0 \int_{-\infty}^{\infty} \exp\left[-\frac{(k-k_0)^2}{2\Delta^2}\right] \exp\left[ikx - i\omega t\right] \frac{\mathrm{d}k}{2\pi}. \tag{1.18}$$

Here $\omega = \omega(k)$, so that the components do not travel at the same velocity.

1. Expand $\omega(k)$ about k_0 to second-order:

   $$\omega(k) \approx \omega_0 + V(k-k_0) + \frac{D}{2}(k-k_0)^2,$$

   where $\omega_0 = \omega(k_0)$, $V = \mathrm{d}\omega/\mathrm{d}k|_{k=k_0}$, and $D = \mathrm{d}V/\mathrm{d}k|_{k=k_0}$. Insert this expansion into equation (1.18) and derive an expression of the amplitude,

   $$A(x,t) = A_0 e^{ik_0x - i\omega_0 t} \int_{-\infty}^{\infty} \exp\left[-\kappa^2\left(\frac{1}{2\Delta^2} + i\frac{Dt}{2}\right) + i\kappa(x - Vt)\right] \frac{\mathrm{d}\kappa}{2\pi}, \tag{1.19}$$

   where $\kappa = k - k_0$.

Modified from an exercise in Thorne and Blandford [2017].

2. Evaluate the integral (1.19) and show that

   $$|A(x,t)| = (A^*A)^{1/2} \propto \exp\left[-\frac{(x-Vt)^2}{2L^2}\right],$$

   where

   $$L = \frac{1}{\Delta}\left[1 + (D\Delta^2 t)^2\right]^{1/2}.$$

3. Interpret this: What is the spatial extent of the packet at $t = 0$? Derive an expression for the distance the packet center travels in the time it takes to double in width. Apply this to ocean swells, for which the dispersion relation is $\omega = \sqrt{gk}$, where g is the gravitational acceleration. A typical wavelength is 100 m. How narrow a spread in wavelengths, Δ/k_0, is required for a wave packet to travel 1 000 km without doubling in size?

2

From Maxwell to Planck to Einstein

2.1 Solution to Maxwell's equations in vacuum

The electromagnetic field $(\boldsymbol{E}, \boldsymbol{B})$ is described by Maxwell's equations, which in Gaussian units are

$$\boldsymbol{\nabla} \times \boldsymbol{E} + \frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} = 0 \tag{2.1}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{2.2}$$

$$\nabla \cdot \boldsymbol{E} = 4\pi\rho \tag{2.3}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} - \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} = \frac{4\pi}{c}\boldsymbol{J}. \tag{2.4}$$

From the source-free equations (2.1) and (2.2), we can introduce the potentials $(\Phi, \boldsymbol{A})$ such that

$$\begin{aligned} \boldsymbol{B} &= \boldsymbol{\nabla} \times \boldsymbol{A}, \\ \boldsymbol{E} &= -\boldsymbol{\nabla}\Phi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}. \end{aligned}$$

In the absence of source charges and currents $(\rho = 0, \boldsymbol{J} = \boldsymbol{0})$, we substitute for the fields $\boldsymbol{E}, \boldsymbol{B}$ in Equation (2.4) to obtain an equation for the potentials,

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right]\boldsymbol{A} + \boldsymbol{\nabla}\left[\frac{1}{c}\frac{\partial \Phi}{\partial t} + \nabla \cdot \boldsymbol{A}\right] = 0. \tag{2.5}$$

The potentials are not uniquely specified: the fields $\boldsymbol{E}, \boldsymbol{B}$ are unchanged if we make the **gauge transformation** $\boldsymbol{A} \to \boldsymbol{A} + \boldsymbol{\nabla}\psi$, $\Phi \to \Phi - c^{-1}\partial_t\psi$, in which ψ is some scalar field. This gives us enough freedom to choose ψ so that the second term in Equation (2.5) vanishes and leaves us with a wave equation for $\boldsymbol{A}$. By substituting for $(\boldsymbol{E}, \boldsymbol{B})$ into Equation (2.3) and applying the same gauge condition, we obtain a wave equation for Φ as well. More generally, we can recognize that $(\Phi, \boldsymbol{A})$ is a four-vector and then we can bring in the machinery of relativity; for now, though, we'll keep time and space separate and use our gauge freedom to force $\Phi = \nabla \cdot \boldsymbol{A} = 0$.

2.2 *Decomposition into modes: photons*

Since $\boldsymbol{A}$ satisfies a wave equation, we can expand the solution into normal modes,

$$\boldsymbol{A}(\boldsymbol{r}, t; \boldsymbol{k}, \boldsymbol{q}) = \alpha_{\boldsymbol{k},\boldsymbol{q}} \boldsymbol{q} e^{i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega t} + \alpha^*_{\boldsymbol{k},\boldsymbol{q}} \boldsymbol{q}^* e^{-i\boldsymbol{k}\cdot\boldsymbol{r} + i\omega t}. \tag{2.6}$$

In this expression, $\boldsymbol{q}$ is a direction vector ($|\boldsymbol{q}| = 1$); because $\boldsymbol{q}$ is complex it also contains phase information[1]. By substituting Equation (2.6) into Equation (2.5) with the condition $\nabla \cdot \boldsymbol{A} = 0$, we determine that we require

$$\begin{aligned} \omega &= ck, & (2.7)\\ \boldsymbol{q} \cdot \boldsymbol{k} &= 0 & (2.8) \end{aligned}$$

[1] By writing $\boldsymbol{q} = |\boldsymbol{q}| e^{i\theta}$, the terms in Eq. (2.6) become $\alpha |\boldsymbol{q}| e^{i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega t + i\theta}$.

to have a solution to the wave equation. The wave therefore propagates with phase velocity c, and the polarization—the direction of $\boldsymbol{q}$—is orthogonal to the direction of propagation $\boldsymbol{k}$.

The energy density of the electromagnetic field is given by $u = (|\boldsymbol{E}|^2 + |\boldsymbol{B}|^2)/(8\pi)$ and the rate of energy transport is given by the Poynting vector, $\boldsymbol{S} = (c/4\pi)\boldsymbol{E} \times \boldsymbol{B}$. Using our solution, Eq. (2.6), and averaging over many cycles gives for a mode $(\boldsymbol{k}, \boldsymbol{q})$ the energy density,

$$u_{\boldsymbol{k},\boldsymbol{q}} = \frac{\omega^2}{2\pi c^2} |\alpha_{\boldsymbol{k},\boldsymbol{q}}|^2, \tag{2.9}$$

and the flux,

$$\boldsymbol{S}_{\boldsymbol{k},\boldsymbol{q}} = \frac{\omega^2}{2\pi c} |\alpha_{\boldsymbol{k},\boldsymbol{q}}|^2 \,\hat{\boldsymbol{k}} = u_{\boldsymbol{k},\boldsymbol{q}} c \,\hat{\boldsymbol{k}}. \tag{2.10}$$

Here $\hat{\boldsymbol{k}}$ is the unit direction vector. The total energy density and flux are found by summing over modes $(\boldsymbol{k}, \boldsymbol{q})$.

EXERCISE 2.1— Define the variables

$$\begin{aligned} \boldsymbol{Q}_{\boldsymbol{k},\boldsymbol{q}} &= \frac{1}{2\sqrt{\pi c^2}} \left[\alpha_{\boldsymbol{k},\boldsymbol{q}} \boldsymbol{q} e^{i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega t} + \alpha^*_{\boldsymbol{k},\boldsymbol{q}} \boldsymbol{q}^* e^{-i\boldsymbol{k}\cdot\boldsymbol{r} + i\omega t} \right] \\ \boldsymbol{P}_{\boldsymbol{k},\boldsymbol{q}} &= \frac{-i\omega}{2\sqrt{\pi c^2}} \left[\alpha_{\boldsymbol{k},\boldsymbol{q}} \boldsymbol{q} e^{i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega t} - \alpha^*_{\boldsymbol{k},\boldsymbol{q}} \boldsymbol{q}^* e^{-i\boldsymbol{k}\cdot\boldsymbol{r} + i\omega t} \right], \end{aligned}$$

and show that the energy density of the mode $\boldsymbol{k}, \boldsymbol{q}$ can be written as

$$u_{\boldsymbol{k},\boldsymbol{q}} = \frac{1}{2} \omega^2 \boldsymbol{Q}^2_{\boldsymbol{k},\boldsymbol{q}} + \frac{1}{2} \boldsymbol{P}^2_{\boldsymbol{k},\boldsymbol{q}} \equiv H_{\boldsymbol{k},\boldsymbol{q}}.$$

Further show that $\boldsymbol{Q}_{\boldsymbol{k},\boldsymbol{q}}$ and $\boldsymbol{P}_{\boldsymbol{k},\boldsymbol{q}}$ obey Hamilton's equations:

$$\frac{\partial H_{\boldsymbol{k},\boldsymbol{q}}}{\partial \boldsymbol{P}_{\boldsymbol{k},\boldsymbol{q}}} = \dot{\boldsymbol{Q}}_{\boldsymbol{k},\boldsymbol{q}}, \qquad \frac{\partial H_{\boldsymbol{k},\boldsymbol{q}}}{\partial \boldsymbol{Q}_{\boldsymbol{k},\boldsymbol{q}}} = -\dot{\boldsymbol{P}}_{\boldsymbol{k},\boldsymbol{q}}.$$

Hence argue that the radiation field is (formally) equivalent to a collection of harmonic oscillators of various $\boldsymbol{k}, \boldsymbol{q}$.

As a computational aid, we'll take our domain to be a box of volume V with periodic boundary conditions. We therefore write $\alpha_{\boldsymbol{k},\boldsymbol{q}} = A_{\boldsymbol{k},\boldsymbol{q}}/\sqrt{V}$, so that we get the correct potential upon integrating over the box's volume. Our general solution may then be written as a sum over modes,

$$\boldsymbol{A}(\boldsymbol{r},t) = \sum_{\boldsymbol{k},\boldsymbol{q}} \left[\frac{A_{\boldsymbol{kq}}\boldsymbol{q}}{\sqrt{V}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}-i\omega t} + \frac{A^*_{\boldsymbol{kq}}\boldsymbol{q}^*}{\sqrt{V}} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}+i\omega t} \right], \tag{2.11}$$

with total energy

$$E = uV = \sum_{\boldsymbol{k},\boldsymbol{q}} |A_{\boldsymbol{kq}}|^2 \frac{\omega^2}{2\pi c^2}. \tag{2.12}$$

At this point, there are several routes to a description of the field in terms of spin-one particles known as photons. A classic method[2] is to construct the Hamiltonian for the electromagnetic field and perform a canonical transformation to the Hamiltonian of a harmonic oscillator (Exercise 2.1). One then imports the quantum mechanical description of the harmonic oscillator (Box 2.1).

[2] W. Heitler. *The Quantum Theory of Radiation*. Dover, 1984

Box 2.1 Quantizing the harmonic oscillator

For the harmonic oscillator Hamiltonian,

$$\hat{H} = \frac{1}{2} m\omega^2 \hat{x}^2 + \frac{1}{2m}\hat{p}^2,$$

we can construct the operator

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + i\frac{\hat{p}}{\sqrt{2m\hbar\omega}}$$

with adjoint

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} - i\frac{\hat{p}}{\sqrt{2m\hbar\omega}}$$

Their commutator is

$$\left[\hat{a},\hat{a}^\dagger\right] = -\frac{i}{2\hbar}[\hat{x},\hat{p}] + \frac{i}{2\hbar}[\hat{p},\hat{x}] = 1 \tag{2.13}$$

where in the last step we used $[\hat{x},\hat{p}] = -[\hat{p},\hat{x}] = i\hbar$. Although the operator $\hat{a}$ is not Hermitian, the operator

$$\hat{N} \equiv \hat{a}^\dagger \hat{a} = \frac{m\omega}{2\hbar}\hat{x}^2 + \frac{\hat{p}^2}{2m\hbar\omega} - \frac{i}{2\hbar}[\hat{p},\hat{x}] \tag{2.14}$$

is. Further, Eq. (2.14) implies that the Hamiltonian of the simple harmonic oscillator can be written as

$$\hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\right).$$

Hence we can find the eigenstates of energy by finding the eigenstates of $\hat{N}$.

Box 2.1 continued

WHAT CAN WE INFER ABOUT THE EIGENVALUES OF $\hat{N}$? Let $|\lambda\rangle$ be an eigenstate with eigenvalue λ. Since $\lambda = \langle\lambda|\hat{N}|\lambda\rangle = (\langle\lambda|\hat{a}^\dagger)(\hat{a}|\lambda\rangle) = |\hat{a}|^2$, all eigenvalues are positive: $\lambda \geq 0$. What about the operator $\hat{a}$ acting on $|\lambda\rangle$? Using the commutator relation (2.13) we have

$$\begin{aligned}\hat{N}\hat{a}|\lambda\rangle &= (\hat{a}^\dagger\hat{a})\hat{a}|\lambda\rangle = (\hat{a}\hat{a}^\dagger - 1)\hat{a}|\lambda\rangle = \hat{a}(\hat{a}^\dagger\hat{a} - 1)|\lambda\rangle \\ &= (\lambda - 1)\hat{a}|\lambda\rangle .\end{aligned}$$

If $|\lambda\rangle$ is an eigenstate of $\hat{N}$ (and hence $\hat{H}$), then $\hat{a}|\lambda\rangle$ is also, with eigenvalue $\lambda - 1$. Since $(\langle\lambda|\hat{a}^\dagger)(\hat{a}|\lambda\rangle) = \lambda$, we can determine the normalization: $\hat{a}|\lambda\rangle = \sqrt{\lambda}|\lambda - 1\rangle$.

By a similar argument, we find that

$$\hat{N}\hat{a}^\dagger|\lambda\rangle = \hat{a}^\dagger(\hat{a}^\dagger\hat{a} + 1)|\lambda\rangle = (\lambda + 1)\hat{a}^\dagger|\lambda\rangle ,$$

with

$$\hat{a}^\dagger|\lambda\rangle = \sqrt{\lambda + 1}|\lambda + 1\rangle .$$

Since all $\lambda \geq 0$, the λ must be integers, so that $\hat{a}|1\rangle = |0\rangle$ and $\hat{a}|0\rangle = 0$, which truncates the descent.

In summary, the oscillator has energy states $E_n = \hbar\omega(n + 1/2)$, where n is the mean number of quanta. The energy of the oscillator changes via the destruction ($\hat{a}$) or creation ($\hat{a}^\dagger$) of quanta.

EXERCISE 2.2— Show that the operator $\hat{N} = \hat{a}^\dagger\hat{a}$ is Hermitian.

Here we'll simply note that numerous phenomena—e.g., the photoelectric effect, Compton scattering, electron-positron production—suggest that the electromagnetic energy is quantized into discrete units called photons, and that the energy of an individual photon is proportional to its frequency. The electromagnetic field is thus specified by giving the occupation numbers $N_{\boldsymbol{kq}}$ for the various modes. A photon labeled by $(\boldsymbol{k}, \boldsymbol{q})$ has momentum $\boldsymbol{p} = \hbar\boldsymbol{k}$ and energy $E = \hbar|\boldsymbol{k}|c = \hbar\omega$. The total energy of the field is then

$$E = \sum_{\boldsymbol{k},\boldsymbol{q}} N_{\boldsymbol{kq}}\hbar\omega.$$

Comparing this expression with Equation (2.12), we find that

$$N_{\boldsymbol{kq}} = |A_{\boldsymbol{kq}}|^2 \frac{\omega}{2\pi\hbar c^2} : \tag{2.15}$$

the occupation number is proportional to the amplitude of the mode. The relation of the energy to $|A_{\boldsymbol{kq}}|^2$, along with the description of the field as a collection of harmonic oscillators (ex. 2.1) suggests writing our field coefficients as operators,

$$\hat{A}_{\boldsymbol{kq}} = \sqrt{\frac{2\pi\hbar c^2}{\omega}}\hat{a}_{\boldsymbol{k},\boldsymbol{q}}, \qquad \hat{A}^\dagger_{\boldsymbol{kq}} = \sqrt{\frac{2\pi\hbar c^2}{\omega}}\hat{a}^\dagger_{\boldsymbol{k},\boldsymbol{q}}$$

where $\hat{a}_{\boldsymbol{k},\boldsymbol{q}}$, $\hat{a}^\dagger_{\boldsymbol{k},\boldsymbol{q}}$ are the lowering and raising operators for each oscillator mode $\boldsymbol{k}$, $\boldsymbol{q}$.

To relate the photon spin to the polarization states, first notice that although $\boldsymbol{q}$ has three components, the constraint (2.8) $\boldsymbol{k}\cdot\boldsymbol{q} = 0$ means only two are independent. Suppose we take our z-axis along the direction of propagation $\hat{\boldsymbol{k}}$ and choose as our basis positive and negative helicity states

$$\begin{aligned} \boldsymbol{q}_+ &= \frac{1}{\sqrt{2}}(\hat{\boldsymbol{e}}_x + i\hat{\boldsymbol{e}}_y), \\ \boldsymbol{q}_- &= \frac{1}{\sqrt{2}}(\hat{\boldsymbol{e}}_x - i\hat{\boldsymbol{e}}_y), \end{aligned}$$

where $\hat{\boldsymbol{e}}$ are unit directional vectors. If we then rotate our coordinate system by an angle θ about $\hat{\boldsymbol{e}}_z$, the polarization basis vectors in the new coordinate system (denoted by a $'$) are

$$\begin{aligned} \boldsymbol{q}'_+ &= e^{i\theta}\boldsymbol{q}_+ \\ \boldsymbol{q}'_- &= e^{-i\theta}\boldsymbol{q}_-. \end{aligned}$$

This transformation under rotation is precisely the behavior of the eigenfunctions of a spin-one particle with its spin axis along $\hat{\boldsymbol{e}}_z$. We therefore identify the quantized excitations of the electromagnetic field—photons—as being spin-one particles.

2.3 *Emission and absorption of photons; Einstein coefficients*

In non-relativistic classical mechanics, the Hamiltonian for a particle in an electromagnetic field is

$$H = \frac{1}{2m}\left[\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}(\boldsymbol{r},t)\right]^2 + e\Phi(\boldsymbol{r},t).$$

Here e is the charge of the particle. You can find a discussion about the appearance of $\boldsymbol{A}$ with the momentum and gauge invariance in any good mechanics text; suffice it to say that the expression in [ ] is gauge-invariant.

Treating the radiation classically for the moment, we translate the classical Hamiltonian translates over to the quantum mechanical operator, which expands to

$$\hat{H} = \frac{\hat{p}^2}{2m} + \left[-\frac{e}{2mc}\left(\hat{\boldsymbol{p}}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot\hat{\boldsymbol{p}}\right) + \frac{e^2}{2mc^2}A^2 + e\Phi\right]. \tag{2.16}$$

Since $\boldsymbol{A}$ depends on position, $\hat{\boldsymbol{p}}$ and $\boldsymbol{A}$ do not commute.

EXERCISE 2.3 — Consider a photon of wavelength λ incident on an atom. Show that in order-of-magnitude,

$$\frac{(e/mc)|\boldsymbol{p}\cdot\boldsymbol{A}|}{(e^2/mc^2)A^2} \sim \frac{E_0}{E}\frac{a_{\mathrm{B}}}{\lambda}\alpha^{-1}$$

where E is the perturbing electric field ($E \sim A/\lambda$), $E_0 = e/a_{\mathrm{B}}^2$ is a typical electric field strength in an atom (a_{B} is the Bohr radius), and $\alpha = e^2/(\hbar c)$ is the fine structure constant. Hence the term $\propto A^2$ in equation (2.16) is typically negligible compared to the terms $\propto \boldsymbol{p}\cdot\boldsymbol{A}$.

As shown in Exercise 2.3, the term $\propto A^2$ can be neglected; and if we work in the transverse, or Coulomb, gauge then $\Phi = 0$. Suppose we have a large number of particles (index ℓ) with position and momentum operators $\hat{\boldsymbol{r}}_\ell$ and $\hat{\boldsymbol{p}}_\ell$: then we can write the Hamiltonian as a sum over ℓ. The first term in the [ ] of eq. (2.16) becomes

$$-\frac{e}{c}\int\left[\frac{1}{2}\sum_\ell \frac{\hat{\boldsymbol{p}}_\ell}{m_\ell}\delta(\boldsymbol{r}-\hat{\boldsymbol{r}}_\ell)+\delta(\boldsymbol{r}-\hat{\boldsymbol{r}}_\ell)\frac{\hat{\boldsymbol{p}}_\ell}{m_\ell}\right]\cdot\boldsymbol{A}(\boldsymbol{r},t)\,\mathrm{d}V \equiv -\frac{e}{c}\int\hat{\boldsymbol{J}}\cdot\boldsymbol{A}\,\mathrm{d}V \tag{2.17}$$

where the term in [ ] is the operator of particle current $\hat{\boldsymbol{J}}$.

We then expand $\boldsymbol{A}$ using equation (2.11) and treat it as a time-dependent harmonic perturbation (Box 2.2); from equation (2.22) we see that the terms $(A_{\boldsymbol{kq}}/\sqrt{V})\boldsymbol{q}e^{i\boldsymbol{k}\cdot\boldsymbol{r}-i\omega t}$ with $\hbar\omega = E_n - E_m$ will induce an upward transition from a state $|m\rangle$ to a state $|n\rangle$ with a rate for each mode $(\boldsymbol{k},\boldsymbol{q})$

$$\begin{aligned}\Gamma^{\boldsymbol{kq}}_{m\to n} &= \frac{2\pi}{\hbar}\frac{e^2}{Vc^2}\left|A_{\boldsymbol{kq}}\right|^2\left|\langle n|\,\boldsymbol{J}_{\boldsymbol{k}}\cdot\boldsymbol{q}\,|m\rangle\right|^2\delta(E_n-E_m-\hbar\omega)\\ &= \frac{4\pi^2e^2}{V\omega}N_{\boldsymbol{kq}}\left|\langle n|\,\boldsymbol{J}_{\boldsymbol{k}}\cdot\boldsymbol{q}\,|m\rangle\right|^2\delta(E_n-E_m-\hbar\omega).\end{aligned} \tag{2.18}$$

In this equation, $\boldsymbol{J}_{\boldsymbol{k}} = \int \mathrm{d}V\,\boldsymbol{J}e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$ is the Fourier transform of the particle current $\hat{\boldsymbol{J}}$. The rate is proportional to the density of photons $N_{\boldsymbol{kq}}/V$ in mode $(\boldsymbol{k},\boldsymbol{q})$.

Box 2.2 follows the treatment in Baym [1990].

Box 2.2 Time-dependent perturbation theory

Suppose we have a system in a state $|\Psi\rangle$ acting under a Hamiltonian $\hat{H}_0$. The system evolves in time according to

$$i\hbar\frac{\partial}{\partial t}|\Psi\rangle = \hat{H}_0|\Psi\rangle\,; \tag{2.19}$$

we wish to analyze the behavior under a perturbation $\hat{V}$. Specifically, we are interested in an oscillatory potential, which we'll increase in amplitude as time increases from $t\to-\infty$:

$$V = \hat{V}e^{\eta t}\left[e^{-i\omega t}+e^{i\omega t}\right]. \tag{2.20}$$

Box 2.2 continued

Here $\eta > 0$ is an arbitrary parameter for that we'll eventually take $\eta \to 0$. One can show that the result doesn't really depend on how the perturbation is turned on; but having the exponential cutoff ensured that the integrals are doable.

To proceed, we first factor out the time dependence from the unperturbed Hamiltonian by writing $|\Psi\rangle = e^{-i\hat{H}_0 t/\hbar}|\psi\rangle$ and substituting into the perturbed equation

$$i\hbar\frac{\partial}{\partial t}|\Psi\rangle = \left(\hat{H}_0 + V\right)|\Psi\rangle ;$$

as a result, we remove the evolution due to $\hat{H}_0$ and obtain

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = e^{i\hat{H}_0 t/\hbar} V e^{-i\hat{H}_0 t/\hbar}|\psi\rangle . \tag{2.21}$$

If we start in an eigenstate $|\psi(t \to -\infty)\rangle = |m\rangle$ of $\hat{H}_0$ with energy E_m, then we can get an approximation to $|\psi\rangle$ by substituting $|\psi\rangle \approx |m\rangle$ on the right-hand side of equation (2.21):

$$|\psi(t)\rangle = -\frac{i}{\hbar}\int_{-\infty}^{t} \mathrm{d}t \; e^{i\hat{H}_0 t/\hbar} V e^{-i\hat{H}_0 t/\hbar}|m\rangle .$$

The amplitude for the system to be in an eigenstate $|n\rangle$ of $\hat{H}_0$ with energy E_n at time t is

$$\langle n|\,\psi\rangle = -\frac{i}{\hbar}\int_{-\infty}^{t} \mathrm{d}t \;\langle n|\, e^{i\hat{H}_0 t/\hbar} V e^{-i\hat{H}_0 t/\hbar}|m\rangle ;$$

furthermore, since $e^{-i\hat{H}_0 t/\hbar}|m\rangle = e^{-iE_m t/\hbar}|m\rangle$ and $\langle n|\, e^{i\hat{H}_0 t/\hbar} = \langle n|\, e^{iE_n t/\hbar}$, the amplitude to be in state $|n\rangle$ at time t becomes

$$\begin{aligned}\langle n|\,\psi\rangle &= -\frac{i}{\hbar}\int_{-\infty}^{t} \mathrm{d}t \left[e^{i(\Delta E - \hbar\omega - i\hbar\eta)t/\hbar} + e^{i(\Delta E + \hbar\omega - i\hbar\eta)t/\hbar}\right]\langle n|\,\hat{V}\,|m\rangle \\ &= -e^{\eta t}\left[\frac{e^{i(\Delta E - \hbar\omega)t/\hbar}}{\Delta E - \hbar\omega - i\hbar\eta} + \frac{e^{i(\Delta E + \hbar\omega)t/\hbar}}{\Delta E + \hbar\omega - i\hbar\eta}\right]\langle n|\,\hat{V}\,|m\rangle .\end{aligned}$$

Here $\Delta E = E_n - E_m$.

The probability for the system to have transitioned from state $|m\rangle$ to state $|n\rangle$ after time t is therefore

$$\begin{aligned}\mathcal{P}_{m\to n}(t) = |\langle n|\,\psi\rangle|^2 = e^{2\eta t}\left|\langle n|\,\hat{V}\,|m\rangle\right|^2 \\ \times\left\{\frac{1}{(\Delta E - \hbar\omega)^2 + \hbar^2\eta^2} + \frac{1}{(\Delta E + \hbar\omega)^2 + \hbar^2\eta^2}\right. \\ + \frac{e^{-2i\omega t}}{[\Delta E - \hbar\omega - i\hbar\eta][\Delta E + \hbar\omega + i\hbar\eta]} \\ \left. + \frac{e^{2i\omega t}}{[\Delta E - \hbar\omega + i\hbar\eta][\Delta E + \hbar\omega - i\hbar\eta]}\right\} .\end{aligned}$$

Box 2.2 continued

Now if we suppose we have a large number of systems (like a collection of atoms upon which we are shining light), then the instantaneous rate at which a system makes a transition is $\Gamma_{m\to n} = \mathrm{d}\mathcal{P}_{m\to n}/\mathrm{d}t$; also, we want the transition rate averaged over many cycles. The oscillatory terms—those containing $e^{\pm 2i\omega t}$—will then average to zero, leaving us with the transition rate

$$\Gamma_{m\to n} = \left|\langle n|\,\hat{V}\,|m\rangle\right|^2 e^{2\eta t} \times \left\{\frac{2\eta}{(\Delta E - \hbar\omega)^2 + \hbar^2\eta^2} + \frac{2\eta}{(\Delta E + \hbar\omega)^2 + \hbar^2\eta^2}\right\}$$

Now it's time to take the limit $\eta \to 0$: clearly the result $\Gamma_{m\to n} = 0$ unless $\hbar\omega = \pm(E_n - E_m)$; in fact,

$$\Gamma_{m\to n} = \frac{2\pi}{\hbar}\left|\langle n|\,\hat{V}\,|m\rangle\right|^2 \left[\delta\left(\Delta E - \hbar\omega\right) + \delta\left(\Delta E + \hbar\omega\right)\right]. \quad (2.22)$$

The first δ-function comes from the $e^{-i\omega t}$ term; the second, from the $e^{i\omega t}$ term. Since our frequencies are positive, the first delta function therefore corresponds to upward transitions $E_n > E_m$, while the second corresponds to downward transitions $E_n < E_m$.

Next, we'd like to include our description of the radiation field as a collection of modes $\{\ldots N_{\boldsymbol{kq}}\ldots\}$: our initial state is then $|m;\ldots N_{\boldsymbol{kq}}\ldots\rangle$; our final state, $|n;\ldots N_{\boldsymbol{kq}} - 1\ldots\rangle$. To make this description consistent with Equation (2.18) we define an operator $\hat{A}_{\boldsymbol{kq}}$ that decreases $N_{\boldsymbol{kq}}$ by one,

$$\left\langle\ldots N_{\boldsymbol{kq}} - 1\ldots\right|\hat{A}_{\boldsymbol{kq}}\left|\ldots N_{\boldsymbol{kq}}\ldots\right\rangle = \sqrt{\frac{2\pi\hbar c^2}{\omega}N_{\boldsymbol{kq}}} \quad (2.23)$$

to within a phase factor that we set to unity[3]. Taking the complex conjugate of Equation (2.23) gives the operator that increases $N_{\boldsymbol{kq}}$ by one,

[3] Which with hindsight we know is okay: photons are bosons

$$\left\langle\ldots N_{\boldsymbol{kq}}\ldots\right|\hat{A}^{\dagger}_{\boldsymbol{kq}}\left|\ldots N_{\boldsymbol{kq}} - 1\ldots\right\rangle = \sqrt{\frac{2\pi\hbar c^2}{\omega}N_{\boldsymbol{kq}}}. \quad (2.24)$$

Notice that with $\hat{A}^{\dagger}_{\boldsymbol{kq}}$, the eigenvalue contains the number of photons in the *final* state, not the number in the initial state.

With the operators $\hat{A}_{\boldsymbol{kq}}$ and $\hat{A}^{\dagger}_{\boldsymbol{kq}}$ the rate for the system to make a transition $|m;\ldots N_{\boldsymbol{kq}}\ldots\rangle \to |n;\ldots N_{\boldsymbol{kq}} - 1\ldots\rangle$ is

$$\begin{aligned}\Gamma^{\boldsymbol{kq}}_{m,N_{\boldsymbol{kq}}\to n,N_{\boldsymbol{kq}}-1} &= \frac{2\pi}{\hbar}\frac{e^2}{Vc^2}\delta(E_n - E_m - \hbar ck)\left|\langle n|\,\boldsymbol{J_k}\cdot\boldsymbol{q}\,|m\rangle\right|^2 \\ &\quad\times\left|\left\langle\ldots N_{\boldsymbol{kq}} - 1\ldots\right|\hat{A}_{\boldsymbol{kq}}\left|\ldots N_{\boldsymbol{kq}}\ldots\right\rangle\right|^2, \\ &= \frac{4\pi^2e^2}{V\omega}N_{\boldsymbol{kq}}\left|\langle n|\,\boldsymbol{J_k}\cdot\boldsymbol{q}\,|m\rangle\right|^2\delta(E_n - E_m - \hbar ck),\end{aligned} \quad (2.25)$$

which is the same as Equation (2.18). This is a description of the *absorption* of a photon $(\boldsymbol{k}, \boldsymbol{q})$. The rate for our system to *emit* a photon $(\boldsymbol{k}, \boldsymbol{q})$, i.e., to make a transition $|n; \ldots N_{\boldsymbol{kq}} \ldots\rangle \to |m; \ldots N_{\boldsymbol{kq}} + 1 \ldots\rangle$, is

$$\begin{aligned} \Gamma^{\boldsymbol{kq}}_{n,N_{\boldsymbol{kq}}\to m,N_{\boldsymbol{kq}}+1} &= \frac{2\pi}{\hbar}\frac{e^2}{c^2V}\delta(E_n - E_m - \hbar ck)\left|\langle m|\,\boldsymbol{J}_{-\boldsymbol{k}}\cdot\boldsymbol{q}^*\,|n\rangle\right|^2 \\ &\quad\times\left|\langle\ldots N_{\boldsymbol{kq}}+1\ldots|\,\hat{A}^{\dagger}_{\boldsymbol{kq}}\,|\ldots N_{\boldsymbol{kq}}\ldots\rangle\right|^2 \\ &= \frac{4\pi^2e^2}{V\omega}(N_{\boldsymbol{kq}}+1)\left|\langle m|\,\boldsymbol{J}_{-\boldsymbol{k}}\cdot\boldsymbol{q}^*\,|n\rangle\right|^2\delta(E_n - E_m - \hbar ck) \end{aligned} \tag{2.26}$$

Notice that while the absorption rate is proportional to $N_{\boldsymbol{kq}}$, the emission rate is proportional to $N_{\boldsymbol{kq}} + 1$; these two terms account for *induced* and *spontaneous* emission, respectively.

The rates for emission and absorption are linked. We can lump the atomic matrix elements into a coefficient and write Equations (2.25) and (2.26) as

$$\Gamma^{\boldsymbol{kq}}_{m,N_{\boldsymbol{kq}}\to n,N_{\boldsymbol{kq}}-1} = b_{mn}N_{\boldsymbol{kq}} \tag{2.27}$$

$$\Gamma^{\boldsymbol{kq}}_{n,N_{\boldsymbol{kq}}\to m,N_{\boldsymbol{kq}}+1} = b_{nm}N_{\boldsymbol{kq}} + a_{nm}. \tag{2.28}$$

From the form of Equations (2.25) and (2.26) we expect that $b_{mn} = b_{nm}$ and also $a_{nm} = b_{nm}$. That this is so can be shown by a statistical argument made by Einstein.

First, since photons have integer spin, they obey Bose-Einstein statistics: the mean occupation number for a given mode ν is

$$\bar{N}_\nu = (e^{\beta h\nu} - 1)^{-1}, \tag{2.29}$$

where $\beta = (k_{\mathrm{B}}T)^{-1}$ is the inverse temperature and $k_{\mathrm{B}} = 1.3806 \times 10^{-16}\,\mathrm{erg\,K^{-1}}$ is the *Boltzmann constant*. Another way to derive this mean occupation number is to consider the radiation field as a collection of harmonic oscillators each with frequency ν. The "levels" for each oscillator are given by $E_n = nh\nu$, and therefore in equilibrium the relative probability of two levels is given by the Boltzmann factor,

$$\frac{\mathcal{P}(N_2)}{\mathcal{P}(N_1)} = e^{-\beta(N_2-N_1)h\nu}.$$

The mean energy for a distribution of photons is therefore

$$\begin{aligned} \bar{E}_\nu &= \frac{\sum_{n=0}(nh\nu)e^{-\beta nh\nu}}{\sum_{n=0}e^{-\beta nh\nu}} \\ &= -\frac{\mathrm{d}}{\mathrm{d}\beta}\ln\left(\sum_{n=0}^{\infty}e^{-n\beta h\nu}\right) \end{aligned}$$

Evaluating the sum as a geometric series, taking the derivative, and equating $\bar{E}_\nu = \bar{N}_\nu h\nu$ gives the desired result.

Now suppose we have a cavity with the radiation in thermal equilibrium; in this cavity are some atoms with two levels, m and n, with an energy difference between the levels $E_n - E_m = \hbar\omega$. The rate of upward transitions is $\bar{N}_m b_{mn} \bar{N}_{\boldsymbol{kq}}$ where $\bar{N}_m$ is the number of atoms in state m; the rate of downward transitions is $\bar{N}_n(b_{nm}\bar{N}_{\boldsymbol{kq}} + a_{nm}) = \bar{N}_m e^{-\beta\hbar\omega}(b_{nm}\bar{N}_{\boldsymbol{kq}} + a_{nm})$ where the Boltzmann factor accounts for the relative likelihood of finding the atom in state n versus m. For simplicity, we are assuming the states are not degenerate. Since in thermal equilibrium the rate of upward transitions must equal the rate of downward transitions,

$$\bar{N}_m b_{mn} \bar{N}_{\boldsymbol{kq}} = \bar{N}_m e^{-\beta\hbar\omega}(b_{nm}\bar{N}_{\boldsymbol{kq}} + a_{nm}),$$

we find after rearranging terms that

$$\bar{N}_{\boldsymbol{kq}} = \frac{a_{nm}/b_{mn}}{e^{\beta\hbar\omega} - b_{nm}/b_{mn}}. \tag{2.30}$$

For $b_{nm} = b_{mn} = a_{nm}$ we recover the Bose-Einstein distribution. Notice that without induced emission we would just get a Maxwell-Boltzmann distribution.

3

A Phenomenological Description of Radiation

3.1 The specific intensity

Having shown that we can describe the electromagnetic field by enumerating photon states, the next task is to go to the limit of large occupation numbers—i.e., many photons per state—and formulate a description of the radiation in terms of intensity and energy flux.

To start, we replace the sum over modes with an integral. We need to ensure that we count states correctly when we do this. Let's take our volume to be a box with sides of length L. (We'll see in a bit that the explicit reference to volume will cancel from our formulae.) Such a box can accommodate wavevectors $|k| > \pi\mathcal{N}/L$, with $\mathcal{N} = 1, 2, \ldots$. Hence the number of modes increases by $\mathrm{d}k = 2\pi/L$ as we increase[1] $\mathcal{N}$ by $\Delta\mathcal{N} = 1$. Extending this argument to all three dimensions, we can make the replacement

[1] The factor of two accounts for the positive and negative values of $\boldsymbol{k}$.

$$\Delta\mathcal{N}_x \Delta\mathcal{N}_y \Delta\mathcal{N}_z \to L^3 \left(\frac{\mathrm{d}k_x}{2\pi}\right)\left(\frac{\mathrm{d}k_y}{2\pi}\right)\left(\frac{\mathrm{d}k_z}{2\pi}\right)$$

and the sum over all modes becomes

$$\frac{1}{V}\sum_{\boldsymbol{k},\boldsymbol{q}} \to \sum_{\boldsymbol{q}} \left(\frac{1}{2\pi}\right)^3 \int \mathrm{d}^3k = \sum_{\boldsymbol{q}} \left(\frac{1}{2\pi}\right)^3 \int k^2\,\mathrm{d}k\,\mathrm{d}\Omega, \tag{3.1}$$

with the volume canceling out. In the last expression we've also converted to spherical coordinates with $\mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$ being a differential of solid angle.

With this change, we can express the energy density, Equation (2.9), as

$$u = \sum_{\boldsymbol{q}} \left(\frac{1}{2\pi}\right)^3 \int k^2 \mathrm{d}k\, \frac{\omega^2}{2\pi c^2} |A_{\boldsymbol{kq}}|^2\,\mathrm{d}\Omega. \tag{3.2}$$

In terms of the occupation number, Equation (2.15), the radiative energy density is

$$u = \sum_{\boldsymbol{q}} \int \frac{h\nu^3}{c^3} N_{\nu\boldsymbol{q}}\, \mathrm{d}\nu\, \mathrm{d}\Omega. \tag{3.3}$$

In this expression we've also changed variables to frequency ν via $k = 2\pi\nu/c$.

It is useful to look at the radiative flux in a small range of frequencies $\mathrm{d}\nu$ traveling into a narrow cone of solid angle $\mathrm{d}\Omega$. We shall call this quantity the *specific intensity* I_ν: from Equations (2.9), (2.10), and (3.3), the specific intensity is related to the occupation numbers via

$$I_\nu\, \mathrm{d}\nu\, \mathrm{d}\Omega \equiv \left[\sum_{\boldsymbol{q}} \frac{h\nu^3}{c^2} N_{\nu\boldsymbol{q}} \right] \mathrm{d}\nu\, \mathrm{d}\Omega. \tag{3.4}$$

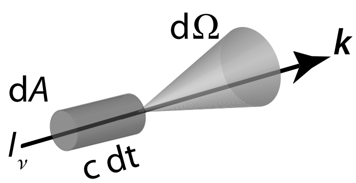

Figure 3.1: The intensity I_ν is the energy in a frequency band $\mathrm{d}\nu$ propagating into a cone about direction $\hat{\boldsymbol{k}}$ incident on area $\mathrm{d}A$ in a time $\mathrm{d}t$.

For unpolarized light $\sum_{\boldsymbol{q}} \to 2$; unless stated otherwise, we'll make this assumption from now on.

EXERCISE 3.1— Suppose that you observe a star with your naked eye under ideal seeing conditions, and suppose that this star is at the limit of what the human eye can detect. Estimate the rate at which photons from this star reach your retina.

For most applications in astronomy, the length over which light travels is much larger than a wavelength; in this case, we are in the geometrical optics limit and we can describe light as traveling along rays. The specific intensity is a useful quantity in this limit because it is conserved along a ray in the absence of interactions with matter (and doppler shifts). This conservation is a consequence from Liouville's theorem that a volume in phase space is conserved along trajectories.

To see this, note that from eq. (3.1) we can write the number of photons as

$$N = \sum_{\boldsymbol{q}} \left(\frac{1}{2\pi}\right)^3 \int \mathrm{d}^3k\, \mathrm{d}^3x\, N_{\boldsymbol{kq}} = \sum_{\boldsymbol{q}} \frac{1}{h^3} \int \mathrm{d}^3p\, \mathrm{d}^3x\, N_{\boldsymbol{kq}}.$$

Suppose a source emits a number N of photons in a volume $\mathrm{d}A\, c\mathrm{d}t$ traveling in a narrow angle $\mathrm{d}\Omega$ about $\boldsymbol{p} = \hbar\boldsymbol{k}$. This defines a small volume of phase space $\mathrm{d}^3p\, \mathrm{d}^3x$, which along the ray is conserved by Liouville's theorem. In the absence of interactions with matter, N is also constant and therefore $N_{\boldsymbol{kq}} \propto I_\nu/\nu^3$ is invariant along the ray. The invariance of $\nu^{-3}I_\nu$ holds true in a relativistic context, whereas I_ν is constant along a ray only in the Newtonian limit.

EXERCISE 3.2— You observe a source that emits a thermal spectrum $B_\nu(T)$ and that is redshifted, with $\lambda_{\mathrm{obs}} = \lambda_{\mathrm{emit}}(1+z)$. Show that the spectrum you observe is Planckian, but with temperature $T/(1+z)$.

3.2 *Moments of the specific intensity*

Just as we define the specific intensity as the energy flux in a frequency interval $d\nu$, we can define the specific energy density

$$u_\nu = \int \sum_{\boldsymbol{q}} \frac{h\nu^3}{c^3} N_\nu \, d\Omega. \tag{3.5}$$

This is just an integral over angle of the specific intensity

$$u_\nu = \frac{1}{c} \int d\Omega \, I_\nu.$$

Now suppose we want the specific flux crossing an area with normal $\hat{\boldsymbol{n}}$. We first multiply the specific intensity by a unitvector $\hat{\boldsymbol{k}}$ along the direction of the ray, and then take the component along $\hat{\boldsymbol{n}}$ and integrate over all directions,

$$F_\nu = \int d\Omega \, I_\nu (\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{n}}) = \int d\Omega \; \cos\theta \, I_\nu. \tag{3.6}$$

The units of F_ν are energy/area/time/frequency.

EXERCISE 3.3 — The Crab nebula is commonly used as a calibration source in X-ray astronomy. Over the band of photon energies (2–10) keV, the spectral distribution is well-approximated by a power-law, $F_\nu \propto \nu^{-2}$, and the fluence in this energy range is $\int F_\nu \, d\nu = 2.4 \times 10^{-8} \, \mathrm{erg\,cm^{-2}\,s^{-1}}$.

Suppose we wish to observe with *Chandra* a source with a similar spectral distribution as the Crab, but with an overall fluence that is 0.001 that of the Crab. Take the collecting area of *Chandra* in the (2–10) keV band to be $340\,\mathrm{cm}^2$. How long of an integration time does one need to collect enough photons to ensure 10% errors on the total count rate?

EXERCISE 3.4 — A typical bright quasar has a flux of $F_\nu = 10\,\mathrm{Jy} = 10 \times 10^{-23}\,\mathrm{erg\,cm^{-2}\,s^{-1}\,Hz^{-1}}$. Suppose that source were observed continuously over 40 yr at the Arecibo radio telescope. Take the spectral distribution to be flat over the antenna bandpass (0.312–0.342) GHz. How would the total energy received over these forty years compare to some everyday expenditure: for example, how would the energy received compare with that required to lift some common weight over some distance?

EXERCISE 3.5 — For the source in exercise 3.4, estimate the photon occupation number. Does it make sense to treat photons individually? You may take the distance to be 1 Gpc and the size of the emitting region to be 0.01 pc.

Notice the pattern. To get u_ν, we multiplied I_ν by a weighting factor $1 = (\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}})^0$ and integrated over angle. To get F_ν, we multiplied I_ν by a weighting factor $(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}})^1 = \cos^1\theta$ and integrated over angle. This procedure—multiply by a power of $\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}}$ and integrate over angle—is formally known as *taking a moment* of the specific intensity. The specific energy density is proportional to the zeroth moment of the intensity; the specific flux is proportional to the first moment of the intensity.

The next moment is related to the stress tensor, which is the momentum flux along direction $\hat{\boldsymbol{n}}$ being transported across an area with normal $\hat{\boldsymbol{m}}$:

$$\mathsf{P}_\nu^{mn} = \frac{1}{c}\int \mathrm{d}\Omega\, I_\nu \left(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{m}}\right)\left(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{n}}\right). \tag{3.7}$$

This is a tensor because it contains two directional vectors, $\hat{\boldsymbol{m}}$ and $\hat{\boldsymbol{n}}$. The factor of c^{-1} comes from momentum being related to frequency as $p = h\nu/c$. The stress tensor P_ν is clearly symmetric: $\mathsf{P}_\nu^{mn} = \mathsf{P}_\nu^{nm}$.

It is often more convenient to work with these moments—u_ν, F_ν, P_ν—of the radiative intensity. The moments, being weighted averages over angle, contain less information about the radiative intensity; the lower-order moments do, however, have a readily interpretable physical meaning. Although formally one can construct higher-order moments, in practice only the first three have any connection with a physical quantity.

3.3 *Thermodynamics of the radiation field*

If the radiation field is in thermal equilibrium, then the occupation numbers satisfy Equation (2.29). Inserting this into Equation (3.4) gives the specific intensity in equilibrium, known as the *Planck* function,

$$I_\nu^{\text{equil.}} \equiv B_\nu = \frac{2h\nu^3}{c^2}\left[\exp\left(\frac{h\nu}{k_\mathrm{B}T}\right) - 1\right]^{-1}. \tag{3.8}$$

Dividing by c and integrating over all frequencies gives the energy density:

$$\begin{aligned} u &= \int \frac{2h\nu^3}{c^3}\frac{1}{e^{h\nu/k_\mathrm{B}T}-1}\,\mathrm{d}\nu\,\mathrm{d}\Omega = \frac{8\pi h}{c^3}\int_0^\infty \frac{\nu^3}{e^{h\nu/k_\mathrm{B}T}-1}\,\mathrm{d}\nu \\ &= \left[\frac{8\pi^5 k_\mathrm{B}^4}{15h^3c^3}\right] T^4 \equiv aT^4. \end{aligned} \tag{3.9}$$

Here $a = 7.566\times10^{-15}\,\mathrm{erg\,cm^{-3}\,K^4}$ is the *radiation constant*.

EXERCISE 3.6— Derive the blackbody spectral distribution with respect to wavelength, B_λ. Show that the peaks for B_ν and B_λ do *not* coincide, but that the peaks of νB_ν and λB_λ do.

At low frequencies ($\nu \ll k_\mathrm{B}T/h$) we can expand Eq. (3.8),

$$B_\nu \approx \frac{2\nu^2}{c^2}k_\mathrm{B}T.$$

In radio astronomy, one often defines a *brightness temperature*, $\Theta = I_\nu c^2/(2\nu^2 k_\mathrm{B}) = I_\lambda \lambda^4/(2ck_\mathrm{B})$.

EXERCISE 3.7— Estimate the brightness temperature for the WKAR broadcast antenna in Okemos. What does the value you obtain tell you about the radiative process?

The net flux, Equation (3.6), vanishes for radiation in thermal equilibrium. This follows from the isotropy of the radiative intensity. If we imagine that the radiation is escaping from a small opening in a *hohlraum*, the integrating only over outward directions gives

$$F = \int_0^{2\pi} \mathrm{d}\varphi \int_0^1 \mathrm{d}(\cos\theta) \int \mathrm{d}\nu\, \frac{2h\nu^3}{c^2} \frac{\cos\theta}{\exp\left(\frac{h\nu}{k_\mathrm{B}T}\right) - 1} = \sigma_\mathrm{SB} T^4. \qquad (3.10)$$

Here $\sigma_\mathrm{SB} = ac/4 = 5.670\times10^{-5}\,\mathrm{erg\,cm^{-2}\,s^{-1}\,K^{-4}}$ is the *Stefan-Boltzmann* constant.

Did you note how we changed variables from θ to $\mu = \cos\theta$? With this change, the integration over 4π steradians is

$$\int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \sin\theta\,\mathrm{d}\theta = \int_0^{2\pi} \mathrm{d}\varphi \int_{-1}^{1} \mathrm{d}\mu.$$

EXERCISE 3.8— Show that the total energy flux in a given frequency interval is proportional to the corresponding area under a curve in a plot of νF_ν against $\log\nu$, and likewise for λF_λ against $\log\lambda$.

For the stress tensor, the off-diagonal components, $\hat{\boldsymbol{m}} \neq \hat{\boldsymbol{n}}$, vanish as well. The diagonal components are all equal; since the rate of momentum transport across a unit area is just the force on that area, which is the pressure, we have

$$\begin{aligned} P_\mathrm{rad} &= \frac{1}{c}\int \mathrm{d}\nu \int \mathrm{d}\Omega\, B_\nu \cos^2\theta = \frac{1}{c}\int B_\nu\,\mathrm{d}\nu \int_0^{2\pi} \mathrm{d}\varphi \int_{-1}^{1} \mu^2\,\mathrm{d}\mu \\ &= \frac{4\pi}{3c}\int B_\nu\,\mathrm{d}\nu = \frac{1}{3}aT^4. \end{aligned} \qquad (3.11)$$

That the pressure is one-third of the energy density is in general true for a relativistic gas.

EXERCISE 3.9 — Suppose you observe a sphere of radius R from a distance D as shown in Fig. 3.2. The emitted intensity I_ν is uniform over the surface, but it is a function of the angle θ between the ray and the normal to the surface. Show that the observed flux F_ν from the entire visible surface of the sphere is

$$F_\nu = \left(\frac{R}{D}\right)^2 \int_0^{2\pi} \int_0^{\pi/2} I(\theta)\ \cos\theta\ \sin\theta\, d\theta\, d\varphi;$$

that is, the integration over the solid angle subtended by the sphere is equivalent to integrating over outward directions from a single point on the surface. Show that if $I_\nu(\theta) = B_\nu$ is a thermal spectrum (and in particular, is independent of θ), then

$$\int F_\nu\, d\nu = \sigma_{SB} T_{eff}^4 \left(\frac{R}{D}\right)^2.$$

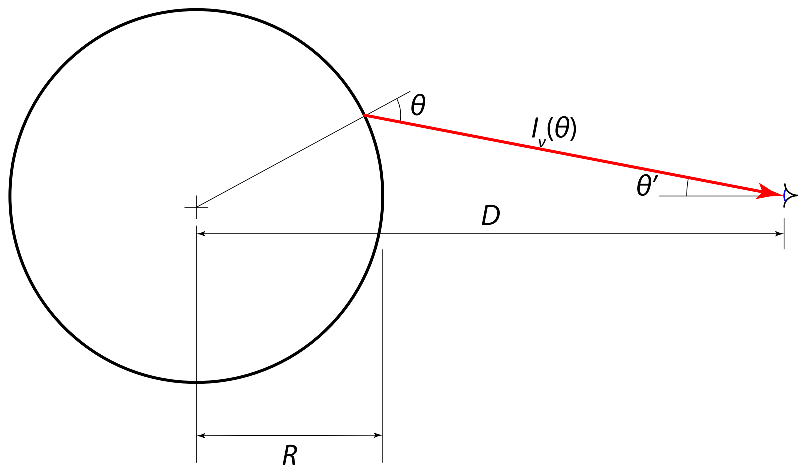

Figure 3.2: The intensity from a sphere observed a distance D away. Here the intensity depends on the angle θ between the ray and the normal to the observer.

4

The Equation of Transfer

We saw in Chapter 2 that the interaction of photons with matter for a given microscopic process connecting levels n and m (with $E_n - E_m = h\nu$) consists of three terms: absorption, with rate $b_{mn}N_{\boldsymbol{kq}}$; stimulated emission, with rate $b_{nm}N_{\boldsymbol{kq}}$; and spontaneous emission, with rate a_{nm}. Here the a and b coefficients represent matrix elements connecting the levels n and m in the matter. We then showed in Chapter 3 how we could describe our radiation field by the intensity I_ν. Now we incorporate the interaction with matter to derive an equation governing the evolution of I_ν as it passes through a medium.

4.1 *Absorption*

We begin with absorption. The rate of absorption for a single atom is proportional to N_ν, and a sample of atoms will absorb in a range of frequencies $\Delta\nu$ about ν: the atoms will have some motion, so there is a Doppler shift; there is an uncertainty principle for the finite lifetime of an excited state; the atom may collide with other atoms; and so on. To account for this spread in frequencies, we introduce a dimensionless function $\varphi(\nu)$ which is peaked about the frequency $\nu = |E_n - E_m|/h$ of the transition. The rate of absorption for one atom is then $\int N_\nu b_{mn}\varphi(\nu)\,\mathrm{d}\nu\,\mathrm{d}\Omega$.

If we have a small volume $\Delta\mathcal{V}$ containing $n_m\Delta\mathcal{V}$ absorbers[1], then the rate at which photons are absorbed by atoms in state m is

$$n_m\Delta\mathcal{V}\frac{1}{4\pi}\int\underbrace{\left(\frac{2\pi b_{mn}c^2}{h\nu^3}\right)}_{\equiv B_{mn}}\underbrace{\left(\frac{2h\nu^3}{c^2}N_\nu\right)}_{I_\nu}\varphi(\nu)\,\mathrm{d}\Omega\,\mathrm{d}\nu$$

$$\equiv\quad n_m\Delta\mathcal{V}\int\frac{B_{mn}}{4\pi}I_\nu\varphi(\nu)\,\mathrm{d}\nu\,\mathrm{d}\Omega. \qquad (4.1)$$

Here we've factored out 4π so that if everything is isotropic the integration over angle yields unity. With this convention, the units of B_{mn} are $\mathrm{cm}^2\,\mathrm{erg}^{-1}$.

Now let's take $\Delta\mathcal{V} = \Delta s\Delta\mathcal{A}$, where s is along the direction $\boldsymbol{k}$ of a ray and $\Delta\mathcal{A}$ is normal to $\boldsymbol{k}$. The incident energy flux into our volume in a

[1] n_m is the number of atoms in state m per unit volume

frequency interval $\mathrm{d}\nu$ and in an angular range $\mathrm{d}\Omega$ about $\boldsymbol{k}$ is then

$$I_\nu \, \mathrm{d}\nu \, \mathrm{d}\Omega \, \Delta\mathcal{A};$$

the rate of energy absorption from this ray in the volume is

$$n_m \left[h\nu \frac{B_{mn}}{4\pi} \varphi(\nu) \right] I_\nu \, \mathrm{d}\nu \, \mathrm{d}\Omega \, \Delta\mathcal{A} \, \Delta s.$$

If we therefore have a ray I_ν incident on a volume $\Delta\mathcal{A}\Delta s$, then its intensity upon exiting the volume will have decreased:

$$I_\nu(s + \Delta s) = I_\nu(s) - n\sigma_\nu I_\nu(s)\Delta s. \tag{4.2}$$

In this expression we've cancelled out the common factors of $\Delta\mathcal{A}\,\mathrm{d}\Omega\,\mathrm{d}\nu$ and introduced

$$\sigma_\nu \equiv \frac{\text{“rate of specific energy absorption”}}{\text{“incident specific flux”}} = \frac{h\nu B_{mn}}{4\pi} \frac{n_m}{n} \varphi(\nu). \tag{4.3}$$

The quantity σ_ν has dimensions of area and is termed the *cross-section*.

If we take the limit of Eq. (4.2) for which $\Delta s \ll I_\nu / |\mathrm{d}I_\nu/\mathrm{d}s|$, i.e., Δs is small on a macroscopic scale while $\Delta s > \lambda$, so that our description makes sense, we then have a differential equation for the intensity,

$$\left. \frac{\mathrm{d}I_\nu}{\mathrm{d}s} \right|_{\text{absorp.}} = -n\sigma_\nu I_\nu. \tag{4.4}$$

It is common to introduce the *opacity* defined via $\rho\kappa_\nu \equiv n\sigma_\nu$, with ρ being the mass density; the opacity has dimension $[\kappa_\nu] \sim \mathrm{cm}^2/\mathrm{g}$. The combination $n\sigma_\nu = \rho\kappa_\nu$ is sometimes denoted as the *extinction coefficient*[2] α_ν.

[2] George B. Rybicki and Alan P. Lightman. *Radiative Processes in Astrophysics*. Wiley, 1979

If σ_ν does not depend on I_ν, the solution to Eq. (4.4) is straightforward: $I_\nu(s) = I_\nu(0)e^{-\tau_\nu}$, where

$$\tau_\nu(s) = \int_0^s \rho\kappa_\nu \mathrm{d}s \tag{4.5}$$

is the *optical depth*. Note that $\rho\kappa_\nu = n\sigma_\nu$ has dimensions of inverse length: we call $\ell_\nu = (n\sigma_\nu)^{-1}$ the *mean free path*. The optical depth is therefore simply the path length measured in units of a photon mean free path.

EXERCISE 4.1— You are observing a star with a ground-based telescope. Suppose that the extinction coefficient $\alpha_\nu = \rho\kappa_\nu$ depends only on the vertical height above ground. Show that in terms of magnitudes, the flux reaching the telescope when the star is at an angle θ from the zenith is

$$m(\theta) = m_0 + k_0 \sec\theta,$$

in which $k_0 = (2.5 \log e)\tau = 1.086\tau$, $\tau = \int \alpha \, \mathrm{d}z$ and m_0 is the magnitude that would be observed in the absence of an atmosphere. In this expression we neglect the curvature of the Earth. The quantity $\sec\theta$ is thus an approximation for the *airmass*.

4.2 *Emission, both spontaneous and stimulated*

For emission, we saw in Section 2.3 that the rate per atom for a downwards transition from level n to m was $a_{nm} + b_{nm}N_\nu$. As with absorption, we allow for the transition occurring over a spread in frequencies by introducing $\varphi(\nu)$, and recast the rate in terms of specific intensity:

$$\text{emission rate} = n_n A_{nm} \varphi(\nu)\, \mathrm{d}\nu \frac{\mathrm{d}\Omega}{4\pi} + n_n B_{nm} I_\nu \varphi(\nu)\, \mathrm{d}\nu \frac{\mathrm{d}\Omega}{4\pi}. \tag{4.6}$$

Here the first term is for spontaneous emission, and the second is for stimulated. This is the rate of photon emission; to get the energy emitted we'll need to multiply the emission rate by $h\nu$.

If we again consider a ray incident on a cylinder of volume $\Delta\mathcal{A}\,\Delta s$, then the gain in intensity over the volume is

$$I_\nu(s+\Delta s) - I_\nu(s) = n_n \Delta s \left[\frac{A_{nm}}{4\pi} h\nu\, \varphi(\nu)\right] + n_n \Delta s \left[\frac{B_{nm}}{4\pi} h\nu\, \varphi(\nu)\right] I_\nu,$$

so that

$$\left.\frac{\mathrm{d}I_\nu}{\mathrm{d}s}\right|_{\text{spon. emission}} = n_n \frac{A_{nm}}{4\pi} h\nu\, \varphi(\nu) = \frac{\rho \epsilon_\nu}{4\pi} \tag{4.7}$$

and

$$\left.\frac{\mathrm{d}I_\nu}{\mathrm{d}s}\right|_{\text{stim. emission}} = n_n \left[\frac{B_{nm}}{4\pi} h\nu\, \varphi(\nu)\right] I_\nu. \tag{4.8}$$

In Equation (4.7), we define an emissivity ϵ_ν with dimension $\left[\mathrm{erg\,s^{-1}\,g^{-1}\,Hz^{-1}}\right]$; the factor of $(4\pi)^{-1}$ makes the right-hand side into a per-steradian quantity. The quantity $\rho\epsilon_\nu/(4\pi)$ is often denoted as [3] j_ν with units of $\left[\mathrm{erg\,s^{-1}\,cm^{-3}\,Hz^{-1}}\right]$.

[3] George B. Rybicki and Alan P. Lightman. *Radiative Processes in Astrophysics*. Wiley, 1979

We can combine the stimulated emission term, Eq. (4.8), with the absorption term, Equations (4.2) and (4.4):

$$\left.\frac{\mathrm{d}I_\nu}{\mathrm{d}s}\right|_{\text{corr. abs.}} = -I_\nu \left[\left(B_{mn} - \frac{n_n}{n_m} B_{nm}\right) \frac{h\nu}{4\pi} \varphi(\nu) \frac{n_m}{n}\right] n. \tag{4.9}$$

The term in $[\cdot]$ is the corrected absorption cross-section $\sigma_\nu^{\text{corr.}}$. Notice also that we have incorporated the abundance of particles in state m, n_m/n, into the definition of σ_ν and ϵ_ν. We denote by n the total number of atoms (in any state): $n = \sum_i n_i$. Likewise, the mass density is $\rho = \sum_i M_i n_i$, where M_i is the mass of species i.

EXERCISE 4.2 — In our original derivation of the Einstein a and b coefficients, Sec. 2.3, we showed that for non-degenerate atomic levels n and m, the coefficients were all equal, $a_{nm} = b_{nm} = b_{mn}$.

1. Generalize this: show that if the levels n and m are degenerate with occupation numbers g_n and g_m, then the relations between the coefficients are

$$\frac{b_{nm}}{b_{mn}} = \frac{g_m}{g_n}, \quad \frac{a_{nm}}{b_{mn}} = \frac{g_m}{g_n}.$$

2. Next, from the definitions of the coefficients B_{nm}, B_{mn}, and A_{nm}, show that

$$\frac{B_{nm}}{B_{mn}} = \frac{g_m}{g_n}, \quad \frac{A_{nm}}{B_{mn}} = \frac{2h\nu^3}{c^2}\frac{g_m}{g_n}.$$

Scattering

The final process to consider is scattering. For *coherent scattering*, also called *elastic scattering*, the photon is redirected into a different direction, but no energy is transferred to the matter. Scattering changes the intensity in two ways: energy is scattered out of the the beam, but energy is also scattered *into* the beam from other directions. The change in intensity due to scattering therefore has not only a negative term, similar to absorption, but also a positive term:

$$\left.\frac{\mathrm{d}I_\nu}{\mathrm{d}s}\right|_{\mathrm{scat.}} = -\rho\kappa_\nu^{\mathrm{sca}} I_\nu + \rho\kappa_\nu^{\mathrm{sca}} \int \Phi(\hat{\boldsymbol{k}}, \hat{\boldsymbol{k}}') I_\nu(\hat{\boldsymbol{k}}')\mathrm{d}\Omega'. \tag{4.10}$$

Here the redistribution function Φ is both normalized, $\int \Phi(\hat{\boldsymbol{k}}, \hat{\boldsymbol{k}}')\,\mathrm{d}\Omega' = 1$, and reversible, $\Phi(\hat{\boldsymbol{k}}, \hat{\boldsymbol{k}}') = \Phi(\hat{\boldsymbol{k}}', \hat{\boldsymbol{k}})$. For isotropic scattering, $\Phi = (4\pi)^{-1}$, so $\int I_\nu \Phi\,\mathrm{d}\Omega = J_\nu$, where $J_\nu = (4\pi)^{-1}\int I_\nu\,\mathrm{d}\Omega$ is the mean intensity. Isotropic scattering simply redistributes the energy over all angles. We'll take this to be the case in the rest of the chapter, so that the $\mathrm{d}I_\nu/\mathrm{d}s|_{\mathrm{scat.}} = -\rho\kappa_\nu^{\mathrm{sca}}(I_\nu - J_\nu)$.

EXERCISE 4.3 — Consider a plasma with absorption opacity $\kappa_\nu^{\mathrm{abs}}$ and scattering opacity $\kappa_\nu^{\mathrm{sca}}$, both of which are constant. A photon is emitted and takes a hop of average length $\ell = \rho^{-1}(\kappa_\nu^{\mathrm{abs}} + \kappa_\nu^{\mathrm{sca}})^{-1}$; at the end of the hop, the photon is either scattered into a random direction for another hop, or else it is absorbed. Show that the average number of hops the photon takes until being absorbed is

$$\langle N_{\mathrm{hop}}\rangle = \frac{\kappa_\nu^{\mathrm{abs}} + \kappa_\nu^{\mathrm{sca}}}{\kappa_\nu^{\mathrm{abs}}}.$$

4.3 Putting everything together: the source function and albedo

We now combine the terms for absorption (corrected for stimulated emission), emission, and (isotropic) scattering into the full differential equation for the specific intensity,

$$\frac{dI_\nu}{ds} = -\rho\left(\kappa_\nu^{\rm abs} + \kappa_\nu^{\rm sca}\right) I_\nu + \rho \frac{\epsilon_\nu}{4\pi} + \rho\kappa_\nu^{\rm sca} J_\nu. \tag{4.11}$$

We can further simplify this equation by defining the optical depth $d\tau_\nu = \rho(\kappa_\nu^{\rm abs} + \kappa_\nu^{\rm sca})ds$ and rewriting Eq. (4.11) as

$$\begin{aligned}\frac{dI_\nu}{d\tau_\nu} &= -I_\nu + \frac{\epsilon_\nu}{4\pi\kappa_\nu^{\rm abs}}\left(1 - \frac{\kappa_\nu^{\rm sca}}{\kappa_\nu^{\rm abs} + \kappa_\nu^{\rm sca}}\right) + \frac{\kappa_\nu^{\rm sca}}{\kappa_\nu^{\rm abs} + \kappa_\nu^{\rm sca}} J_\nu \\ &\equiv -I_\nu + \underbrace{\frac{\epsilon_\nu}{4\pi\kappa_\nu^{\rm abs}}(1 - \mathcal{A}_\nu) + \mathcal{A}_\nu J_\nu}_{\equiv S_\nu}.\end{aligned} \tag{4.12}$$

The relative importance of scattering is measured by the single-scattering *albedo*, $\mathcal{A}_\nu \equiv \kappa_\nu^{\rm sca}/(\kappa_\nu^{\rm abs} + \kappa_\nu^{\rm sca})$. In Eq. (4.12) we've also introduced the *source function* S_ν.

In the presence of scattering, the source function

$$\begin{aligned}S_\nu &= \frac{\epsilon_\nu}{4\pi\kappa_\nu^{\rm abs}}(1 - \mathcal{A}_\nu) + \mathcal{A}_\nu J_\nu \\ &= \frac{\epsilon_\nu}{4\pi\kappa_\nu^{\rm abs}}(1 - \mathcal{A}_\nu) + \mathcal{A}_\nu \frac{1}{4\pi}\int I_\nu\, d\Omega\end{aligned} \tag{4.13}$$

depends on the integral of I_ν over angle via J_ν, so that equation (4.12) is an *integro-differential* equation and in general does not have a closed-form solution. If scattering is absent ($\mathcal{A}_\nu = 0$), so that $S_\nu = \epsilon_\nu/(4\pi\kappa_\nu^{\rm abs})$ is a known function of τ_ν, then we can formally solve equation (4.12):

$$I_\nu(\tau_\nu) = I_\nu(0)\exp(-\tau_\nu) + \int_0^{\tau_\nu} S_\nu(\tau_\nu)\exp(t - \tau_\nu)\, dt. \tag{4.14}$$

EXERCISE 4.4 — Suppose we have a box containing two-level atoms. The levels are in thermal equilibrium at temperature T.

1. What is the source function S_ν?
2. Now suppose a ray passes through our box. The intensity is Planckian at temperature T_γ, i.e., $I_\nu = B_\nu(T_\gamma)$, but $T_\gamma \neq T$. What is dI_ν/ds if $T_\gamma > T$? If $T_\gamma < T$? If $T_\gamma = T$? Give a "intuitive" physical explanation for this.

EXERCISE 4.5 — Suppose we have an ionized cloud of uniform temperature $T = 10^4$ K, electron number density $n_e = 2000\,\mathrm{cm}^{-3}$, and radius $R = 0.6$ pc. You observe this cloud in the radio over a frequency range $(10–10^4)$ MHz. The primary interaction between radiation and matter in the cloud is *free-free*, or *bremsstrahlung*, absorption with coefficient

$$\rho\kappa_\nu^{\mathrm{ff}} \approx 6.56 \times 10^{-2} n_e^2 T^{-3/2} \nu^{-2}\,\mathrm{cm}^{-1}.$$

Here n_e is in units of cm^{-3}, T is in units of K, and ν is in units of Hz. Assume that collisions in the cloud are sufficient to maintain the electrons and ions in local thermodynamic equilibrium (LTE).

1. Find the frequency ν_0 at which $\tau_\nu = 1$ for a line of sight through the center of the cloud.
2. What is the source function S_ν? Make an approximation for the source function appropriate for the range of observed frequencies.
3. Expand the equation of transfer in the limit $\tau_\nu \ll 1$, and get an approximate expression for I_ν as a function of ν. Do the same for the case $\tau_\nu \gg 1$. Make a schematic plot of I_ν as a function of ν over the range of frequencies observed. Indicate on the plot the frequency ranges in which the emission is optically thin and optically thick and indicate how I_ν scales with ν in each of these regimes.

4.4 *Diffusion Approximation and the Rosseland Mean Opacity*

At large optical depth, such as deep in a stellar interior, the radiation field is in thermal equilibrium, so that $I_\nu = S_\nu = B_\nu$. To understand this, consider the formal solution, Equation (4.14): at large τ_ν, $I_\nu \to S_\nu$. If the matter is in local thermodynamic equilibrium, so that all levels follow a Boltzmann distribution, then $\epsilon_\nu/(4\pi\kappa_\nu^{\mathrm{abs}}) = B_\nu$. In addition, if we are at very large optical depth, then conditions over the scale of a mean free path should not vary much, and the radiation field should be nearly isotropic; we therefore expect that $J_\nu = I_\nu$ and $\mathrm{d}I_\nu/\mathrm{d}\tau_\nu \to 0$. Under these conditions, the equation of transfer becomes

$$0 \approx \frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} \approx (1 - \mathcal{A}_\nu)(B_\nu - I_\nu).$$

Thus $B_\nu = I_\nu = J_\nu$, and the source function becomes

$$S_\nu = B_\nu(1 - \mathcal{A}_\nu) + \mathcal{A}_\nu B_\nu = B_\nu.$$

If the radiation field is perfectly isotropic there is no flux, however, so we must have some small anisotropy. Let's imagine the photon performing a random walk. At very large optical depth, the temperature and density will only vary slightly over the length of a hop ℓ. Let's imagine a small cube of material, with the size of this cube being ℓ. Because we are so very nearly isotropic and in thermal equilibrium, the flux through any one face of this cube must be $(c/6)u$, where u is the radiation energy density. Now suppose we have two adjacent cubes, with the common face of the cubes being at $x = 0$. The flux across the face has contributions from photons emitted at $x - \ell$ and $x + \ell$, so the net flux is

$$\begin{aligned} F &\approx \frac{c}{6}u(x-\ell) - \frac{c}{6}u(x+\ell) \\ &\approx -\frac{1}{3}c\ell\frac{\mathrm{d}u}{\mathrm{d}x}. \end{aligned} \tag{4.15}$$

This is a diffusion equation with coefficient $c/(3\rho\kappa)$. Our derivation is very crude, as it neglects the variation in cross section with the properties of the ambient medium and with the photon frequency. Nonetheless, this is basically the correct scenario; heat diffuses with a coefficient given by some suitably defined average over all sources of opacity.

TO COMPUTE THE FLUX IN A MORE RIGOROUS FASHION, let's write I_ν as B_ν plus a correction,

$$I_\nu = B_\nu(T) + I_\nu^{(1)}(\hat{\boldsymbol{k}}). \tag{4.16}$$

The superscript $^{(1)}$ reminds us this is a first-order correction. Now, let $\mu = \cos\theta$ be the direction cosine between our ray $\hat{\boldsymbol{k}}$ and the gradient of I_ν: that is,

$$\frac{\mathrm{d}}{\mathrm{d}s} = \hat{\boldsymbol{k}} \cdot \boldsymbol{\nabla}.$$

Substituting this and the expansion for I_ν, Eq. (4.16), into the steady-state equation of transfer, Eq. (4.12) and keeping the lowest order terms on both sides of the equation gives

$$\frac{1}{\rho\kappa_\nu}\hat{\boldsymbol{k}} \cdot \boldsymbol{\nabla} B_\nu = S_\nu - (B_\nu + I_\nu^{(1)});$$

upon setting the term $S_\nu - B_\nu = 0$ on the right-hand side we obtain

$$I_\nu^{(1)} = -\frac{1}{\rho\kappa_\nu}\hat{\boldsymbol{k}} \cdot \boldsymbol{\nabla} B_\nu = -\frac{1}{\rho\kappa_\nu}\frac{\partial B_\nu}{\partial T}\hat{\boldsymbol{k}} \cdot \boldsymbol{\nabla} T. \tag{4.17}$$

This is anisotropic: the energy transport is largest in the direction "down" the temperature gradient. Let's get the net flux crossing an area with normal $\hat{\boldsymbol{n}}$: multiply equation (4.17) by $\hat{\boldsymbol{k}}$ to get the flux; and then take the component along a direction $\hat{\boldsymbol{n}}$; then replace the two dot products by the angle cosine μ, and integrate over $\mathrm{d}\Omega = 2\pi\mathrm{d}\mu$ to obtain

$$\boldsymbol{F}_\nu = -\int_{-1}^{1} \frac{1}{\rho\kappa_\nu}\left(\frac{\partial B_\nu}{\partial T}\boldsymbol{\nabla} T\right) 2\pi\mu^2 \mathrm{d}\mu = -\frac{4\pi}{3}\frac{1}{\rho}\left[\frac{1}{\kappa_\nu}\frac{\partial B_\nu}{\partial T}\right]\boldsymbol{\nabla} T. \tag{4.18}$$

The quantity in [ ] deserves a closer look. First, suppose κ_ν is independent of frequency. Then equation (4.18) means that the energy transport is greatest at the frequency where $\partial B_\nu/\partial T$ is maximum, and *not* at the peak of the Planck spectrum.

Let us define the *Rosseland mean opacity* as

$$\kappa_{\mathrm{R}} \equiv \left[\frac{\int \mathrm{d}\nu\, \kappa_\nu^{-1} (\partial B_\nu / \partial T)}{\int \mathrm{d}\nu\, (\partial B_\nu / \partial T)} \right]^{-1}.$$

We can use this to integrate equation (4.18) to obtain the total radiative flux,

$$\boldsymbol{F} = -\frac{4\pi}{3} \frac{1}{\rho \kappa_{\mathrm{R}}} \boldsymbol{\nabla} \left[\int \mathrm{d}\nu\, B_\nu \right] = -\frac{1}{3} \frac{c}{\rho \kappa_{\mathrm{R}}} \boldsymbol{\nabla} a T^4. \qquad (4.19)$$

This is just our formula for radiation diffusion (eq. [4.15]) that we obtained from physical arguments, but now we have an expression for the effective opacity κ_{R}.

EXERCISE 4.6 — Model a star as a spherical cavity, radius R, filled with thermal radiation at temperature T.

1. Suppose the star is suddenly rendered transparent, so that all the radiation streams out radially. Make an estimate for the flux during this calamitous event.
2. Now suppose each photon has to random-walk its way out, with each step being of length ℓ. How many steps are required before the photon has a reasonable chance of reaching the surface? What is the total distance covered by the photon during its walk, and how long does this walk take?
3. Use your answers in part 2 to estimate the flux, and show that it agrees in scaling with equation (4.19) if we set $\ell = 1/(\rho \kappa_{\mathrm{R}})$.

4.5 *Moments of the transfer equation, and the Eddington approximation*

Until now, we've been writing the LHS of the transfer equation as $\mathrm{d}I_\nu/\mathrm{d}s$, where s is some distance along the path of the ray. We want to make this more general, since we'll want to compute I_ν for many different paths. As an example, consider a thin, plane-parallel atmosphere (planet or star), so that all physical quantities depend on height z above some reference point. We can still define an optical depth τ_ν with respect to z:

$$\tau_\nu = \int_z^\infty \rho \left(\kappa_\nu^{\mathrm{abs}} + \kappa_\nu^{\mathrm{sca}} \right) \mathrm{d}z'; \qquad (4.20)$$

for a ray traveling along direction $\hat{\boldsymbol{k}}$ with $\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{z}} = \mu$ and $\mathrm{d}z = \mu \mathrm{d}s$, the equation of transfer becomes

$$\mu \frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = I_\nu - S_\nu. \qquad (4.21)$$

Note the change of sign, which comes from our orientation of coordinates, Eq. (4.20).

Now, you may have noticed that with isotropic scattering the source function doesn't depend on angle. It might then occur to you to average Eq. (4.21) over angle: defining the first moment of I_ν as

$$H_\nu \equiv \frac{1}{4\pi}\int \mu I_\nu \,\mathrm{d}\Omega = \frac{1}{2}\int_{-1}^{1} \mu I_\nu \,\mathrm{d}\mu,$$

we obtain

$$\frac{\mathrm{d}H_\nu}{\mathrm{d}\tau_\nu} = J_\nu - S_\nu. \tag{4.22}$$

The right-hand side is now a simple function of J_ν, but this comes at the cost of an extra quantity H_ν that is related to J_ν in some complicated fashion. We can get another equation in terms of H_ν by multiplying Eq. (4.21) by μ and integrating over all angles:

$$\frac{\mathrm{d}K_\nu}{\mathrm{d}\tau_\nu} = H_\nu. \tag{4.23}$$

Here

$$K_\nu \equiv \frac{1}{4\pi}\int \mu^2 I_\nu \,\mathrm{d}\Omega = \frac{1}{2}\int_{-1}^{1} \mu^2 I_\nu \,\mathrm{d}\mu,$$

and the term with the source function vanishes because it is odd in μ.

So far, this mathematical jiggery-pokery doesn't really help, however; we've generated an additional equation at the cost of yet another variable K_ν, so that we still have more variables than equations. We could continue this procedure of multiplying Eq. (4.21) by successive powers of μ and averaging over angle; in so doing we would generate a series of equations containing increasingly higher moments of the radiation field. We would always have more variables, however, than equations; in order for this approach to help, we need a condition[4] for truncating this expansion.

[4] Known as a *closure* relation.

A classic closure scheme, due to Eddington, is to assert that $K_\nu = J_\nu/3$ is true everywhere. Recall that in thermodynamical equilibrium J_ν, H_ν, and K_ν are related[5] to the specific energy density, flux, and pressure:

[5] cf. §3.3

$$\begin{aligned} u_\nu &= \frac{4\pi}{c} J_\nu, \\ F_\nu &= 4\pi H_\nu, \\ P_\nu &= \frac{4\pi}{c} K_\nu. \end{aligned} \tag{4.24}$$

For thermal radiation the pressure is 1/3 of the energy density, so that $K_\nu = J_\nu/3$. In general the intensity $I_\nu \neq B_\nu$ is *not* thermal; the *Eddington approximation* is to assert that $K_\nu = J_\nu/3$ holds even where the radiation field isn't in equilibrium. With this condition, Equations (4.22) and (4.23) form a closed and solvable set. This closure relation is commonly used in low-accuracy models of stellar atmospheres. As explored in exercise 4.7, the Eddington approximation is equivalent to treating the anisotropy of the radiation field as being linear in μ.

EXERCISE 4.7— Suppose we expand our radiation field into multipoles: that is,

$$I_\nu(\mu) = \sum_{n=0}^{\infty} I_\nu^{(n)} P_n(\mu),$$

where P_n is the Legendre polynomial of order n and $I_\nu^{(n)}$ is a coefficient. Show that the Eddington approximation is equivalent to dropping all terms of order $n = 2$ and higher in this expansion.

EXERCISE 4.8— Another classic approximation in stellar atmospheres is to write the intensity as a sum of two streams, one upward and one downward.

$$I_\nu(\mu) = I_\nu^+ \delta\left(\mu - \frac{1}{\sqrt{3}}\right) + I_\nu^- \delta\left(\mu + \frac{1}{\sqrt{3}}\right). \tag{4.25}$$

Here δ refers to the Dirac delta function. Show that in this approximation, the moments of the transfer equation are

$$\begin{aligned} J_\nu &= \frac{1}{2}\left(I_\nu^+ + I_\nu^-\right) \\ H_\nu &= \frac{1}{2\sqrt{3}}\left(I_\nu^+ - I_\nu^-\right) \\ K_\nu &= \frac{1}{6}\left(I_\nu^+ + I_\nu^-\right). \end{aligned} \tag{4.26}$$

Also show that the definition (Eq. 4.25) ensures that the Eddington approximation is automatically satisfied.

4.6 *A grey atmosphere*

As a worked example of the Eddington approximation, we'll consider the idealized case of a grey atmosphere in *local thermodynamic equilibrium*. By "grey," we mean that $\kappa_\nu^{\rm abs}$ and $\kappa_\nu^{\rm sca}$ are independent of frequency. By local thermodynamic equilibrium, we mean that the energy levels in the matter are in a thermal distribution, so that $\epsilon_\nu/4\pi\kappa_\nu^{\rm abs} = B_\nu$ and the source function is $S_\nu = B_\nu(1-\mathcal{A}) + \mathcal{A}J_\nu$. Note that this does not necessarily imply that the radiation is in thermal equilibrium with the matter.

If our atmosphere is in steady-state, then there is no net energy exchange between matter and radiation when we integrate emission and absorption over all frequencies and angles:

$$\int \left(\frac{\epsilon_\nu}{4\pi} - \kappa_\nu^{\rm abs} J_\nu\right)\, d\nu = 0,$$

which implies that $\int B_\nu\, d\nu = \int J_\nu\, d\nu$. Note that this does not necessarily imply that $J_\nu = B_\nu$. We can use this to simplify our equation for H_ν, Eq. (4.22):

$$\int \frac{dH_\nu}{d\tau_\nu}\, d\nu = 0,$$

so $H = \int H_\nu \, \mathrm{d}\nu$ is constant throughout the atmosphere.

If H is constant, then we can integrate Eq. (4.23) over all frequencies and then find $K = \int K_\nu \, \mathrm{d}\nu = H(\tau + \tau_0)$. Now we can use our closure condition, $K = J/3$, to eliminate J in our original transfer equation (4.21). Notice that since $\int B_\nu \, \mathrm{d}\nu = \int J_\nu \, \mathrm{d}\nu = J$, the source function integrated over all frequencies is

$$\int S_\nu \, \mathrm{d}\nu = \int B_\nu (1 - \mathcal{A}) + \mathcal{A} J_\nu \, \mathrm{d}\nu = J = 3H(\tau + \tau_0).$$

Substituting this into Eq. (4.21) and integrating over all frequency gives

$$\mu \frac{\mathrm{d}I}{\mathrm{d}\tau} = I - 3H(\tau + \tau_0). \tag{4.27}$$

We can integrate Eq. (4.27) over τ. We are interested in the radiation emerging from great depth in the atmosphere, so our integration is from $\tau \to \infty$ to τ. As before, we write $I = e^{\tau/\mu}\mathfrak{I}(\tau)$, with $\mathfrak{I} \sim e^{-\tau/\mu}$ as $\tau \to \infty$; substituting this into Eq. (4.27) and canceling common factors gives

$$\frac{\mathrm{d}\mathfrak{I}}{\mathrm{d}\tau} = -\frac{3H}{\mu} e^{-\tau/\mu}(\tau + \tau_0).$$

Upon integrating from a given depth τ inwards, we obtain

$$I(\tau) = 3H\left(\tau + \mu + \tau_0\right). \tag{4.28}$$

To determine τ_0 we require that at $\tau = 0$ the integral over all outward-bound rays gives the net flux:

$$2\pi \int_0^1 \mu I \, \mathrm{d}\mu = 6\pi H \int_0^1 \mu(\mu + \tau_0) \, \mathrm{d}\mu = F = 4\pi H, \tag{4.29}$$

which fixes $\tau_0 = 2/3$.

Since the flux $F = 4\pi H$ is constant, we can set $F = \sigma_{\mathrm{SB}} T_{\mathrm{eff}}^4$. Here $\sigma_{\mathrm{SB}} = ac/4$ is the *Stefan-Boltzmann* constant. Since the angle-averaged intensity, when integrated over all frequencies, is $J = B$ and $B = acT^4/(4\pi) = \sigma_{\mathrm{SB}} T^4/\pi$ (see Eq. [4.24] and [3.9]), our equation for the moment K becomes

$$\frac{\sigma_{\mathrm{SB}} T^4}{3\pi} = \frac{J}{3} = K = \frac{\sigma_{\mathrm{SB}} T_{\mathrm{eff}}^4}{4\pi} \left(\tau + \frac{2}{3}\right),$$

thus giving the temperature as a function of optical depth:

$$T^4(\tau) = \frac{3}{4} T_{\mathrm{eff}}^4 \left(\tau + \frac{2}{3}\right). \tag{4.30}$$

Thus $T = T_{\mathrm{eff}}$ at $\tau^{2/3}$. What is the probability that a photon emitted at $\tau = 2/3$ will escape without being absorbed or scattered?

EXERCISE 4.9 — For a grey atmosphere, find the specific intensity as a function of angle $\arccos(\mu)$ between the normal to the surface and the direction to the observer.

1. What is the ratio of the intensity between the center of the sun and the edge? You should find it reduced; that is, the limb of the sun appears darker than the center.
2. What happens for a star that is sufficiently far away that it is no longer resolved? What is the net flux emitted towards a distant observer in this case?

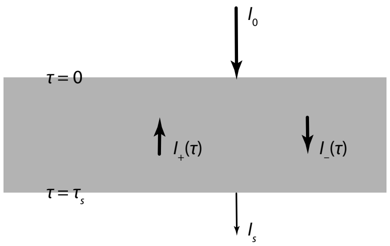

Figure 4.1: An irradiated slab.

EXERCISE 4.10 — Suppose we have a plane-parallel slab with a pure scattering gray opacity as shown in Fig. 4.1. The slab's top face is illuminated uniformly with incident intensity I_0. The total optical depth through the slab is τ_s. Use the two-stream approximation to find, as functions of τ_s, the intensity that emerges from the bottom of the slab I_s, as well as the intensity that is reflected (re-emerges) from the top side, $I^+(\tau = 0)$.

1. Demonstrate that the flux H is constant. Derive an expression for $J(\tau)$ in terms of H and I^-.
2. Write expressions for the mean intensity at the top and bottom of the slab, namely $J(\tau = 0)$ and $J(\tau = \tau_s)$.
3. Use the expression for $J(\tau)$, part 1, and the boundary conditions, part 2, to solve for $I_+(\tau = 0)$ and I_s.
4. Now show that your expression makes sense in the limits $\tau_s \to 0$ and $\tau_s \to \infty$.

EXERCISE 4.11 — In this problem we'll consider a planet with a grey atmosphere that is being irradiated by its host star. Let the star have radius $R_\star$ and effective temperature $T_\star$, and let the star be a distance D from the planet.

1. Show that the incident intensity on the planet is $(\sigma_{SB}/\pi)WT_\star^4$, where $W = (R_\star/D)^2$.
2. Solve the transfer equation (Eq. 4.21) for a grey atmosphere. Begin by taking moments of the equation. As before, argue that the flux H is constant, and show that
$$J(\tau) = 3H\tau + J_0.$$
Since H is constant, write $H = (4\pi)^{-1}\sigma_{SB}T_{\rm int}^4$. Then use the two-stream relations (Eqn. 4.26) to express J_0 in terms of H and I^-. Finally, set $J = (\sigma_{SB}/\pi)T^4(\tau)$ and I^- to the incident intensity to get an expression for $T(\tau)$ in terms of $T_\star$ and $T_{\rm int}$.
3. Qualitatively describe the temperature structure of the atmosphere for $WT_\star \gg T_{\rm int}$. How does it compare to the case of negligible irradiation?

ONCE WE HAVE THE THERMAL STRUCTURE, WE CAN FIND THE EMERGENT SPECTRUM. To get the spectral distribution, start with equation (4.21) and (assuming the atmosphere has some absorption so that the matter and radiation can come into equilibrium) insert $S_\nu = B_\nu(T)$; solving for I_ν at $\tau = 0$ then gives

$$I_\nu(\mu, \tau = 0) = \frac{1}{\mu} \int_0^\infty B_\nu \left[T(\tau)\right] \, e^{-\tau/\mu} \, \mathrm{d}\tau. \qquad (4.31)$$

A plot of the spectral distribution for the emergent flux for $\mu = 1$ is shown (*open circles*) in Fig. 4.2. For comparison, a plot of the Planck distribution (*solid line*) is also shown. Both distributions are normalized to the total flux. You will note that the emergent spectrum does not exactly match the Planck distribution (although it is close), even though the opacity is grey.

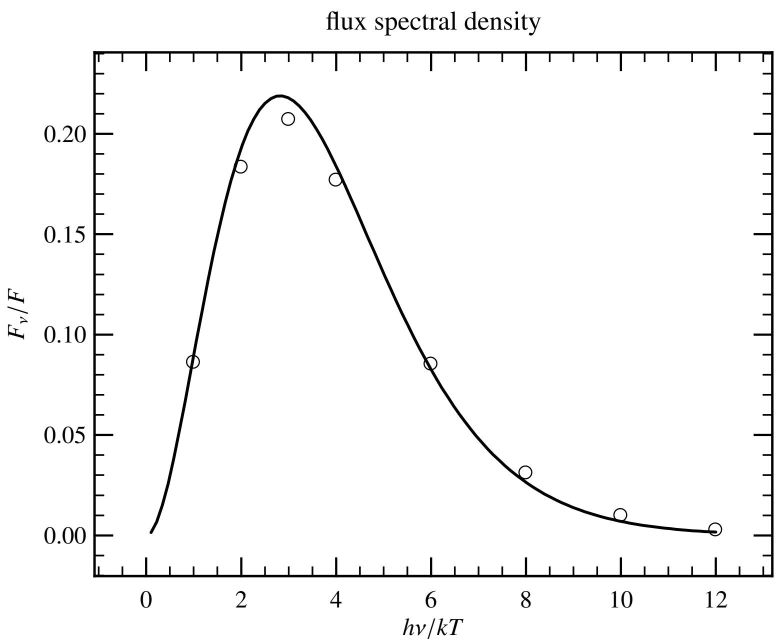

Figure 4.2: Spectral distribution from a grey atmosphere. The open circles are from Chandrasekhar, *Radiative Transfer*; the solid line is the Planck distribution.

5

Simple Radiating Systems

Now that we've completed our description of the radiation field and described the equation of transfer, the next task is to investigate various radiative processes. In this chapter, we describe some simple classical systems; namely low-energy scattering from free electrons and Rayleigh scattering. We shall also look at how signals are modified by propagation through a plasma. We begin by revisiting Maxwell's equations in the presence of sources.

5.1 The fields of a moving source

In Chapter 2, we saw how Maxwell's equations could be combined into an expression for the potentials $(\Phi, \boldsymbol{A})$; if we now retain the source terms $(\rho, \boldsymbol{j})$, we reduce the four Maxwell equations into two equations for $(\Phi, \boldsymbol{A})$:

$$\begin{aligned} \left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right]\Phi - \boldsymbol{\nabla}\left[\frac{1}{c}\frac{\partial \Phi}{\partial t} + \nabla\cdot\boldsymbol{A}\right] &= 4\pi\rho, \\ \left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right]\boldsymbol{A} + \boldsymbol{\nabla}\left[\frac{1}{c}\frac{\partial \Phi}{\partial t} + \nabla\cdot\boldsymbol{A}\right] &= \frac{4\pi}{c}\boldsymbol{j}. \end{aligned} \tag{5.1}$$

Using the gauge freedom in choosing the potentials, we can set

$$\frac{1}{c}\frac{\partial \Phi}{\partial t} + \nabla\cdot\boldsymbol{A} = 0,$$

thereby giving us the more compact equation

$$\left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right)\left[\begin{array}{c}\Phi\\ \boldsymbol{A}\end{array}\right] = 4\pi\left[\begin{array}{c}\rho\\ \boldsymbol{j}/c\end{array}\right]. \tag{5.2}$$

The operator in () on the left is often denoted as $\Box^2$; it is the space-time version of the Laplacian operator and is called the *d'Alambertian*.

To solve Eq. (5.2), we first find the Green function $G(\boldsymbol{x}, t; \boldsymbol{x}', t')$ with the property

$$\Box^2 G = 4\pi\delta(\boldsymbol{x} - \boldsymbol{x}')\delta(t - t');$$

the general solution is then

$$\begin{bmatrix} \Phi \\ \boldsymbol{A} \end{bmatrix} = \int G(\boldsymbol{x}, t; \boldsymbol{x}', t') \begin{bmatrix} \rho(\boldsymbol{x}', t') \\ \boldsymbol{j}(\boldsymbol{x}', t')/c \end{bmatrix} \mathrm{d}^3\boldsymbol{x}'\mathrm{d}t'. \tag{5.3}$$

Note that for $\boldsymbol{x} \neq \boldsymbol{x}'$ and $t \neq t'$, $\Box^2 G = 0$.

Let us expand $\Box^2$ in spherical coordinates for $\boldsymbol{x} \neq \boldsymbol{x}'$, $t \neq t'$, and take G to only depend on $r = |\boldsymbol{x} - \boldsymbol{x}'|$: then

$$\Box^2 G = \frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} - \frac{1}{r}\frac{\partial^2}{\partial r^2}(rG) = 0.$$

Away from $r = 0$, we can multiply this equation by r and recover the wave equation; thus the solution is

$$G = \frac{1}{r}\left[f_+(t - r/c) + f_-(t + r/c)\right].$$

Outgoing waves are represented by f_+, incoming by f_-.

Since we are considering sources, we only keep the f_+ term, which represents outgoing waves.

To pin down the form of f_+, we note that as we approach $r \to 0$, the term r/c becomes negligible compared to t. In that case, we expect the time derivatives to be small compared to the spatial derivatives, so that as we approach the origin our equation for G becomes

$$\Box^2 G\big|_{r\to 0} \to -\nabla^2\left(\frac{f_+(t)}{r}\right) = 4\pi\delta(\boldsymbol{r})\delta(t' - t).$$

But this is just Poisson's equation for a point particle at the origin with a funny "charge" $\delta(t' - t)$. We know the solution:

$$\frac{f_+(t)}{r} = \frac{\delta(t' - t)}{r}.$$

Now t here is really $t - r/c$ with r being really small; making this replacement gives us the retarded Green function,

$$G_+(\boldsymbol{x}, t; \boldsymbol{x}', t') = \frac{\delta[t' - (t - |\boldsymbol{x} - \boldsymbol{x}'|/c)]}{|\boldsymbol{x} - \boldsymbol{x}'|}. \tag{5.4}$$

Note that G_+ is non-zero only if t' lies on the past light-cone for point $(\boldsymbol{x}, t)$.

Substituting G_+ into Eq. (5.3) and taking the integral over t' gives,

$$\begin{bmatrix} \Phi \\ \boldsymbol{A} \end{bmatrix} = \int \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} \begin{bmatrix} \rho(\boldsymbol{x}', t - r/c) \\ \boldsymbol{j}(\boldsymbol{x}', t - r/c)/c \end{bmatrix} \mathrm{d}^3\boldsymbol{x}'. \tag{5.5}$$

Intuitively, this says that the contribution from a source a distance r away occurs when a photon has had time to traverse the distance from that source.

Now, suppose we have a single point particle of charge q moving on a path $\boldsymbol{\xi}(\tau)$ with velocity $\boldsymbol{u}(\tau) = \mathrm{d}\boldsymbol{\xi}/\mathrm{d}\tau$. The charge density and current density are then

$$\begin{bmatrix} \rho(\boldsymbol{x}', t - r/c) \\ \boldsymbol{j}(\boldsymbol{x}', t - r/c)/c \end{bmatrix} = \int \begin{bmatrix} q \\ q\boldsymbol{u}(\tau)/c \end{bmatrix} \delta[\boldsymbol{x}' - \boldsymbol{\xi}(\tau)]\delta\left[\tau - \left(t - \frac{|\boldsymbol{x} - \boldsymbol{x}'|}{c}\right)\right] \mathrm{d}\tau. \tag{5.6}$$

Substituting this equation for the sources into Eq. (5.5) and taking the integral with respect to $\mathrm{d}^3\boldsymbol{x}'$ gives

$$\begin{bmatrix} \Phi \\ \boldsymbol{A} \end{bmatrix} = \int \frac{1}{|\boldsymbol{x} - \boldsymbol{\xi}(\tau)|} \begin{bmatrix} q \\ q\boldsymbol{u}(\tau)/c \end{bmatrix} \delta\left[\tau - \left(t - \frac{|\boldsymbol{x} - \boldsymbol{\xi}(\tau)|}{c}\right)\right] \mathrm{d}\tau.$$

Time to change variables: let $\boldsymbol{r}(\tau) = \boldsymbol{x} - \boldsymbol{\xi}(\tau)$, and let $\tau' = \tau - (t - |\boldsymbol{r}|/c)$. Then $\mathrm{d}\tau' = \mathrm{d}\tau(1 + \dot{r}/c)$, and using $2r\dot{r} = 2\boldsymbol{r} \cdot \dot{\boldsymbol{r}} = -2\boldsymbol{r} \cdot \boldsymbol{u}$, we can finish the integral over τ' to obtain

$$\Phi(\boldsymbol{x}, t) = \left[\frac{q}{r(\tau)(1 - \hat{\boldsymbol{r}} \cdot \boldsymbol{u}/c)}\right]_{\tau = t - r(\tau)/c} \tag{5.7}$$

$$\boldsymbol{A}(\boldsymbol{x}, t) = \left[\frac{q\boldsymbol{u}(\tau)/c}{r(\tau)(1 - \hat{\boldsymbol{r}} \cdot \boldsymbol{u}/c)}\right]_{\tau = t - r(\tau)/c}. \tag{5.8}$$

Here $\hat{\boldsymbol{r}} = \boldsymbol{r}/|\boldsymbol{r}|$ is a unit directional vector along $\boldsymbol{r}$; it points from the location of the source at time τ to the location where the fields are to be evaluated.

The potentials [Equations (5.7) and (5.8)] have a part that depends on the particles position and velocity at retarded time $t - r/c$, which one might have expected on analogy with electrostatics, and a factor in the denominator that depends on $\boldsymbol{u}/c$, which is a bit less intuitive. Note the effect of $\hat{\boldsymbol{r}} \cdot \boldsymbol{u}$: if the particle is moving relativistically, then the potentials are quite large for directions in front of the particles' line of motion.

The fields can be found by straightforward, albeit tedious, differentiation. Defining $\boldsymbol{\beta} = \boldsymbol{u}/c$ and $\kappa = 1 - \hat{\boldsymbol{r}} \cdot \boldsymbol{\beta}$, the fields from a moving particle of charge q can be expressed as

$$\boldsymbol{E}(\boldsymbol{x}, t) = \left[\frac{q(1 - \beta^2)}{\kappa^3 r^2}(\hat{\boldsymbol{r}} - \boldsymbol{\beta}) + \frac{q}{c\kappa^3 r}\hat{\boldsymbol{r}} \times \left\{(\hat{\boldsymbol{r}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\right\}\right]_{\tau = t - r(\tau)/c} \tag{5.9}$$

$$\boldsymbol{B}(\boldsymbol{x}, t) = [\hat{\boldsymbol{r}} \times \boldsymbol{E}(\boldsymbol{x}, t)]_{\tau = t - r(\tau)/c} \tag{5.10}$$

Bear in mind that $\boldsymbol{r}$, $\hat{\boldsymbol{r}}$, and $\boldsymbol{\beta}$ are all functions of τ.

There are two terms in the expression for $\boldsymbol{E}$, and they scale differently with r. The first term goes as q/r^2, just like the electrostatic version. Note the direction, however: instead of pointing along $\hat{\boldsymbol{r}}$, that is, to the position at the retarded time, it points along $\hat{\boldsymbol{r}} - \boldsymbol{\beta}$, which is away from the position the particle would have at time t if β were constant. It is as if the electric field "anticipates" the motion of the particle.

The second term falls off as r^{-1}, so it is the dominant term sufficiently far from the source and is therefore the radiation field. This term is proportional to the acceleration $\dot{\boldsymbol{\beta}}$ of the particle. Notice that when this term dominates, $\boldsymbol{E}$ and $\boldsymbol{B}$ are both perpendicular to $\hat{\boldsymbol{r}}$, and $\boldsymbol{B}$ is perpendicular to $\boldsymbol{E}$. In the non-relativistic limit, $|\boldsymbol{\beta}| \ll 1$, let θ be the angle between $\dot{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{r}}$. Then

$$|\boldsymbol{B}| = |\boldsymbol{E}| \simeq \frac{q}{c^2 r} |\dot{\boldsymbol{u}}| \sin\theta;$$

$\boldsymbol{E}$ lies in the plane defined by $\hat{\boldsymbol{r}}$ and $\dot{\boldsymbol{\beta}}$ and is perpendicular to $\hat{\boldsymbol{r}}$. The radiation fields are maximum in a direction perpendicular to the acceleration.

EXERCISE 5.1— The schematic (5.1) shows the trajectory $\boldsymbol{\xi}(\tau)$ of a charged particle. The arcs show loci of constant retarded times from the point "O", and the spacing between arcs is unity. On this schematic, indicate the "near" electric field at point "O"; for the other 7 points (all located at the same light travel time from the particle), indicate the directions of the radiation $\boldsymbol{E}$ and $\boldsymbol{B}$ fields.

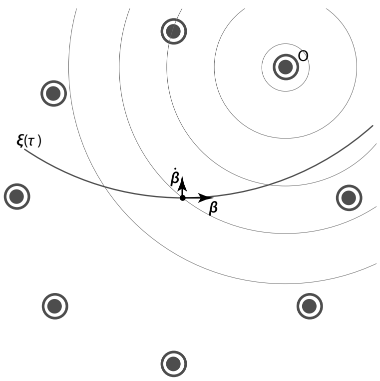

Figure 5.1: Schematic for exercise 5.1.

The flux can be found by computing the Poynting vector:

$$\boldsymbol{S} = \frac{c}{4\pi}\boldsymbol{E}\times\boldsymbol{B} = \frac{c}{4\pi}|\boldsymbol{E}|^2\,\hat{\boldsymbol{r}} = \frac{q^2}{4\pi c^3 r^2}|\dot{\boldsymbol{u}}|^2 \sin^2\theta\,\hat{\boldsymbol{r}}.$$

To get the total power emitted, we encase our charge in a sphere of radius r, centered on the particle, with the axis along $\dot{\boldsymbol{u}}$. In this case the flux is normal to the sphere, so the total power is

$$P = \int |\boldsymbol{S}|\, r^2 \mathrm{d}\Omega = \frac{q^2}{4\pi c^3}|\dot{\boldsymbol{u}}|^2 \left[2\pi \int_{-1}^{1} (1-\mu^2)\mathrm{d}\mu\right] = \frac{2q^2}{3c^3}|\dot{\boldsymbol{u}}|^2, \qquad (5.11)$$

a result known as *Larmor's formula*.

5.2 *Thomson scattering*

As an application of Larmor's formula, let's consider a free electron sitting in space, which is irradiated by low-frequency radiation. The electron will accelerate because of the electric field; as a result of this acceleration, the electron will then radiate.

EXERCISE 5.2— Why can we neglect the magnetic field when computing the acceleration of the charge?

The equation of motion of the electron is

$$m_e \dot{\boldsymbol{u}} = q_e E e^{i\omega t} \boldsymbol{\xi}, \qquad (5.12)$$

where $\boldsymbol{\xi}$ is the polarization direction of the electric field (we'll assume plane polarization) and q_e is the electron charge. From Eq. (5.11), the average power emitted by this charge over a cycle is

$$\langle P \rangle = \frac{1}{3} \frac{q_e^4}{m_e^2 c^3} E^2;$$

if we compare this with the incident flux, averaged over a cycle, $\langle S_{\rm inc.} \rangle = c|E^2|/8\pi$, we find the total cross-section for *Thomson scattering*:

$$\sigma_{\rm Th} = \frac{\langle P \rangle}{\langle S_{\rm inc.} \rangle} = \frac{8\pi}{3} \left(\frac{q_e^2}{m_e c^2} \right)^2 = 0.665 \times 10^{-24}\ {\rm cm}^2. \quad (5.13)$$

The quantity in parentheses is known as the *classical electron radius*.

The scattered radiation is polarized. To see how this works, let's set up our coordinates so that the incident light is traveling in the z-direction. The electron is at the origin, and the vector $\hat{\boldsymbol{r}}$ to the observer is in the z-y plane, as shown in Fig. 5.2. Then we may choose the polarization vectors of our incident ray as being along the x- and y-axes.

For incident radiation with polarization $\boldsymbol{\xi}_1$, we can combine equation (5.12) for the motion of the electron with equation (5.9) for the electric field (for non-relativistic electrons, $|\boldsymbol{\beta}| \ll 1$); since $\hat{\boldsymbol{r}}$ is perpendicular to $\boldsymbol{\xi}_1$, the electric field is

$$\boldsymbol{E}_1' = -\frac{e^2}{mc^2 r} E_1 e^{i\omega t}\, \boldsymbol{\xi}_1'$$

and the scattered flux along $\hat{\boldsymbol{r}}$ is

$$\boldsymbol{S}_1 = \frac{c E_1^2}{8\pi r^2} \left(\frac{e^2}{m_e c^2} \right)^2 \hat{\boldsymbol{r}}.$$

A factor of $1/2$ comes from averaging $|\Re(e^{i\omega t})|^2 = \cos^2 \omega t$ over several oscillation cycles.

For incident radiation with polarization $\boldsymbol{\xi}_2$ along y, the cross-product of $\hat{\boldsymbol{r}}$ and $\dot{\boldsymbol{\beta}}$ picks up a factor of $\cos\theta$, where θ is the angle between the incident and scattered rays. As a result, the scattered electric field is

$$\boldsymbol{E}_2' = -\frac{e^2}{mc^2 r} E_2 e^{i\omega t} \cos\theta\, \boldsymbol{\xi}_2'$$

and the scattered flux along $\hat{\boldsymbol{r}}$ is

$$\boldsymbol{S}_2 = \frac{c E_2^2}{8\pi r^2} \left(\frac{e^2}{m_e c^2} \right)^2 \cos^2\theta\, \hat{\boldsymbol{r}}.$$

If the incident radiation is unpolarized, so that $E_1 = E_2$, then the degree of polarization for the scattered radiation is

$$\Pi \equiv \frac{|S_1| - |S_2|}{|S_1| + |S_2|} = \frac{1 - \cos^2\theta}{1 + \cos^2\theta}. \quad (5.14)$$

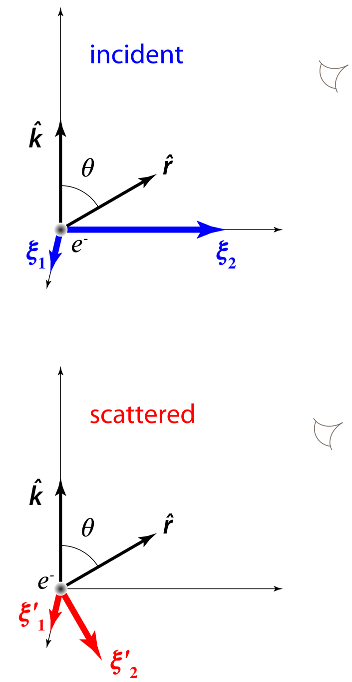

Figure 5.2: Schematic of Thomson scattering.

5.3 The classical oscillator

Suppose we have a classical charged harmonic oscillator, $\boldsymbol{x}(t) = \boldsymbol{x}_0 e^{i\omega t}$, of charge q_e. The instantaneous power emitted by the oscillator is

$$P(t) = \frac{2}{3}\frac{q_e^2}{c^3}|\dot{\boldsymbol{u}}|^2, \tag{5.15}$$

which when averaged over a cycle is

$$\langle P(t)\rangle = \frac{q_e^2}{3c^3}x_0^2\omega^4, \tag{5.16}$$

since $\dot{\boldsymbol{u}} = -\omega^2\boldsymbol{x}_0 e^{i\omega t}$. Since the oscillator is radiating, it is losing energy and is damped. In mechanics, the loss of energy goes as $\boldsymbol{F}\cdot\boldsymbol{u}$; we can get a term that has this form by integrating equation (5.15) by parts over a cycle:

$$-\int_{t_1}^{t_2}\mathrm{d}t\,\frac{2}{3}\frac{q_e^2}{c^3}\dot{\boldsymbol{u}}\cdot\dot{\boldsymbol{u}} = -\frac{2}{3}\frac{q_e^2}{c^3}\dot{\boldsymbol{u}}\cdot\boldsymbol{u}\Big|_{t_1}^{t_2} + \frac{2}{3}\frac{q_e^2}{c^3}\int_{t_1}^{t_2}\mathrm{d}t\,\ddot{\boldsymbol{u}}\cdot\boldsymbol{u}.$$

The first term vanishes and we can therefore identify the radiation damping term as

$$\boldsymbol{F}_{\mathrm{rad}} = \frac{2}{3}\frac{q_e^2}{c^3}\ddot{\boldsymbol{u}} = -m\left(\frac{2q_e^2\omega^2}{3c^3m}\right)\boldsymbol{u} \equiv -m\gamma\boldsymbol{u}$$

with the term in parentheses being the damping constant γ.

Now let our oscillator's "natural" frequency be ω_0 and drive the oscillator with an electric field $\boldsymbol{E}e^{i\omega t}$; the equation of motion for the oscillator is then

$$m\ddot{\boldsymbol{x}} = -m\omega_0^2\boldsymbol{x} + q_e\boldsymbol{E}e^{i\omega t} - m\gamma\dot{\boldsymbol{x}}. \tag{5.17}$$

Substituting a trial function $\boldsymbol{x} \propto e^{i\omega t}$ gives

$$\boldsymbol{x} = \frac{q_e}{m}\frac{\boldsymbol{E}e^{i\omega t}}{(\omega_0^2-\omega^2)+i\omega\gamma}. \tag{5.18}$$

Taking the second derivative w.r.t. time of $\boldsymbol{x}$, substituting into eq. (5.11), and averaging over a cycle gives the power radiated by the oscillator,

$$\langle P(t)\rangle = \frac{q_e^4\omega^4E^2}{3c^3m^2}\frac{1}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}.$$

Dividing $\langle P(t)\rangle$ by the incident average flux, $cE^2/(8\pi)$, gives the cross-section,

$$\sigma = \left(\frac{8\pi}{3}\frac{q_e^4}{m^2c^4}\right)\frac{\omega^4}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}. \tag{5.19}$$

For $\omega \gg \omega_0, \gamma$ this reduces to the Thomson cross-section for $q_e = e$. We'll next explore the cases $\omega \ll \omega_0$ and $\omega \approx \omega_0$.

Rayleigh scattering

For $\omega \ll \omega_0$, the cross-section (5.19) for scattering becomes

$$\sigma_{\text{Ray}} \simeq \left(\frac{8\pi}{3}\frac{q_e^4}{m^2c^4}\right)\left(\frac{\omega}{\omega_0}\right)^4. \tag{5.20}$$

This is important in planetary atmospheres: the strong frequency dependence accounts for the blue sky. Physically, the scattering is caused by the polarization of molecules induced by the electric field.

Of course, this model is really crude: can we really calculate the polarization of air molecules this way? What should we use for the charge q_e—is it e? and what for the mass m? It turns, out, amazingly enough, that we don't need to know them to determine the cross-section and the polarization. In the limit that we go to very low frequency, then from Eq. (5.18) we have the induced polarization per unit volume,

$$\boldsymbol{P} = nq_e\boldsymbol{x} \approx \frac{nq_e^2}{m\omega_0^2}\boldsymbol{E},$$

where n is the number of molecules per unit volume. The electric displacement is therefore

$$\boldsymbol{D} = \boldsymbol{E} + 4\pi\boldsymbol{P} = \varepsilon\boldsymbol{E}$$

with permitivity

$$\varepsilon = 1 + \frac{4\pi nq_e^2}{m\omega_0^2}. \tag{5.21}$$

The effective velocity of light in such a medium is $c/\sqrt{\varepsilon}$, so that the index of refraction is $N = \sqrt{\varepsilon}$. We can therefore express $m\omega_0^2/q^2$ in terms of the index of refraction of air; doing so and substituting back into Eq. (5.20) gives (to lowest order)

$$\sigma_{\text{Ray}} \simeq \frac{2}{3\pi n^2}\left(\frac{2\pi}{\lambda}\right)^4 |N-1|^2. \tag{5.22}$$

As advertised, this expression does not involve the charges or masses of our oscillators, but we do need a measurement of the index of refraction. For a standard atmosphere with density $n = 2.7\times10^{19}\,\text{cm}^{-3}$ and index of refraction $N-1 \approx 2.93\times10^{-4}$, we find that the mean free path, $\ell = (n\sigma_{\text{Ray}})^{-1}$, is 187 km for red light ($\lambda = 650$ nm) and 30 km for violet light ($\lambda = 410$ nm). Consult Jackson[1] for a detailed calculation and Feynman et al.[2] for an intuitive one.

[1] John D. Jackson. *Classical Electrodynamics*. Wiley, 2d edition, 1975

[2] Richard P. Feynman, Robert B. Leighton, and Matthew Sands. *The Feynman Lectures on Physics*. Addison-Wesley, 1989

The resonant oscillator

Now, for $\omega \approx \omega_0$, we can expand $(\omega_0^2-\omega^2)^2 \approx 4\omega_0^2(\omega_0-\omega)^2$; furthermore, we identify $2q_e^2\omega_0^2/(3c^3m) = \gamma$ and equation (5.19) becomes

$$\sigma = \pi\left(\frac{q_e^2}{mc}\right)\frac{\gamma}{(\omega_0-\omega)^2+(\gamma/2)^2}. \tag{5.23}$$

The line profile is Lorentzian, with a width γ. In terms of wavelength, the width of the line is

$$\Delta\lambda = \gamma \left|\frac{d\lambda}{d\omega}\right|_{\omega=\omega_0} = \frac{2\pi c}{\omega_0^2}\gamma = \frac{4\pi}{3}\left(\frac{q_e^2}{mc^2}\right) = 1.2 \times 10^{-5}\,\text{nm}.$$

If we apply this model to an atomic transition, we get an estimate of the natural line width. This width is independent of the transition frequency[3], and it is extremely narrow compared to the width from other interactions and from doppler broadening.

[3] It is just the classical electron radius

EXERCISE 5.3 — Consider the transition from the $n = 3$ level to the $n = 2$ level in hydrogen.

1. What is the wavelength of this transition?
2. From the linewidth $\Delta\lambda$ given above, estimate the mean lifetime of the $n = 3$ level against spontaneous de-excitation to the $n = 2$ level.

6
Plasmas

6.1 What is a plasma?

A *plamsa* is defined as a gas of charged particles in which the kinetic energy of a typical particle is much greater than the potential energy due to its nearest neighbors.

Screening and the Debye Length

Imagine a typical charged particle in a plasma. Very close to the particle, we expect the electrostatic potential to be that of an isolated charge $\Phi = q/r$. Far from the particle, there will be many other particles surrounding it, and the potential is *screened*. For example, a positive ion will tend to attract electrons to be somewhat, on average, closer to it than other ions: we say that the ion *polarizes* the plasma. As a result of this polarization, the potential of any particular ion should go to zero much faster than $1/r$ due to the "screening" from the enhanced density of opposite charges around it.

Let's consider a plasma having many ion species, each with charge Z_i, and elections. About any selected ion j, particles will arrange themselves according to Boltzmann's law,

$$n_i(r) = n_{i0} \exp\left[-\frac{Z_i e\Phi(r)}{kT}\right]. \tag{6.1}$$

Here n_{i0} is the density of particle i far from the charge j, and r is the distance between particles i and j. (A similar equation holds for the electrons, with Z replaced by -1.) To solve for the potential, we can use Poisson's equation,

$$\nabla^2\Phi = -4\pi \sum_i Z_i e n_i(r) + 4\pi e n_e(r). \tag{6.2}$$

Our assumption is that the term in the exponential of Eq. (6.1) is small, so we may expand it to first order in Φ and substitute that expansion into

Eq. (6.2) to obtain in spherical geometry

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) = -4\pi e\left[\sum_i n_{i0}Z_i\left(1-\frac{Z_i e\Phi}{kT}\right) - n_{e0}\left(1+\frac{e\Phi}{kT}\right)\right].$$

The overall charge neutrality of the plasma implies that $n_{e0} = \sum_i Z_i n_{i0}$; using this to simplify the above equation gives

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) = \left[\frac{4\pi e^2}{kT}\sum_i n_{i0}\left(Z_i^2 + Z_i\right)\right]\Phi \equiv \lambda_{\rm D}^{-2}\Phi. \tag{6.3}$$

The quantity in [ ] has dimensions of reciprocal length squared and we define it as $(1/\lambda_{\rm D})^2$ with $\lambda_{\rm D}$ being called the *Debye length*.

Multiplying equation (6.3) by r, integrating twice, and determining the constant of integration from the condition that as $r \to 0$, $\Phi \to Z_j e/r$ gives the self-consistent potential

$$\Phi = \frac{Z_j e}{r}\exp\left(-\frac{r}{\lambda_{\rm D}}\right). \tag{6.4}$$

The Debye length $\lambda_{\rm D}$ determines the size of the screening cloud around the ion.

In order for the above derivation to be valid, we require that $\lambda_{\rm D} \gg a$, where a is the mean ion spacing; otherwise, there won't be any charges in our cloud to screen the potential! Equivalently, we require the number of particles in a sphere of radius $\lambda_{\rm D}$ to be large,

$$\frac{4\pi}{3}\lambda_{\rm D}^3\sum_i n_i \gg 1. \tag{6.5}$$

This condition must hold if we are to treat the gas as an (ideal) plasma[1].

[1] In high energy density physics, the definition of a plasma is expanded to include cases for which interactions are important

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EXERCISE 6.1— Defining the mean inter-ion spacing a via $4\pi a^3 n/3 = 1$, show that Eq. (6.5) implies that $kT \gg e^2/a$.

6.2 *Propagation of waves through a plasma*

Dispersion in a cold plasma

Suppose that we have a plane wave propagating through a medium containing free electrons with uniform density n_e. The electric field will cause the electrons to oscillate, cf. Eq. (5.12). We'll take the plasma to be cold, so that thermal velocities are small, and we'll assume that the amount of power scattered (Thomson scattering) is also negligible. Finally, we'll ignore collisions in the plasma, variations in the plane of the wave, and oscillations of the ions: as a result the plasma remains neutral everywhere.

The back-and-forth sloshing of the electrons means that there is an alternating current in the plasma, $\boldsymbol{j} = -en_e\boldsymbol{u}$. Since we assume that there is no bunching of electrons, $\nabla \cdot \boldsymbol{E} = 0$; then taking the time derivative of equation (2.4), using equation (2.1) to eliminate $\partial_t \boldsymbol{B}$, and expanding[2] $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{E}) = \boldsymbol{\nabla}(\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E} = \nabla^2 \boldsymbol{E}$ gives

[2] cf. Box 6.1

$$\left(\nabla^2 - \frac{1}{c^2}\partial_t^2\right)\boldsymbol{E} = \frac{4\pi}{c^2}\partial_t \boldsymbol{j} = \frac{4\pi n_e e^2}{m_e c^2}\boldsymbol{E}. \tag{6.6}$$

In this equation we have used $\partial_t \boldsymbol{u} = -e\boldsymbol{E}/m_e$, with m_e being the electron mass. Using a trial solution $\boldsymbol{E} = \boldsymbol{E_k} e^{i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega t}$ gives a *dispersion relation*,

$$c^2 k^2 = \omega^2 - \omega_p^2, \tag{6.7}$$

where

$$\omega_p^2 = \frac{4\pi n_e e^2}{m_e}$$

is the *plasma frequency*.

Box 6.1 Tensors and index notation

A powerful notation when working with tensors is to use the rule that repeated indices are summed over. For example, if x_i, y_j represent vectors in a Euclidian space with components $[x_1, x_2, x_3]$ and $[y_1, y_2, y_3]$, respectively, then the dot product of the vectors is $x_i y_i \equiv x_1 y_1 + x_2 y_2 + x_3 y_3$.

In working with vectors, two useful symbols are the Kronecker delta, defined by

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}, \tag{6.8}$$

and the Levi-Civita symbol, defined by

$$\varepsilon_{ijk} = \begin{cases} 1 & i, j, k \text{ are a cyclic permutation of } 1, 2, 3 \\ -1 & i, j, k \text{ are an anti-cyclic permutation of } 1, 2, 3 \\ 0 & \text{if any indices are identical} \end{cases} \tag{6.9}$$

By a "cyclic permutation of 1, 2, 3", we mean $\{1, 2, 3\}$, $\{3, 1, 2\}$, or $\{2, 3, 1\}$; by "anti-cyclic", we mean $\{2, 1, 3\}$, $\{3, 2, 1\}$, or $\{1, 3, 2\}$—that is, any combination obtained from $\{1, 2, 3\}$ by a single exchange of indices.

In terms of the Levi-Civita symbol, the i^{th} component of the cross product of two vectors $\boldsymbol{a}$, $\boldsymbol{b}$ is written

$$[\boldsymbol{a} \times \boldsymbol{b}]_i = \varepsilon_{ijk} a_j b_k.$$

For example, if $i = 1$, $\varepsilon_{ijk} a_j b_k = a_2 b_3 - a_3 b_2$.

Box 6.1 continued

EXERCISE 6.2 — Show that

$$\varepsilon_{ijk}\varepsilon_{lmn} = \delta_{il}\delta_{jm}\delta_{kn} + \delta_{kl}\delta_{im}\delta_{jn} + \delta_{jl}\delta_{km}\delta_{in} - \delta_{jl}\delta_{im}\delta_{kn} - \delta_{kl}\delta_{jm}\delta_{in} - \delta_{il}\delta_{km}\delta_{jn}.$$

EXERCISE 6.3 — Use the index notation along with the symbols ε_{ijk} and δ_{ij} and the result of exercise 6.2 to prove the following relations.

1. $\nabla \cdot (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{b} \cdot (\boldsymbol{\nabla} \times \boldsymbol{a}) - \boldsymbol{a} \cdot (\boldsymbol{\nabla} \times \boldsymbol{b})$
2. $\boldsymbol{\nabla} \times (\boldsymbol{a} \times \boldsymbol{b}) = (\boldsymbol{b} \cdot \boldsymbol{\nabla})\, \boldsymbol{a} - \boldsymbol{b}\,(\nabla \cdot \boldsymbol{a}) + \boldsymbol{a}\,(\nabla \cdot \boldsymbol{b}) - (\boldsymbol{a} \cdot \boldsymbol{\nabla})\, \boldsymbol{b}$
3. $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{a}) = \boldsymbol{\nabla}\,(\nabla \cdot \boldsymbol{a}) - \nabla^2 \boldsymbol{a}$

For $\omega < \omega_p$, the wavevector $\boldsymbol{k}$ becomes imaginary, and the wave *evanesces* over a lengthscale $\sim (\pi m_e c^2/n_e e^2)^{1/2}$. This is analogous to the skin depth in a conductor: the charges move to short out the electric field. For $\omega > \omega_p$ the group velocity $v_g = \partial\omega/\partial k$ depends on frequency:

$$v_g = c\left[1 - \left(\frac{\omega_p}{\omega}\right)^2\right]^{1/2} < c. \tag{6.10}$$

Higher frequencies travel faster.

EXERCISE 6.4 — Pulsars are magnetized neutron stars that emit a broad spectrum of radiation into a narrow beam. As the neutron star rotates, the beam is swept into and out of the observer's field of view, thereby creating pulses. Suppose you observe the pulses from a particular neutron star over a range of radio frequencies. Show that the time of arrival t_A of the pulses changes with frequency as

$$\frac{dt_A}{d\nu} = -\frac{e^2}{\pi m_e c \nu^3}\mathcal{D},$$

where the *dispersion measure*

$$\mathcal{D} = \int n_e\, d\ell$$

is the integrated column of free electrons along the line of sight to the pulsar. Show that the delay time between two observed frequencies is

$$\Delta t = 8.3\,\text{ms}\;\mathcal{D}\frac{\Delta\nu}{\nu^3}$$

for $\mathcal{D}$ in units of pc cm^{-3} and ν, $\Delta\nu$ in units of GHz.

Dispersion in a cold, magnetized plasma

Now we'll expand the discussion in the previous section to the more general case of a cold, magnetized plasma. We shall again ignore collisions and the motion of ions. We'll relax, however, our assumption that the electron density is uniform (although it will turn out that that is still a valid assumption).

First, we need to review how electrons move under a combined electric and magnetic field (see Fig. 6.1):

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = -\frac{e}{m_e}\boldsymbol{E} - \frac{e}{m_e c}\boldsymbol{u}\times\boldsymbol{B}. \qquad (6.11)$$

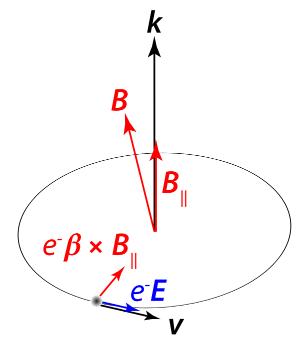


Figure 6.1: Schematic of the response of an electron in a static magnetic field **B** to a wave propagating along direction **k**.

Here **B** is the sum of the static and the wave fields. As we argued before, however, for non-relativistic electrons the contribution from the wave's **B** field is $\sim u/c$ smaller than than that from the wave's **E** field. Further, if the only appreciable motion is due to the wave's **E** field, which is perpendicular to **k**, then only $B_{||} = \boldsymbol{B}\cdot\boldsymbol{k}/|\boldsymbol{k}|$ is important. This is equivalent to stating that non-relativistic electrons will only move a small fraction of a wavelength in one oscillation cycle, so we can neglect motion along **k**.

If we are at a fixed point, we can look for oscillatory solutions at the wave frequency ω; however, if we take our z-axis to be along **k**, then $\mathrm{d}u_x/\mathrm{d}t$ depends on u_y and vice versa. To get around this, recall that we expect the electron to move in a circular fashion, in which case the x- and y-components of the velocity are $\pi/2$ out of phase. This suggests that we choose for basis vectors the right- and left-handed helical vectors:

$$\boldsymbol{u}_\pm = u_0\hat{\boldsymbol{e}}_\pm$$

where[3]

$$\hat{\boldsymbol{e}}_+ = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{e}}_x + i\hat{\boldsymbol{e}}_y),$$
$$\hat{\boldsymbol{e}}_- = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{e}}_x - i\hat{\boldsymbol{e}}_y).$$

[3] cf. §2.2

If we therefore write **u** and **E** in terms of modes $u_\pm\hat{\boldsymbol{e}}_\pm e^{-i\omega t}$, $E_\pm\hat{\boldsymbol{e}}_\pm e^{-i\omega t}$ and substitue into Equation (6.11) we find that

$$u_\pm = -i\frac{e}{m_e(\omega \mp \omega_{\mathrm{L}})}E_\pm. \qquad (6.12)$$

Here

$$\omega_{\mathrm{L}} = \frac{eB_{||}}{m_e c}$$

is the electron Larmor frequency. The current induced by the electric field is thus

$$j_\pm = -n_e e u_\pm = i\frac{n_e e^2}{m_e(\omega \mp \omega_{\mathrm{L}})}E_\pm = \sigma_\pm E_\pm$$

where $\sigma_\pm$ is the electrical conductivity. The factor of i in σ implies that the current is out of phase with the oscillatory electric field.

EXERCISE 6.5 — Show that Eq. (6.12) implies that the time-averaged work done by the electric field is zero for $\omega \neq \omega_{\mathrm{L}}$. That is, in the absence of collisions, there is no dissipation.

Now that we have the electronic response, we can look for solutions to the equation of charge continuity, $\partial_t(\rho_e) + \nabla \cdot \boldsymbol{j} = 0$. Here $\rho_e = Zen_i - en_e$ is the combined charge density; in the absence of perturbations from the electromagnetic wave the plasma is neutral, $\rho_e = 0$. Assuming a $e^{-i\omega t}$ response for ρ_e, we obtain

$$\rho_e = i\frac{n_e e^2}{m_e(\omega \mp \omega_{\rm L})\omega}\boldsymbol{k} \cdot \boldsymbol{E}.$$

Inserting this into Gauss's law, $\nabla \cdot \boldsymbol{E} = 4\pi\rho_e$, implies

$$i\boldsymbol{k} \cdot \boldsymbol{E} = \frac{4\pi n_e e^2}{m_e\omega(\omega \mp \omega_{\rm L})} i\boldsymbol{k} \cdot \boldsymbol{E} = \frac{\omega_p^2}{\omega(\omega \mp \omega_{\rm L})} i\boldsymbol{k} \cdot \boldsymbol{E},$$

where $\omega_p = 4\pi n_e e^2/m_e$ is again the electron plasma frequency. This is equivalent to writing

$$\varepsilon \nabla \cdot \boldsymbol{E} \equiv \left[1 - \frac{\omega_p^2}{\omega(\omega \mp \omega_{\rm L})}\right] \nabla \cdot \boldsymbol{E} = 0$$

with ε being the dielectric constant. Since $\varepsilon \neq 0$ in general, we require $\boldsymbol{k} \cdot \boldsymbol{E} = 0$: that is, the wave is transverse and therefore $\rho_e = 0$; there is no bunching of excess charge and the plasma remains neutral. In that case, $\boldsymbol{k} \cdot \boldsymbol{j} = 0$: the currents are purely transverse as well.

With $\nabla \cdot \boldsymbol{E} = 0$, we insert our trial function $\boldsymbol{E} = \boldsymbol{E}_\pm e^{i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega t}$ into Eq. (6.6) and obtain a dispersion relation for right(left)-circularly polarized waves:

$$c^2 k_\pm^2 = \omega^2 \left[1 - \frac{\omega_p^2}{\omega(\omega \mp \omega_{\rm L})}\right] = \varepsilon\omega^2. \tag{6.13}$$

For $\omega \gg \omega_{\rm L}$, we recover our previous dispersion relation, Eq. (6.7). At higher frequencies, $\omega \gg \omega_p, \omega_{\rm L}$ we can expand ε and write the dispersion relation as

$$k_\pm = \frac{\omega}{c} - \underbrace{\frac{\omega_p^2}{2\omega c}}_{=\Delta k_0} \mp \underbrace{\frac{\omega_p^2 \omega_{\rm L}}{2c\omega^2}}_{=\Delta k_\pm}.$$

EXERCISE 6.6 — Suppose we have a plane-polarized wave,

$$\boldsymbol{E} = \frac{1}{\sqrt{2}} \left(\hat{\boldsymbol{e}}_+ + \hat{\boldsymbol{e}}_- \right) E_k e^{i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega t}$$

traversing a magnetized plasma in the z-direction. Show that after going a length ℓ, the right(left)-circular polarization components will have a phase

$$-\int_0^\ell \Delta k_0 \, \mathrm{d}z \mp \int_0^\ell \Delta k_\pm \, \mathrm{d}z$$

relative to what they would have had in the absence of the plasma. Show that the electric vector after going a length ℓ has a polarization

$$\cos\psi \hat{\boldsymbol{e}}_x + \sin\psi \hat{\boldsymbol{e}}_y$$

where

$$\psi = \frac{e^3}{2\pi m_e^2 c^2 \nu^2} \underbrace{\int_0^\ell n_e B_{||} \, \mathrm{d}z}_{\equiv \mathcal{R}} .$$

In other words, the polarization vector has rotated by an angle ψ, and this angle depends on frequency. Thus measurements of the plane of polarization can be used to infer $\mathcal{R}$, the *rotation measure*, which provides information on the integrated line-of-sight strength of the magnetic field.

7
Bremsstrahlung Radiation

7.1 Collisions in a plasma

To begin, let's imagine a light particle (electron) colliding with a much heavier, fixed particle (an ion), as illustrated in Figure 7.1. (This picture also applies to a pseudo particle of reduced mass scattering in a fixed potential.) Let the impact parameter be b, and the mass of the incident particle is μ. For Coulomb interactions, the force on the particle is $(q_1q_2/r^2)\hat{\boldsymbol{r}}$. The incident momentum is p_0. Now by assumption, in our plasma most of the interactions are weak (potential energy is much less than kinetic), so let's treat the deflection of the particle as a perturbation. That is, we shall assume that $p_0 = \text{const}$ and that the effect of the interaction is to produce a perpendicular (to p_0) component of the momentum $p_\perp$. The total change in $p_\perp$ is then

$$p_\perp = \int_{-\infty}^{\infty} \mathrm{d}t\, \frac{q_1q_2}{r^2}\sin\theta, \tag{7.1}$$

where $\sin\theta = b/r$ is the angle that the radial vector makes with the horizontal. Substituting $r = b/\sin\theta$ and $\mathrm{d}t = -\mu b\,\mathrm{d}\theta/p_0/\sin^2\theta$, we have

$$p_\perp = -\int_0^{\pi} \sin\theta\,\mathrm{d}\theta\, \frac{\mu}{p_0}\frac{q_1q_2}{b},$$

leading to the intuitive result

$$\frac{p_0p_\perp}{2\mu} = \frac{q_1q_2}{b}. \tag{7.2}$$

Clearly a large angle scattering occurs if $p_\perp \geq p_0$, or

$$b \leq b_0 \equiv \frac{2\mu q_1q_2}{p_0^2}; \tag{7.3}$$

our perturbative approach is therefore only valid for $b \gg b_0$.

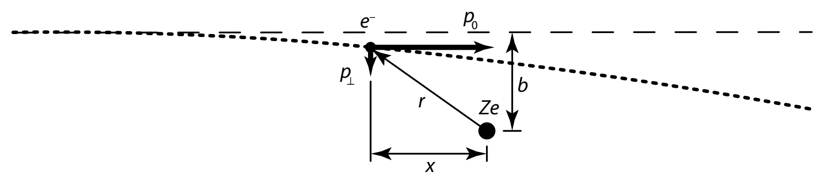


Figure 7.1: Geometry for scattering problem.

7.2 Emissivity

To calculate the emissivity, we start with the acceleration of an electron; according to Eq. (7.1ff) this is

$$|\dot{\boldsymbol{v}}| = \frac{1}{m_e}\left|\frac{\mathrm{d}p}{\mathrm{d}t}\right| = \frac{Ze^2}{m_e r^2} = \frac{Ze^2}{m_e b^2}\frac{1}{1+v^2t^2/b^2}. \tag{7.4}$$

If we substitute the maximum value of $|\dot{\boldsymbol{v}}|$ into Larmor's formula, Eq. (5.11), the maximum power emitted is

$$P(b) = \frac{2}{3}\frac{e^2}{c^3}|\dot{\boldsymbol{v}}|^2 = \frac{2}{3}\frac{Z^2e^6}{m_e^2c^3b^4}. \tag{7.5}$$

EXERCISE 7.1— Plot $P(t;b)$. Set the origin $t=0$ to be the point of closest approach.

If we take $t = 0$ to correspond to when the electron is at closest approach, then most of the acceleration occurs in a range of times $-b/v < t < b/v$.

WHAT ARE THE FREQUENCIES AT WHICH THIS POWER IS RADIATED? If we take the Fourier transform of the acceleration, then by Parseval's theorem the total energy emitted during the encounter is

$$\frac{2}{3}\frac{e^2}{c^3}\int_{-\infty}^{\infty}|\dot{v}|^2\,\mathrm{d}t = \frac{2}{3}\frac{e^2}{c^3}\int_{-\infty}^{\infty}|\dot{v}_\nu|^2\,\mathrm{d}\nu = \frac{4}{3}\frac{e^2}{c^3}\int_0^{\infty}\dot{v}_\nu^*\dot{v}_\nu\,\mathrm{d}\nu, \tag{7.6}$$

where

$$\dot{v}_\nu = \int_{-\infty}^{\infty}\dot{v}e^{2\pi i\nu t}\,\mathrm{d}t$$

is the Fourier transform of the acceleration. In the last equality of equation (7.6), we use $\dot{v}_{-\nu} = \dot{v}_\nu^*$ since v is real. From equation (7.6), we identify the energy emitted per frequency as

$$\frac{4}{3}\frac{e^2}{c^3}\dot{v}_\nu^*\dot{v}_\nu$$

for $\nu > 0$.

Taking the Fourier transform of the acceleration (7.4) gives

$$\dot{v}_\nu = \frac{Ze^2}{m_e b v}\int_{-\infty}^{\infty}\frac{\exp(2\pi i\nu b\xi/v)}{1+\xi^2}\,\mathrm{d}\xi = \pi\frac{Ze^2}{m_e b v}\exp(-2\pi\nu b/v), \qquad (7.7)$$

with the energy spectral density

$$\frac{4\pi^2}{3}\frac{Z^2e^6}{m_e^2c^3v^2b^2}\exp(-4\pi\nu b/v). \qquad (7.8)$$

The power is distributed over a broad range of frequencies up to a cutoff $\nu_{\max} \sim v/b$.

To evaluate the integral in eq. (7.7), we extend our domain of integration to the complex plane. In the upper half plane, the integrand vanishes as $|\xi| \to \infty$, so we may close the integral along a semi-circle in the upper plane. Because the integrand has a pole at $\xi = i$ that is enclosed by the contour, we can use the theorem on residues to evaluate the integral,

$$\oint \frac{\exp(2\pi i\nu b\xi/v)}{1+\xi^2}\,\mathrm{d}\xi = 2\pi i\left[(\xi - i)\frac{\exp(2\pi i\nu b\xi/v)}{(\xi+i)(\xi-i)}\right]_{\xi=i} = 2\pi i\frac{\exp(-2\pi\nu b/v)}{2i}.$$

EXERCISE 7.2 — Consider the Fourier transform $G(\omega)$ of a function $g(t)$. Show that if $g(t)$ is some peaked function with width σ—i.e., $g = g(t/\sigma)$—then the width of $G(\omega)$ is σ^{-1}. For definiteness, you may set $g(t) = (\sqrt{2\pi}\sigma)^{-1}\exp\left[-t^2/(2\sigma^2)\right]$.

TO GET THE TOTAL EMISSIVITY FROM A PLASMA, we must next integrate over a distribution of impact parameters b and collision rates for a thermal (Maxwell-Boltzmann) distribution of electron velocities. In a range of impact parameters $(b, b+\mathrm{d}b)$, an ion sees a current (number of electrons per second)

$$n_e \times v \times \frac{4\pi}{(2\pi k_\mathrm{B}T/m_e)^{3/2}}\exp\left[-\frac{m_e v^2}{2k_\mathrm{B}T}\right] v^2\mathrm{d}v \times 2\pi b\,\mathrm{d}b.$$

If we then multiply this expression by the spectral density (7.8) and the ion density n_I and integrate over all relevant velocities and impact parameters, we obtain the emissivity per unit volume,

$$\begin{aligned}\rho\epsilon_\nu^{\mathrm{ff}} &= \frac{8\sqrt{2}\pi^2}{3}n_I n_e\frac{Z^2e^6}{m_e^2c^3}\left(\frac{m_e}{k_\mathrm{B}T}\right)^{3/2}\\ &\quad\times\int_{v_{\min}}^{v_{\max}}\exp\left(-\frac{m_e v^2}{2k_\mathrm{B}T}\right)v\,\mathrm{d}v\int_{b_{\min}}^{b_{\max}}\frac{\mathrm{d}b}{b}\exp\left(-\frac{4\pi\nu b}{v}\right).\end{aligned}$$

For the limits of the integrals, note that we assumed non-relativistic electrons, so $v_{\max} \ll c$. For a thermal distribution, as long as $k_\mathrm{B}T \ll m_e c^2 \approx 5\,\mathrm{GK}$, the exponential term is very small and we can safely take $v_{\max} \to \infty$. At a frequency ν, there must be enough energy to emit at least one photon, so $mv_{\min}^2/2 = h\nu$. At large impact parameters, the potential is screened, so the $b_{\max} < \lambda_\mathrm{D}$. For the minimum impact parameter, we assumed a small-angle ($p_\perp/p_0 \ll 1$) collision, so that $m_e v^2/2 \gg Ze^2/b$. As v increases, $b_{\min}$ decreases; but in order to be in the classical regime, we also require that $m_e v b > \hbar$ (uncertainty principle) for $m_e v^2/2 > 4Z^2e^4m_e/\hbar^2 = 8Z^2\,\mathrm{Ry}$, where $1\,\mathrm{Ry} = 13.6\,\mathrm{eV}$. Since $m_e v^2/2 \simeq k_\mathrm{B}T$, for $k_\mathrm{B}T > Z^2\,\mathrm{Ry}$ our cutoff for b is set by the uncertainty principle.

In fact, the energy must be large enough to emit several photons if we are to be in the classical regime

We can therefore define two parameters that describe the integration: $u = h\nu/k_B T$ and $\eta = k_B T/Z^2$ Ry. We then change the variables of integration to $x = m_e v^2/2k_B T$ and $y = 4\pi\nu b/v$ so that

$$\rho\epsilon_\nu^{\mathrm{ff}} \approx \frac{8\sqrt{2}\pi^2}{3} n_I n_e \frac{Z^2 e^6}{m_e^2 c^3} \left(\frac{m_e}{k_B T}\right)^{1/2} \underbrace{\left[\int_u^\infty e^{-x}\,\mathrm{d}x \int_{y_{\min}(x;u,\eta)}^\infty \frac{e^{-y}}{y}\,\mathrm{d}y\right]}_{\equiv I}.$$

For $u \ll 1$ and $\eta > 1$ ($k_B T > Z^2$ Ry), the minimum impact parameter is set by the uncertainty principle, so that

$$y_{\min} = \frac{4\pi\nu b_{\min}}{v} = \frac{4\pi\nu\hbar}{m_e v^2} = \frac{u}{x}.$$

For $u \ll 1 < x$, the integral over y is

$$\int_{u/x}^\infty \frac{e^{-y}}{y}\,\mathrm{d}y \approx \ln\left(\frac{x}{\zeta u}\right),$$

where $\zeta \simeq 1.78$. The integral over x is then

The exponential function $E_1(z)$ is defined by

$$E_1(z) = \int_z^\infty e^{-x}/x\,\mathrm{d}x;$$

for $z \ll 1$, $E_1(z) \approx -\gamma - \ln(z)$ where $\gamma = 0.5772\ldots$ is *Euler's constant*. For convenience we define $\zeta = e^\gamma$.

$$\int_u^\infty e^{-x}\left[\ln x - \ln(\zeta u)\right]\,\mathrm{d}x;$$

doing the first integral by parts and setting $e^{-u} \approx 1$ gives

$$I \approx \ln\left(\frac{1}{\zeta^2}\frac{k_B T}{h\nu}\right).$$

To within a factor of order unity, this is the expression in the "small-angle, U.P. region" of Novikov and Thorne[1].

[1] I. D. Novikov and K. S. Thorne. Astrophysics of black holes. In C. Dewitt and B. S. Dewitt, editors, *Black Holes (Les Astres Occlus)*, pages 343–450, 1973

For $\eta < 1$ ($k_B T < Z^2$ Ry), if $u = h\nu/k_B T \ll 1$, then large-angle scatterings are unimportant, and $y_{\min} = (2u/\eta^{1/2})x^{-3/2}$. Since $x \sim 1$ where the integrands are large, we have again that $y_{\min} \ll 1$, so the integral becomes

$$I \approx \int_u^\infty e^{-x}\left[\frac{3}{2}\ln x + \ln\left(\frac{\eta^{1/2}}{2\zeta u}\right)\right]\,\mathrm{d}x \approx \ln\left[\frac{1}{2\zeta^{5/2}}\left(\frac{k_B T}{Z^2\,\mathrm{Ry}}\right)^{1/2}\left(\frac{k_B T}{h\nu}\right)\right].$$

To within a factor of order unity, this is the expression in the "small-angle, classical region" of Novikov and Thorne [1973].

For $u \gtrsim 1$, the contribution to the integral I comes from where $x \approx u$, so that $I = \text{const.} \times e^{-h\nu/k_B T}$. We therefore introduce the *velocity-averaged Gaunt factor* via

$$I = \frac{4}{\pi\sqrt{3}}\exp\left(-\frac{h\nu}{k_B T}\right)\bar{g}_\nu^{\mathrm{ff}}.$$

The velocity-averaged Gaunt factor $\bar{g}_\nu^{\mathrm{ff}}$ contains the details about the integration and is a slowly (logarithmically) varying function of ν. The free-free emissivity can then be expressed as

The energy spectral density (7.8) is $\propto b^{-2}$, but the number of electrons with a given impact parameter is $\propto b\,\mathrm{d}b$, so that the integration over all scatters has a characteristic $\int \mathrm{d}b/b \approx \ln(b_{\max}/b_{\min})$. This "Coulomb logarithm" typically appears when calculating collision rates in a plasma.

$$\rho\epsilon_\nu = \frac{32\pi^2}{3}\left(\frac{2}{3\pi}\right)^{1/2}\frac{Z^2 e^6}{m_e^2 c^3} n_I n_e \left(\frac{m_e}{k_B T}\right)^{1/2}\exp\left(-\frac{h\nu}{k_B T}\right)\bar{g}_\nu^{\mathrm{ff}}. \qquad (7.9)$$

The factor of $T^{-1/2}$ is because there is a factor of v^{-1} that appears in the integration (the collision time is $\sim\ b/v$). Note that the emissivity, as a function of frequency, is roughly constant (aside from the slow variation in $\bar{g}_\nu^{\rm ff}$) up to $\nu \approx k_{\rm B}T/h$ and then decreases exponentially.

EXERCISE 7.3 — In an HII region, mean temperatures are $T \approx 8\,000$ K; electron densities vary considerably from region to region, but the mean is $n_e \approx 1.4\,\mathrm{cm}^{-3}$ [Spitzer, 1978].

1. What is the condition for the gas to act as a plasma? Is this condition satisfied? What is the Debye screening length?
2. Suppose we have an electron with the mean energy of the plasma on a head-on collision course with an ion. What is the closest approach of two particles? How does it compare with the minimum distance set by the uncertainty principle?
3. Look up an appropriate expression for the Gaunt factor for emission in the GHz and in the optical.
4. Derive an expression for the opacity $\kappa_\nu^{\rm ff}$.

8
Relativity

8.1 Overview

The equation of motion of a particle, as described by Newtonian mechanics, has several interesting properties.

1. It is invariant if we pick a different origin for our coordinates: $t' = t + t_0$, $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{x}_0$. There is no privileged location in space or time.

2. It is invariant we rotate our spatial coordinates. For example, a rotation $\mathbf{R}$ about the z-axis by an angle θ transforms a vector $\boldsymbol{u}$ into $\boldsymbol{u}'$:

$$\boldsymbol{u}' = \mathbf{R}\boldsymbol{u} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}. \qquad (8.1)$$

   You can verify that this rotation leaves the dot product $\boldsymbol{u}\cdot\boldsymbol{v}$ unchanged; that is, $\boldsymbol{u}'\cdot\boldsymbol{v}' = (\mathbf{R}\boldsymbol{u})\cdot(\mathbf{R}\boldsymbol{v}) = \boldsymbol{u}\cdot\boldsymbol{v}$. As a result, the norm of a vector $|\boldsymbol{u}| = \sqrt{\boldsymbol{u}\cdot\boldsymbol{u}}$—its length—is unchanged by rotations.

3. It is invariant if we change to a frame moving with constant velocity $\boldsymbol{V}$: that is,

$$\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{V}t. \qquad (8.2)$$

This is called a *Galilean* transformation

Implicit in the third transformation is that intervals of time Δt are universal, i.e., frame-independent.

Maxwell's equations are invariant under the first two transformations, but not the third: the equations have traveling wave solutions with a constant propagation velocity c, so under a Galilean transformation the equations are altered. There are two possible resolutions.

1. The equations of mechanics are invariant under transformation 3. In this case, Maxwell's equations hold only in one, privileged, coordinate system, and it is possible experimentally to determine one's velocity with respect to this privileged frame. This has been conclusively demonstrated to not be so.

2. The equations of motion are not invariant under transformation 3, and must be reformulated to preserve the constancy of c. Einstein showed that transformation 3 implicitly assumes that different observers can agree on whether two events are simultaneous, and that this is in general not possible.

The failure of simultaneity means that any coordinate transformation involves mixing temporal and spatial coordinates: instead of specifying events by spatial vectors and a universal time, we must instead specify the coordinates of an event with a four-vector

$$x^{\mu} = \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}$$

In this discussion, we choose our units so that $c = 1$.

Here the superscript μ takes on values $[0, \ldots, 3]$ with $x^0 = t$.

What is the "length" of x^{μ}? We certainly want the spatial part to look like a Euclidian norm ($\sqrt{\boldsymbol{x} \cdot \boldsymbol{x}}$) so that it will be invariant under spatial rotations. We also need to ensure that a measurement of the speed of light, $|\mathrm{d}\boldsymbol{x}/\mathrm{d}t| = 1$, holds in all inertial frames. The four-vector between two events in spacetime is $\Delta x^{\mu} = (\Delta t, \Delta \boldsymbol{x})$. If the quantity $-(\Delta t)^2 + (\Delta \boldsymbol{x})^2$ between two events is the same in all inertial frames, then the speed of light will be the same in all frames, since along the path of a photon $-(\Delta t)^2 + (\Delta \boldsymbol{x})^2 = 0$.

Suppose an observer in an inertial reference frame carries a clock, which is stationary in her frame. The interval between two successive ticks of the clock is just $\mathrm{d}s^2 = -\mathrm{d}t^2$. We therefore define the proper time $\mathrm{d}\tau^2 = -\mathrm{d}s^2$ as the time measured by a clock carried by an observer at rest in a given inertial frame. Requiring that $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}\boldsymbol{x}^2$ be the same in all inertial reference frames requires that different observers will in general not agree on the time elapsed or the spatial separation of two events, as illustrated by the following example.

Suppose we have a clock at rest in frame O. The time between successive ticks is $\Delta t = \Delta \tau$. In frame O', the clock has velocity $\boldsymbol{v}' = \mathrm{d}\boldsymbol{x}'/\mathrm{d}t'$. In O', the proper time is

$$\Delta \tau'^2 = \Delta t'^2 - |\Delta \boldsymbol{x}'|^2 = \Delta t'^2 \left(1 - |\boldsymbol{v}'|^2\right) = \Delta \tau^2 = \Delta t^2$$

In frame O', the time between the two events is $\Delta t' = \Delta t/\sqrt{1 - v^2} > \Delta t$: an observer in O' finds the clock running slower than one in O.

8.2 The Lorentz transformation

The norm, or length, of a four-vector is

$$s^2 = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 \tag{8.3}$$

Although this norm is not positive-definite, it does satisfy the parallelogram identity,

$$\|a^\mu + b^\mu\|^2 + \|a^\mu - b^\mu\|^2 = 2\,\|a^\mu\|^2 + 2\,\|b^\mu\|^2\,;$$

which allows us to define a bilinear, symmetric *inner product* between two four-vectors a^μ and b^ν as

$$\begin{aligned} \langle a^\mu, b^\mu \rangle &= \frac{1}{4}\left[\|a^\mu + b^\mu\|^2 - \|a^\mu - b^\mu\|^2\right] \\ &= \eta_{\mu\nu} a^\mu b^\nu \end{aligned} \tag{8.4}$$

where $\eta_{00} = -1$, $\eta_{0i} = \eta_{i0} = 0, i = 1,2,3$, and $\eta_{ij} = \delta_{ij}$, $\forall i,j = 1,2,3$. Here the object $\eta_{\mu\nu}$ is known as the *metric tensor*. We may therefore express the norm of a vector as $s^2 = \eta_{\mu\nu} x^\mu x^\nu$.

We use latin subscripts to mean only the spatial components, and greek indices to refer to all four components. Also, we use the convention that a repeated index, one upstairs and one downstairs, is to be summed over: for example,

$$\eta_{\alpha\beta} x^\beta \equiv \eta_{\alpha 0} x^0 + \eta_{\alpha 1} x^1 + \eta_{\alpha 2} x^2 + \eta_{\alpha 3} x^3.$$

In Euclidian geometry, $\eta_{ij} = \mathrm{diag}[1,1,1]$ and the inner product is just the familiar dot product.

Our task, then, is to find a replacement for the Galilean transformation (8.2) between inertial frames that leaves s^2 invariant; such a transformation will preserve the constancy of the speed of light. A general coordinate transformation is

$$x'^\alpha = L^\alpha{}_\beta x^\beta + a^\alpha, \tag{8.5}$$

where $L^\alpha{}_\beta$ is a 4×4 matrix and a^α is a constant four-vector. We can set $a^\alpha = 0$, since it just sets the origin. To keep $\mathrm{d}s^2$ invariant, we require that

$$\mathrm{d}s'^2 = \eta_{\alpha\beta}\left(L^\alpha{}_\gamma \mathrm{d}x^\gamma\right)\left(L^\beta{}_\delta \mathrm{d}x^\delta\right) = \left(\eta_{\alpha\beta} L^\alpha{}_\gamma L^\beta{}_\delta\right)\mathrm{d}x^\gamma \mathrm{d}x^\delta = \mathrm{d}s^2,$$

so that

$$\eta_{\alpha\beta} L^\alpha{}_\gamma L^\beta{}_\delta = \eta_{\gamma\delta}. \tag{8.6}$$

What transformations satisfy this condition? Spatial rotations, in which

$$L^i{}_0 = L^0{}_i = 0, \quad L^0{}_0 = 1, \quad L^i{}_j = R^i{}_j,$$

where R^i_j is a rotation matrix[1] clearly leave $\mathrm{d}s^2$ invariant. A more interesting case is transforming from a frame $\mathcal{O}$ in which a particle is at rest, $\mathrm{d}\boldsymbol{x} = 0$, to a frame $\mathcal{O}'$ in which the particle moves with velocity $\boldsymbol{v} = \mathrm{d}\boldsymbol{x}'/\mathrm{d}t'$. This implies that

[1] cf. Eq. (8.1)

$$\begin{aligned} \mathrm{d}t' &= L^0{}_0 \mathrm{d}t + L^0{}_i \mathrm{d}x^i = L^0{}_0 \mathrm{d}t \quad \text{and} \\ \mathrm{d}x'^i &= L^i{}_0 \mathrm{d}t + L^i{}_j \mathrm{d}x^j = L^i{}_0 \mathrm{d}t. \end{aligned}$$

Dividing the second expression by the first implies that

$$v'^i = \frac{\mathrm{d}x'^i}{\mathrm{d}t'} = \frac{L^i{}_0}{L^0{}_0}. \tag{8.7}$$

Then applying Eq. (8.6) for η_{00} gives

$$\eta_{\alpha\beta} L^\alpha{}_0 L^\beta{}_0 = \sum_{i=1}^{3}\left(L^i{}_0\right)^2 - \left(L^0{}_0\right)^2 = \left(|v|^2 - 1\right)\left(L^0{}_0\right)^2 = \eta_{00} = -1. \tag{8.8}$$

Hence $L^0{}_0 \equiv \gamma = 1/\sqrt{1-|v|^2}$, and $L^i{}_0 = \gamma v^i$.

We also require that if we boost *back* from frame $\mathcal{O}'$ frame $\mathcal{O}$—that is, we apply a transformation with $\boldsymbol{v} \to -\boldsymbol{v}$—then we recover our original coordinates (Exercise 8.1). If we align our frames so that their relative motion is along the x-axis, then we can determine that $L^0{}_1 = L^1{}_0 = \gamma v$ and $L^1{}_1 = L^0{}_0 = \gamma$. We therefore have the *Lorentz transformation* to a frame moving with velocity $\boldsymbol{v} = v\hat{\boldsymbol{e}}_x$,

$$L^\alpha{}_\beta(\boldsymbol{v} = v\hat{\boldsymbol{e}}_x) = \begin{bmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{8.9}$$

EXERCISE 8.1 — Show that the if we replace $\boldsymbol{v}$ by $-\boldsymbol{v}$ in Eq. (8.9), we obtain the inverse Lorentz transformation; that is, show that $L^\alpha{}_\beta(-\boldsymbol{v})L^\beta{}_\gamma(\boldsymbol{v}) = \delta^\alpha{}_\gamma$.

Finally, we can then rotate frame $\mathcal{O}'$ relative to frame $\mathcal{O}$, which gives us the general form for a boost to an arbitrary $\boldsymbol{v}$:

$$L^\alpha{}_\beta(\boldsymbol{v}) = \begin{bmatrix} \gamma & \gamma v_x & \gamma v_y & \gamma v_z \\ \gamma v_x & 1 + v_x^2(\gamma-1)/|v|^2 & v_x v_y(\gamma-1)/|v|^2 & v_x v_z(\gamma-1)/|v|^2 \\ \gamma v_y & v_y v_x(\gamma-1)/|v|^2 & 1 + v_y^2(\gamma-1)/|v|^2 & v_y v_z(\gamma-1)/|v|^2 \\ \gamma v_z & v_z v_x(\gamma-1)/|v|^2 & v_z v_y(\gamma-1)/|v|^2 & 1 + v_z^2(\gamma-1)/|v|^2 \end{bmatrix} \tag{8.10}$$

Again, this is the boost from a frame in which a particle is at rest to a frame in which the particle has velocity $\boldsymbol{v}$.

EXERCISE 8.2 — Einstein rides on a rocket traveling at high speed, while Lorentz measures the length of the rocket as it flies by. Afterwards, they meet to discuss the experiment. Lorentz explains how his experimental apparatus marked off the positions of the front and rear of the rocket at the same given time. Einstein replies that he was watching Lorentz make his measurements of the positions of the front and rear of the rocket. How would Einstein describe Lorentz's measurement? Show explicitly that if the rest frame length of the rocket is L, then Lorentz will measure the length as L/γ.

Box 8.1 Contravariant and covariant

You may have wondered why we make a distinction between indices that are superscripted and indices that are subscripted. Let's begin by expanding a four-vector as

$$\vec{x} = \sum_{\alpha=0}^{3} x^\alpha \vec{e}_\alpha.$$

Here the x^α are termed the *contravariant* components of the vec-

Box 8.1 continued

tor when expanded in terms of the basis $\vec{e}_\alpha$. The basis vectors satisfy the relation

$$\vec{e}_\alpha \cdot \vec{e}_\beta = \eta_{\alpha\beta}.$$

The *covariant* components of $\vec{x}$, denoted with a subscripted index, are defined by

$$x_\alpha = \vec{x} \cdot \vec{e}_\alpha = \eta_{\alpha\beta} x^\beta.$$

Notice that $x_0 = -x^0$ and $x_i = x^i$ for $i = 1, 2, 3$. The sign change for the time-like component is something not found in a Euclidian geometry.

To see why the distinction matters, suppose we evaluate a gradient

$$\nabla = \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right) \tag{8.11}$$

in a frame $\mathcal{R}$. We then wish to express ∇' in a frame $\mathcal{S}$, denoted by $'$, in which the origin of frame $\mathcal{R}$ moves with velocity v. For concreteness, let v be along the x-axis. A four-vector in frame $\mathcal{S}$ has components

$$x'^\alpha = (\gamma x^0 + \gamma v x^1, \gamma v x^0 + \gamma x^1, x^2, x^3). \tag{8.12}$$

Using the chain rule for derivatives, we can write

$$\nabla'_\alpha = \frac{\partial}{\partial x^\beta} \frac{\partial x^\beta}{\partial x'^\alpha}.$$

To evaluate this, we need the inverse transformation, $x^\alpha = (\gamma x'^0 - \gamma v x'^1, -\gamma v x'^0 + \gamma x'^1, x'^2, x'^3)$, so that $\partial x^0 / \partial x'^0 = \gamma$, $\partial x^0 / \partial x'^1 = -\gamma v$, and so forth. We find that

$$\nabla' = \left(\gamma \frac{\partial}{\partial x^0} - \gamma v \frac{\partial}{\partial x^1}, -\gamma v \frac{\partial}{\partial x^0} + \gamma \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right). \tag{8.13}$$

Comparing this with the expression in the unprimed frame (8.11) shows that it does not transform in the same manner as a four-vector (8.12). If however, we "raise" the index by multiplying by $\eta^{\alpha\beta} = \mathrm{diag}[-1, 1, 1, 1]$, we find that

$$\begin{aligned} \nabla'^\alpha &= \eta^{\alpha\beta} \nabla'_\beta \\ &= \left[\gamma \left(-\frac{\partial}{\partial x^0} \right) + \gamma v \frac{\partial}{\partial x^1}, \gamma v \left(-\frac{\partial}{\partial x^0} \right) + \gamma \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right], \end{aligned}$$

which is equivalent to $L^\mu{}_\nu \nabla^\nu = L^\mu{}_\nu \eta^{\nu\delta} \nabla_\delta$.

Box 8.1 continued

So if the gradient is not a vector, than what is it? To understand that, suppose we have a curve $\mathcal{C}(\zeta)$. Here ζ is a measure of distance along the curve. For example, if $\mathcal{C}$ is a worldline of a massive particle, then ζ could be the proper time. Suppose we wish to know how a scalar function φ changes along the curve. We would calculate the directional derivative

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\zeta} = \frac{\partial \varphi}{\partial x^\alpha} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\zeta}\bigg|_{x^\alpha_\mathcal{C}},$$

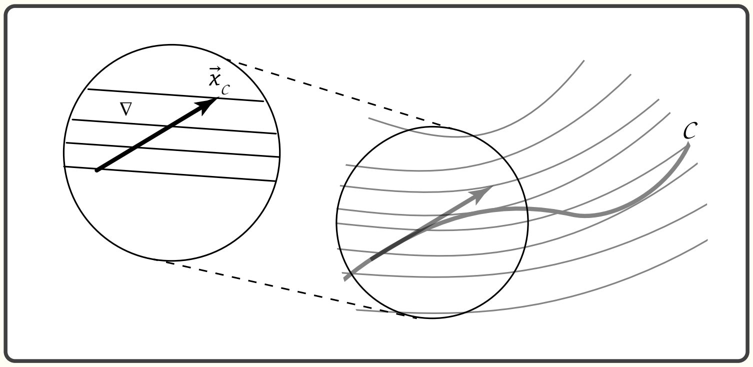
where $x^\alpha_\mathcal{C}(\zeta)$ are the coordinates along the curve. Notice that

$$\vec{x}_\mathcal{C} = \frac{\mathrm{d}x^\alpha_\mathcal{C}}{\mathrm{d}\zeta}\vec{e}_\alpha$$

is the tangent vector to the curve $\mathcal{C}$ and $\partial\varphi/\partial x^\alpha = \nabla_\alpha$. Hence,

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\zeta} = \vec{\nabla}\cdot\vec{x}_\mathcal{C}.$$

The gradient therefore *combines* with a vector to generate a scalar. Such an object is called a *one-form* in differential geometry.



8.3 Kinematics

Now that we have our rules for how coordinates transform, let's develop the four-vector kinematical quantities. The first difficulty we encounter is that $\mathrm{d}\boldsymbol{x}/\mathrm{d}t$ is not a four-vector.[2] Since $\mathrm{d}x^\alpha$ is a four-vector, we need to divide it by a scalar—something that is the same in all frames. The obvious candidate is $\mathrm{d}\tau$, which gives use the *four-velocity*

$$u^\alpha \equiv \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} = \begin{bmatrix} \gamma \\ \gamma\boldsymbol{u} \end{bmatrix}. \tag{8.14}$$

Here $\boldsymbol{u} = \mathrm{d}\boldsymbol{x}/\mathrm{d}t$ is the ordinary velocity in three-space. In the rest frame of the particle, $u^\alpha = (1, \mathbf{0})$.

[2] that is,

$$\frac{\mathrm{d}\boldsymbol{x}'}{\mathrm{d}t'} \neq L^\alpha{}_\beta \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}$$

EXERCISE 8.3 — Suppose we trace out the spacetime path of an object (known as a *worldline*) by recording its coordinate four-vector as it moves. Show that the four-velocity u^α is the unit tangent four-vector to the object's worldline.

Suppose we have a particle that is accelerating. At a given instant of time, we can boost to a momentarily comoving rest frame (MCRF). Over an interval $\Delta\tau$, the particles four-velocity will change by $\Delta u^\alpha = (0, \Delta\boldsymbol{u})$, which is itself a four-vector. If we then multiply by the mass m of the object, measured in the rest frame of the particle,[3] and divide by $\mathrm{d}\tau$, then in the MCRF this four-vector

$$\frac{\mathrm{d}p^\alpha}{\mathrm{d}\tau} \equiv m\,\frac{\mathrm{d}u^\alpha}{\mathrm{d}\tau} \tag{8.15}$$

has components

$$\begin{bmatrix} 0 \\ m\,\mathrm{d}\boldsymbol{u}/\mathrm{d}t \end{bmatrix} = \begin{bmatrix} 0 \\ \boldsymbol{F} \end{bmatrix}, \tag{8.16}$$

where $\boldsymbol{F}$ is the applied (Newtonian) force.

[3] This long-winded definition of rest mass is needed so that m is a scalar (same in all frames).

EXERCISE 8.4 — Show that the four-acceleration is orthogonal to the four-velocity: that is, show that $\eta_{\alpha\beta}u^\alpha \mathrm{d}u^\beta/\mathrm{d}\tau = 0$.

If we boost equation (8.15) from the MCRF to one in which the particle has velocity $\boldsymbol{u}$, we obtain the equation

$$\gamma\frac{\mathrm{d}p^\alpha}{\mathrm{d}t} = \begin{bmatrix} \gamma\boldsymbol{u}\cdot\boldsymbol{F} \\ \boldsymbol{F} + \boldsymbol{u}(\boldsymbol{u}\cdot\boldsymbol{F})(\gamma-1)/u^2 \end{bmatrix}.$$

Notice the component $\mathrm{d}p^0/\mathrm{d}t = \boldsymbol{u}\cdot\boldsymbol{F}$: this is just the rate that work is done on the object; it equals the rate of change of the object's energy. It makes sense to identify $p^0 = mu^0 = \gamma m$ as the energy of the particle. The momentum four-vector is then

$$p^\alpha = \begin{bmatrix} \gamma m \\ \gamma m\boldsymbol{u} \end{bmatrix} = \begin{bmatrix} E \\ \boldsymbol{p} \end{bmatrix}$$

At low velocities, $E = \gamma m \approx m + mu^2/2$. The length of the four-momentum is

$$\eta_{\alpha\beta}p^\alpha p^\beta = -E^2 + p^2 = \gamma^2(-m^2 + m^2u^2) = -m^2.$$

The rest mass m is thus indeed an invariant, and in the rest frame of the particle, $E = m$. Photons travel at velocity c, and have momentum $p = E$; hence for a photon, $\eta_{\alpha\beta}p^\alpha p^\beta = 0$.

EXERCISE 8.5 — Suppose we have a particle with 4-momentum p^α, and an observer moving with 4-velocity u^α. Show that the energy of the particle, as measured by the observer, is $E = -\eta_{\alpha\beta}p^\alpha u^\beta$.

In order for the four-momentum to be useful, all observers must agree on conservation of energy and momentum. Suppose we observe a process among a group of particles $i = 1, \ldots, N$. The net change in four-momentum, as viewed in a different frame, is

$$\sum_{i=1}^{N} \left(\Delta p'^{\alpha}\right)_i = L^{\alpha}{}_{\beta} \sum_{i=1}^{N} \left(\Delta p^{\beta}\right)_i .$$

If momentum and energy are conserved in one inertial frame (i.e., $\sum \Delta p^{\beta} = (0, \mathbf{0})$), they are conserved in all inertial frames.

EXERCISE 8.6 — We argued that for low-frequency radiation, electron scattering would be coherent: the scattered radiation would be at essentially the same frequency as the incident. Show this explicitly: consider a photon of wavelength λ incident on an electron at rest. The photon is scattered to an angle θ with the original momentum, and the wavelength after the scattering is λ'. Compute $\lambda' - \lambda$ as a function of electron mass m_e and scattering angle θ. *Hint:* the algebra is easier if you set $\hbar = c = 1$; equate the initial and final four-momenta, $p^{\mu}_{\gamma,i} + p^{\mu}_{e,i} = p^{\mu}_{\gamma,f} + p^{\mu}_{e,f}$; and then solve for $p^{\mu}_{e,f}$ and compute the absolute value of both sides of the equations using $|p^{\mu}_{e,f}|^2 = -m_e^2$.

8.4 Aberration and Doppler Shift

In this section I use a prime (′) to denote the source frame $\mathcal{S}$.

Suppose we have a frame $\mathcal{S}$ in which a particle moves with velocity $\boldsymbol{V}' = \mathrm{d}\boldsymbol{x}'/\mathrm{d}t'$. What is its velocity $\boldsymbol{V}$ in a frame $\mathcal{R}$, in which an observer sees the origin of $\mathcal{S}$ moving with velocity $\boldsymbol{u}$? To answer, we note that the displacement $(\mathrm{d}t', \mathrm{d}\boldsymbol{x}')$ is a four-vector, so according to the transformation (8.10) in frame $\mathcal{R}$ the differential coordinate four-vector is

$$\begin{bmatrix} \mathrm{d}t \\ \mathrm{d}\boldsymbol{x} \end{bmatrix} = \begin{bmatrix} \gamma(\mathrm{d}t' + \mathrm{d}\boldsymbol{x}' \cdot \boldsymbol{u}) \\ \gamma \boldsymbol{u}\mathrm{d}t' + \mathrm{d}\boldsymbol{x}' + (\mathrm{d}\boldsymbol{x}' \cdot \boldsymbol{u})\frac{\gamma - 1}{|u|^2}\boldsymbol{u} \end{bmatrix}.$$

Hence in frame $\mathcal{R}$,

$$\begin{aligned} \boldsymbol{V} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} &= \frac{\gamma \boldsymbol{u}\mathrm{d}t' + \mathrm{d}\boldsymbol{x}' + (\mathrm{d}\boldsymbol{x}' \cdot \boldsymbol{u})(\gamma - 1)/|u|^2\, \boldsymbol{u}}{\gamma(\mathrm{d}t' + \mathrm{d}\boldsymbol{x}' \cdot \boldsymbol{u})} \\ &= \frac{\gamma \boldsymbol{u} + \boldsymbol{V}' + (\gamma - 1)(\boldsymbol{V}' \cdot \boldsymbol{u})/|u|^2 \boldsymbol{u}}{\gamma(1 + \boldsymbol{V}' \cdot \boldsymbol{u})}. \end{aligned} \tag{8.17}$$

A useful way of writing this is to have $\boldsymbol{V} = (V_{\|}, \boldsymbol{V}_{\perp})$, in which $V_{\|} = \boldsymbol{V} \cdot \boldsymbol{u}/|u|$ and $\boldsymbol{V}_{\perp} = \boldsymbol{V} - V_{\|}\boldsymbol{u}/|u|$. Making this substitution gives

$$V_{\|} = \frac{u + V'_{\|}}{1 + uV'_{\|}} \tag{8.18}$$

$$\boldsymbol{V}_{\perp} = \frac{\boldsymbol{V}'_{\perp}}{\gamma(1 + uV'_{\|})} \tag{8.19}$$

Now suppose in $\mathcal{S}$ our source is emitting photons isotropically: $V' = 1$. Let θ' be the angle between a photon and $\boldsymbol{u}$ in frame $\mathcal{S}$. The a receiver in frame $\mathcal{R}$ will observe the angle to be

$$\cos\theta = \frac{u + \cos\theta'}{1 + u\cos\theta'} \tag{8.20}$$

$$\tan\theta = \frac{|\boldsymbol{V}_\perp|}{V_\parallel} = \frac{\sin\theta'}{\gamma(u + \cos\theta')}. \tag{8.21}$$

Notice what happens if the source is relativistic with $u \approx 1, \gamma \gg 1$: a photon emitted at right angles to $\boldsymbol{u}$, $\theta' = \pi/2$ in the source frame will be observed in frame $\mathcal{R}$ to be at an angle $\sim \gamma^{-1}$. That is, the radiation emitted by a source traveling at relativistic velocity is beamed into a narrow cone of opening half-angle $\gamma^{-1} \ll 1$ about the direction of motion.

In addition to aberration, the frequency of the photons received in frame $\mathcal{R}$ is altered by two effects. The first is the change in elapsed time between the emission of successive wave crests: $\Delta t = \gamma \Delta t' = \gamma/\nu'$. In addition, there is the delay caused by the difference in path length between one wave crest and the next. In frame $\mathcal{R}$, this additional path length is $-\Delta t\, u\cos\theta$, where we orient our frame so that a positive velocity is towards the observer[4]. As a result the received frequency is

$$\nu = (\Delta t)^{-1}_{\text{rec.}} = \frac{1}{\Delta t(1 - u\cos\theta)} = \frac{\nu'}{\gamma(1 - u\cos\theta)}. \tag{8.22}$$

[4] This matches the definition used in Rybicki and Lightman [1979]; it is typical in astronomy, however, to define a positive velocity to be away from the observer.

This is the relativistic Doppler shift. If reduces to the classical expression in the limit $u \ll 1$. Note that in form it isn't symmetrical, as the right-hand side has expressions in both frame $\mathcal{S}$ (ν') and frame $\mathcal{R}$ ($\cos\theta$). This is easily remedied by the aberration formulae, Eq. (8.20) and (8.21); see the exercises.

EXERCISE 8.7 — Recast the formula for the received frequency ν, Eq. (8.22), in terms of ν' and θ'. Find an expression for the inverse doppler shift, namely, find an expression for ν' in terms of ν, θ, and u.

EXERCISE 8.8 — A source emits photons isotropically in its rest frame. For rays emitted in the source frame at angles $30^\circ, 60^\circ, 90^\circ, \ldots 330^\circ$, indicate how the rays would be observed in a frame in which the source is traveling in the x-direction with velocity $u = 0.3c$.

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9 Synchrotron Radiation

Magnetic fields are ubiquitous in the universe. At low energies, the helical motion of a particle in a magnetic field produces emission at the cyclotron frequency $\omega_B = qB/mc$. When the particle is relativistic, however, the beaming of the radiation produces emission over a broad range of frequencies. The acceleration of particles to relativistic energies occurs in many environments, including supernova remnants, and the emission from such particles in a magnetic field is call *synchrotron emission*.

9.1 Overview

Let's start with an electron with velocity in the plane perpendicular to the direction of the (uniform) magnetic field. In the absence of an electric field, the (relativistic) equation of motion is (cf. § 6.2)

$$\gamma \frac{\mathrm{d}m\boldsymbol{v}}{\mathrm{d}t} = e\boldsymbol{\beta} \times \boldsymbol{B},$$

where $\boldsymbol{\beta} = \boldsymbol{v}/c$. Because the acceleration is at right-angles to the velocity, β and therefore γ are constant. The electron gyrates in uniform circular motion with frequency

$$\omega_B = \frac{eB}{\gamma mc},$$

which reduces to the electron cyclotron frequency for $\gamma = 1$. Using Eq. (5.9), we compute the radiation electric field generated by the gyrating electron as shown in Figure 9.1. The magnetic field points along $\hat{\boldsymbol{z}}$; the center of gyration will be at the origin; and the observer lies in the $\hat{\boldsymbol{x}}$ direction at great distance.

As the particle energy γmc^2 increases, the relativistic aberration shapes the electric field into a sharp pulse observed when the electron is traveling along our line of sight—along $\hat{\boldsymbol{x}}$, in this case. On either side of the pulse, the electric field vanishes when $\hat{\boldsymbol{r}} - \boldsymbol{\beta} \parallel \dot{\boldsymbol{\beta}}$. Measuring the angle from the line of sight, we see that since $\dot{\boldsymbol{\beta}}$ is at right angles to $\boldsymbol{\beta}$, the angle at which the field vanishes is $\arccos(\beta) \to 1/\gamma$ at large γ.

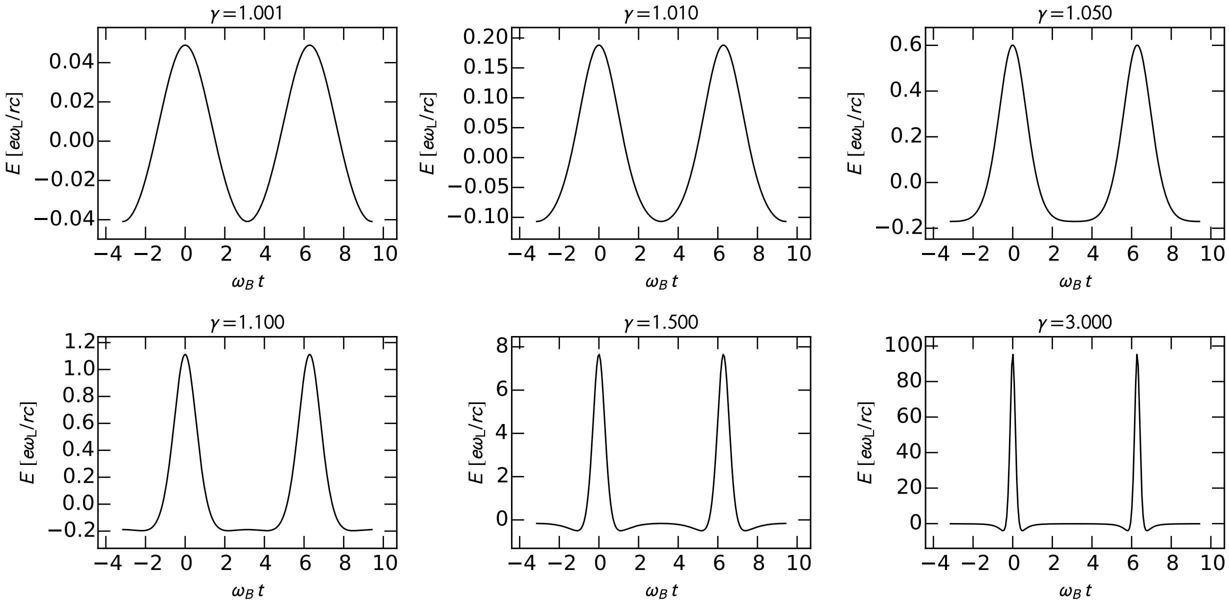

Figure 9.1: Radiation electric field for a particle moving in the xy plane. The magnetic field points along $\hat{\boldsymbol{z}}$, the center of gyration is at the origin, and the observer lies at great distance along the $\hat{\boldsymbol{x}}$ direction.

Such a sharp pulse in the electric field implies that the received power will be distributed over a broad range of frequencies. Rather than compute the spectrum directly from the Fourier transform of the electric field, we'll give a more heuristic description, following Rybicki and Lightman [1979]. We approximate the angle over which the pulse lasts as $\Delta\theta \approx 2/\gamma$. In general, the electron has a component of momentum $p_{\|}$ along $\boldsymbol{B}$ in addition to the component $\boldsymbol{p}_{\perp}$ in the plane perpendicular to the field. Define the *pitch angle* α as the angle between $\boldsymbol{p}$ and $\boldsymbol{B}$; then $|\boldsymbol{p}_{\perp}| = p\sin\alpha$. We first need to determine the time needed for the electron to turn through an angle $\Delta\theta$. To do this, we construct a unit tangent vector $\hat{\boldsymbol{\tau}}$ along the trajectory; then the time for $\hat{\boldsymbol{\tau}}$ to turn through an angle $\Delta\theta$ is $\Delta t = \Delta\theta/|\mathrm{d}\hat{\boldsymbol{\tau}}/\mathrm{d}t|$. The unit tangent vector is just

$$\hat{\boldsymbol{\tau}} = \frac{\boldsymbol{v}}{v} = \left(\frac{v_{\|}}{v}, \frac{\boldsymbol{v}_{\perp}}{v}\right);$$

since $v_{\|}$ is constant, $\mathrm{d}\hat{\boldsymbol{\tau}}/\mathrm{d}t = v^{-1}\mathrm{d}\boldsymbol{v}_{\perp}/\mathrm{d}t = \omega_B \sin\alpha$ and

$$\Delta t = \frac{2}{\gamma\omega_B \sin\alpha}.$$

In the time Δt, the electron moves towards us a distance $\Delta s \approx v\Delta t$; hence the arrival time between the start of the pulse and its end is

$$\Delta t_{\mathrm{A}} \approx \Delta t\left(1 - \frac{v}{c}\right) \approx \frac{1}{\gamma^3\omega_B \sin\alpha}.$$

We therefore expect the power to be distributed over a broad range of frequencies up to a critical frequency $\omega_{\mathrm{c}} \approx \Delta t_{\mathrm{A}}^{-1} = \gamma^3\omega_B \sin\alpha$. Since we'll be interested in averaging over pitch angles α and we aren't computing the spectral shape in detail, we'll define[1] $\omega_c = \gamma^3\omega_B = \gamma^2\omega_L$.

[1] Note that Rybicki and Lightman [1979] normalize the critical frequency as

$$\omega_c \equiv \frac{3}{2}\gamma^3\omega_B \sin\alpha.$$

EXERCISE 9.1— A "typical" galactic magnetic field strength is $B = 10\,\mu$G.

1. What is the electron cyclotron frequency for this B?
2. For radio observations in the GHz range, what is a typical value of γ for the electrons? Would you expect these electrons to have a thermal or non-thermal distribution?
3. In actuality, our emitted radiation would be a discrete series of frequencies rather than a continuous distribution. How good is our approximation of a smooth frequency distribution? *Hint:* What is the spacing between harmonics?

Although not immediately obvious from Equation (5.9), for large γ the electric field depends on the angle θ between $\hat{\boldsymbol{r}}$ and $\boldsymbol{\beta}$ through the combination $\gamma\theta = \gamma\omega_B t$, as illustrated in Fig. 9.2. If we set $t = 0$ to be when the electric field is at maximum, then the doppler shift over the pulse implies that $t = \gamma^2 t_A$, so that the received electric field depends on the time as

$$\gamma\omega_B t \approx \gamma^3\omega_B t_A \approx \omega_c t_A.$$

Hence the electric field is $\boldsymbol{E} \propto f(\omega_c t)$. Here f is some as-yet-unspecified function of $\omega_c t$.

Since the electric field is a function of $\omega_c t$, its Fourier transform is

$$\tilde{\boldsymbol{E}}(\omega) = F\left(\frac{\omega}{\omega_c}\right).$$

That is, the observed electric field, and hence the observed power, is distributed over frequencies as a function of ω/ω_c. We can write the spectral distribution of the power as

$$P_\omega(\gamma) = C\varphi\left(\frac{\omega}{\omega_c}\right).$$

Here C is a as-yet-undetermined constant and $\int_0^\infty \varphi\,d\omega = 1$.

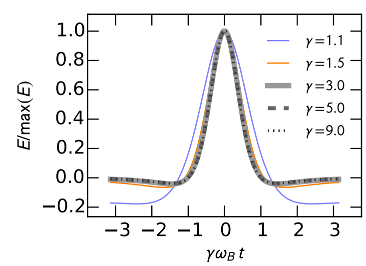

Figure 9.2: The electric field, scaled to its maximum value, as a function of $\gamma\omega_B t$.

TO FIX C, we need to normalize our spectral distribution by the total power emitted. Here we need to make a brief digression to modify Larmor's formula, which contains the Newtonian acceleration. First, as was done in the derivation leading up to Eq. 8.16, we define the four-acceleration $a^\alpha = du^\alpha/d\tau$. This is a four-vector, since $d\tau$ is a Lorentz scalar and du^α is the differential of the four-velocity. Next we boost to a momentarily comoving rest frame (MCRF) of our particle. In this frame

$$du^\alpha = \begin{bmatrix} 0 \\ d\boldsymbol{u} \end{bmatrix}$$

and $\mathrm{d}\tau = \mathrm{d}t$; hence the four-acceleration is just

$$a^\alpha = \begin{bmatrix} 0 \\ \boldsymbol{a} \end{bmatrix},$$

where $\boldsymbol{a}$ is the Newtonian acceleration. As a result, the total power emitted is

$$P = \frac{2}{3}\frac{e^2}{c^3}|\boldsymbol{a}\cdot\boldsymbol{a}| = \frac{2}{3}\frac{e^2}{c^3}\left(\eta_{\mu\nu}a^\mu a^\nu\right)$$

and is therefore the same in all frames.

Now to evaluate the acceleration $\boldsymbol{a}$. Here we hit a small obstacle: in the MCRF, the force due to the magnetic field vanishes. The acceleration is instead due to an electric field that appears in this frame. Denoting the MCRF by a "′", the fields in the MCRF are

$$\boldsymbol{E}' = \gamma\left(\boldsymbol{E} + \boldsymbol{\beta}\times\boldsymbol{B}\right) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\boldsymbol{E}\right) \tag{9.1}$$

$$\boldsymbol{B}' = \gamma\left(\boldsymbol{B} - \boldsymbol{\beta}\times\boldsymbol{E}\right) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\boldsymbol{B}\right); \tag{9.2}$$

since $\boldsymbol{E} = \boldsymbol{0}$, the particle's acceleration in the MCRF is

$$\boldsymbol{a}' = \gamma\frac{e}{m}\boldsymbol{\beta}\times\boldsymbol{B} = \gamma\frac{e}{m}\beta B\sin\alpha.$$

For large γ, $\beta \approx 1$, and the total power emitted is

$$P(\gamma) = \frac{2}{3}\gamma^2\frac{e^4B^2\sin^2\alpha}{m^2c^3}.$$

If we assume that the pitch angle α is randomly distributed, then averaging over angles gives

$$\begin{aligned}
\langle P(\gamma)\rangle &= \frac{2}{3}\gamma^2\frac{e^4B^2}{m^2c^3}\left[\frac{1}{4\pi}\int\sin^2\alpha\,\mathrm{d}\Omega\right] \\
&= \left(\frac{2}{3}\right)^2\frac{e^4B^2}{m^2c^3} = \frac{4}{3}\left[\frac{8\pi}{3}\frac{e^4}{m^2c^4}\right]c\left[\frac{B^2}{8\pi}\right]\gamma^2 \\
&= \frac{4}{3}\sigma_\mathrm{T}cU_B\gamma^2.
\end{aligned} \tag{9.3}$$

Here σ_T is the Thomson scattering cross section and U_B is the energy density of the magnetic field.

We can equate equation (9.3) with the integral over all frequencies of the Fourier transform,

$$P(\gamma) = \int_0^\infty C\varphi\left(\frac{\omega}{\omega_\mathrm{c}}\right)\mathrm{d}\omega = C\omega_\mathrm{c} = C\gamma^2\omega_\mathrm{L}. \tag{9.4}$$

Comparing Equations (9.3) and (9.4) fixes C and we have

$$P_\omega(\gamma) = \frac{4}{3}\frac{\sigma_\mathrm{T}cU_B}{\omega_\mathrm{L}}\varphi\left(\frac{\omega}{\omega_\mathrm{c}}\right). \tag{9.5}$$

This is the spectral distribution for electrons at a single energy γmc^2.

WE CAN DETERMINE THE FORM OF THE SPECTRUM for a population of electrons even without knowing the precise functional form of $\varphi(\omega/\omega_c)$. The electrons are non-thermal (cf. Exercise 9.1), and their distribution with energy can often be described as a power-law,

$$n(\gamma)\,d\gamma = n_0\gamma^{-p}\,d\gamma, \tag{9.6}$$

over a large ranges of energies γmc^2. Here n_0 is a normalizing constant. Typically $p \approx 2.5$ and we may take $\gamma_{max} \to \infty$. To get the total power output at frequency ω, we multiply P_ω by $n(\gamma)$ and integrate over all γ:

$$P_\omega = \frac{4}{3}\frac{\sigma_T c U_B}{\omega_L} n_0 \int_{\gamma_{min}}^{\infty} \gamma^{-p}\varphi\left(\frac{\omega}{\omega_c}\right)\,d\gamma.$$

Changing variables to $\xi = \omega/\omega_c = \omega/(\gamma^2\omega_L)$

$$\begin{aligned} P_\omega &= \frac{2}{3}\frac{\sigma_T c U_B}{\omega_L} n_0 \left(\frac{\omega}{\omega_L}\right)^{-(p-1)/2} \int_0^{\xi_{max}(\omega)} \xi^{(p-3)/2}\varphi(\xi)\,d\xi \\ &\approx \frac{2}{3}\frac{\sigma_T c U_B}{\omega_L} n_0 \left(\frac{\omega}{\omega_L}\right)^{-(p-1)/2}. \end{aligned} \tag{9.7}$$

The bounds of the integral depend on ω; but, if $\varphi \to 0$ for both large and small ξ, we can approximate $\xi_{max} \to \infty$. As a crude approximation, we can take $\varphi(\xi) = \delta(\xi - 1)$: that is, we approximate the spectrum for electrons with energy γmc^2 as a sharp spike at $\omega = \omega_c$, so that the integral is unity.

The important point is that for a power-law distribution of electrons with index p—$n(\gamma) \propto \gamma^{-p}$—the synchrotron spectrum is a power-law with index $(p-1)/2$. For $p \approx 2.5$, typical for many sources, $P_\omega \propto \omega^{-0.75}$. This spectrum is steeper than thermal bremsstrahlung, for example.

AN EXAMPLE OF A SYNCHROTRON-EMITTING SOURCE is the active galaxy Cygnus A. This system, shown in Fig. 9.3 is at a distance 230 Mpc ($z = 0.057$) and has two radio lobes astride the central galaxy (the dot between the lobes). The emission follows a power-law down to ≈ 10 MHz. Table 9.1 lists some measurements of the flux density S_ν.

Table 9.1: Flux density measurements for Cygnus A.

ν [MHz]	S_ν [mJy]	error [mJy]
74	1.66×10^7	4.95×10^6
178	8.10×10^6	2.40×10^6
178	8.70×10^6	3.90×10^6
1400	7.40×10^5	2.25×10^5
1400	8.58×10^5	2.55×10^5

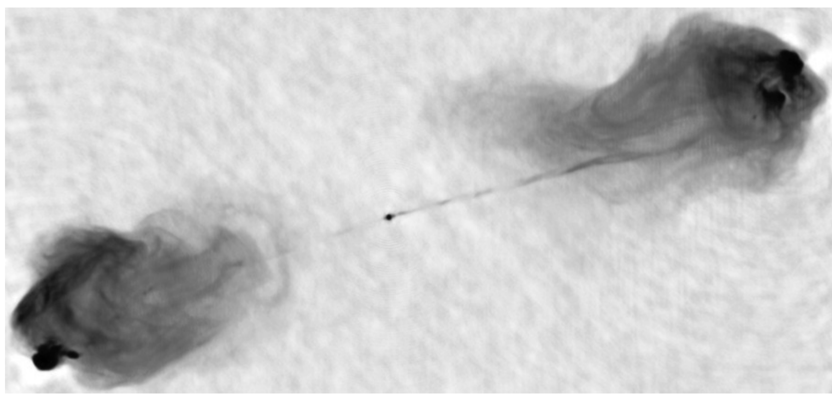

Figure 9.3: Image of Cygnus A, made from continuum observations at 5 GHz with 0.5″ resolution. *Image courtesy of NRAO/AUI.*

EXERCISE 9.2 — Let's explore the energetics of the lobes in Cygnus A.

1. Assume the electrons are at a uniform density with a power-law distribution of energies, $n(\gamma) = n_0\gamma^{-p}$, for $\gamma > \gamma_{\rm min}$. The minimum energy $\gamma_{\rm min}$ is unknown, but show that it should be $\gamma_{\rm min} < (10\,{\rm MHz}/\omega_{\rm L})^{1/2}$. Here $\omega_{\rm L} = eB/mc$ is the Larmor frequency. Then show the energy density of the electrons is
$$U_e = \frac{\gamma_{\rm min}^{2-p}}{p-2} n_0 mc^2.$$
2. Show that synchrotron spectrum (Eq. [9.7]) can be expressed as
$$P_\omega = K n_0 B^{(p+1)/2} \omega^{-(p-1)/2}$$
where K is a constant.
3. If we make a measurement of the flux density S_ω at a given frequency $\omega_\star$, show that
$$n_0 = \left[\frac{4\pi D^2 S_\omega}{VK}\omega_\star^{(p-1)/2}\right] B^{-(p+1)/2}.$$
Here D is the distance to the source and V is the volume of the lobes, and the term in [ ] is a measurable constant. Use this expression for n_0 to get an expression for U_e in terms of B. How does U_e change with increasing B? Argue that for a given P_ω, there is a magnetic field for which $U_e + U_B$ is a minimum. Find the ratio U_e/U_B at this minimum. Estimate p from Table 9.1, and show that $U_e \approx U_B$ if we assume the energy is at a minimum.
4. Solve for B. Estimate the volume of each lobe as a sphere of radius 30 kpc and assume $U_e = U_B$. Your answer should be a reasonable number, i.e., μG is a good unit of magnetic field strength.
5. For this magnetic field strength, find the total energy of the two lobes. Compare this to the energy radiated by stars in a typical galaxy over a year.
6. For this magnetic field strength, estimate the total synchrotron luminosity. How long would it take to radiate away the energy stored in the two lobes?
7. How does the electron lifetime scale with γ? That is, for each energy, estimate how long it would take for those electrons to radiate away their energy. If there were no input of energy into the system, qualitatively argue how the spectrum would evolve in time.

9.2 *Synchrotron Absorption*

The emission from a cloud of synchrotron-emitting particles is, in the absence of backlighting,
$$I_\nu(\tau_\nu) = S_\nu\,(1 - \tau_\nu)\,.$$
Here S_ν is the source function, and since we have a non-thermal distribution of particles, $S_\nu = \varepsilon_\nu/(4\pi\kappa_\nu) \neq B_\nu$. The most direct way to realize this is that $B_\nu = B_\nu(T)$, and for a power-law distribution of electrons, temperature is not defined.

To compute the opacity, we use the formalism of Ch. 4. specifically the Einstein A and B coefficients. Recall that the emissivity is[2]

[2] cf. Eq. (4.7)

$$\frac{\rho\epsilon_\nu}{4\pi} = n_n \frac{A_{nm}}{4\pi} h\nu\, \varphi(\nu);$$

and the opacity is[3]

[3] cf. Eqn. (4.8) and (4.9)

$$\rho\kappa_\nu = (n_m B_{mn} - n_n B_{nm}) \frac{h\nu}{4\pi}\varphi(\nu).$$

The coefficients are related as[4]

[4] Exercise 4.2

$$\frac{B_{nm}}{B_{mn}} = \frac{g_m}{g_n}, \quad \frac{A_{nm}}{B_{nm}} = \frac{2h\nu^3}{c^2}.$$

In these equations, m denotes the lower energy state and n the upper.

Because we have a continuum of electron energies, we must sum over all pairs of upper and lower energy states separated by $h\nu$. The electrons are free particles, so one has to integrate over their phase space: if $f(E)$ is a distribution function, then

$$N = \frac{1}{h^3} \int \mathrm{d}V\, \mathrm{d}^3 p\, f(E),$$

so for an isotropic distribution of relativistic electrons ($E = pc$),

$$n = \frac{N}{V} = \frac{8\pi}{(hc)^3} \int E^2 f(E)\, \mathrm{d}E.$$

Notice this implies that $g_m = g(E_m) = 8\pi E_m^2/(hc)^3$. In the above relations we make the replacement $n_m = n(E_m) = g(E_m)f(E_m)$. Since we have a continuum of energies, rather than discrete levels, let's denote transitions from higher to lower energies with a $\downarrow$, and upward transitions with an $\uparrow$. That is, A_{nm} and B_{nm} become $A_\downarrow$ and $B_\downarrow$, and B_{mn} becomes $B_\uparrow$. Further, we'll denote the higher energy as simply E, and the lower energy as E'.

For a power-law distribution of electrons with $f(E) = E^{-p-2}$, we combine the factor of E^2 from the density of states and write $n(E) = n_0 E^{-p}$.

To sum over all transitions, we write the profile function as $\varphi(\nu) = \delta(E - E' - h\nu)$. The power emitted per electron at energy E is then

$$P_\nu(E) = \int A_\downarrow h\nu \delta(E - E' - h\nu)\, \mathrm{d}E' = A_\downarrow h\nu. \tag{9.8}$$

The opacity is

$$\begin{aligned}
\rho\kappa_\nu &= \int\int \left[\frac{n(E')}{g(E')} - \frac{n(E)}{g(E)}\right] \frac{B_\downarrow}{4\pi} \delta(E - E' - h\nu) h\nu\, g(E)\, \mathrm{d}E\, \mathrm{d}E' \\
&= \int \left[\frac{n(E-h\nu)}{g(E-h\nu)} - \frac{n(E)}{g(E)}\right] \frac{B_\downarrow}{4\pi} h\nu\, g(E)\, \mathrm{d}E \\
&= \int \left[\frac{n(E-h\nu)}{(E-h\nu)^2} - \frac{n(E)}{E^2}\right] (A_\downarrow h\nu) \frac{c^2}{8\pi h\nu^3} E^2\, \mathrm{d}E.
\end{aligned}$$

Here we've used the relations between the A and B coefficients to express the opacity in terms of $A_\downarrow$. Now we use Equation (9.8) to eliminate $A_\downarrow h\nu$ and, since we are in the classical limit with $h\nu \ll E$, expand the term in [ ] to first order in $h\nu$ to obtain

$$\rho\kappa_\nu = \frac{c^2}{8\pi\nu^2}\int \mathrm{d}E\, E^2 \frac{\mathrm{d}}{\mathrm{d}E}\left[\frac{n(E)}{E^2}\right] P_\nu(E).$$

Substituting for $P_\nu(E)$ from Equation (9.5), writing $E = \gamma mc^2$, and changing variables to $\xi = \nu/(\gamma^2\nu_L)$ gives

$$\rho\kappa_\nu \propto \frac{c\sigma_T U_B}{m\nu_L^3} n_0 \left(\frac{\nu}{\nu_L}\right)^{-(p+4)/2}.$$

The opacity increases at low frequencies, so there is a transition frequency below which the source becomes optically thick: the source function is

$$S_\nu = \frac{\epsilon_\nu}{4\pi\kappa_\nu} \propto \nu_L^2 \left(\frac{\nu}{\nu_L}\right)^{5/2}.$$

In the optically thick regime, the spectrum does not depend on p, but the slope is 5/2, rather than 2 as for Rayleigh-Jeans emission.

10
Spectral Lines

10.1 Ionization Balance and Level Populations

Suppose we have a reaction, $A + B + \ldots \to C + D + \ldots$. For example, we might consider the ionization of hydrogen,

$$\mathrm{H} \to \mathrm{H}^{+} + e. \tag{10.1}$$

When this reaction comes into equilibrium, we are at a maximum in entropy, and the condition for equilibrium is that the energy cost, at constant entropy, to run the reaction in the forward direction is the same as to run the reaction in reverse. This can be expressed in terms of chemical potentials as

$$\mu_A + \mu_B + \ldots \to \mu_C + \mu_D + \ldots \tag{10.2}$$

Note in this formalism that a reaction $2A \to B$ would be expressed as $2\mu_A = \mu_B$.

To use Eq. (10.2), both sides must be on the same energy scale. To ionize hydrogen is an endothermic process; the left hand side of Eq. (10.1) is at a lower energy and we therefore subtract the binding energy $Q = 13.6\,\mathrm{eV}$ so that the energy zero-point is the same on both sides:

$$\mu_0 - Q = \mu_+ + \mu_-, \tag{10.3}$$

in which the subscripts 0, $+$, and $-$ denote H, H^+, and e, respectively. To solve this equation to find the abundance of ionized hydrogen, we then need an expression for the chemical potentials.

In statistical equilibrium, we can describe a system of particles by a distribution function $f(\boldsymbol{p}, \boldsymbol{x})\, \mathrm{d}^3p\, \mathrm{d}^3x$, such that the number of particles is

$$N = \int \mathrm{d}^3p\, \mathrm{d}^3x\, f(\boldsymbol{p}, \boldsymbol{x}), \tag{10.4}$$

where the integration is over the phase spaces of momentum and position coordinates $(\boldsymbol{p}, \boldsymbol{x})$. In an ideal gas, the particles do not interact. In such a case, the distribution function $f = f(\boldsymbol{p})$ does not depend on position. The integration over d^3x just gives a factor of the volume, so the number density is $n = \int \mathrm{d}^3p\, f(\boldsymbol{p})$.

The distribution function is

$$f(p) = \frac{g}{h^3}\left[\exp\left(\frac{\epsilon - \mu}{k_B T}\right) \pm 1\right]^{-1}. \tag{10.5}$$

Here the $+$ sign is for fermions (half-integral spin) and the $-$ sign is for bosons (integral spin). The factor g is the degeneracy of states with energy E. For example, $g = 2$ for a spin-1/2 particle.

To explore the non-degenerate limit, take[1] $\Lambda \equiv \exp(\mu/k_B T) \ll 1$. Further, let's look at an isotropic system, so that $\mathrm{d}^3p = 4\pi p^2\,\mathrm{d}p$. Then

$$n(\Lambda, T) = \frac{4\pi g}{h^3}\int \frac{p^2\,\mathrm{d}p}{\Lambda^{-1}\exp\left(\epsilon/k_B T\right) \pm 1} \approx \frac{4\pi\Lambda g}{h^3}\int \exp\left(-\frac{\epsilon}{k_B T}\right) p^2\,\mathrm{d}p.$$

This has the form of a Maxwell-Boltzmann gas. For a non-relativistic system, write $p^2\,dp = m(2m\epsilon)^{1/2}\,d\epsilon$ and make the substitution $x = \epsilon/(k_B T)$ to obtain

$$n(\mu, T) = \frac{4\pi\Lambda g}{h^3}\sqrt{2}(m k_B T)^{3/2}\int_0^\infty x^{1/2}e^{-x}\,dx = \Lambda\left[g\left(\frac{2\pi m k_B T}{h^2}\right)^{3/2}\right].$$

Solving this equation for μ gives

$$\mu = k_B T \ln\Lambda = k_B T \ln\left[\frac{n}{g}\left(\frac{h^2}{2\pi m k_B T}\right)^{3/2}\right]. \tag{10.6}$$

We can now use this expression to find the ionization balance of hydrogen, Eq. (10.3).

Inserting Eq. (10.6) into Eq. (10.3) and rearranging terms gives the *Saha equation*,

$$\frac{n_+ n_-}{n_0} = \frac{g_+ g_-}{g_0}\left(\frac{m_- k_B T}{2\pi\hbar^2}\right)^{3/2}\exp\left(-\frac{Q}{k_B T}\right). \tag{10.7}$$

The number density of all hydrogen in the gas is $n_0 + n_+ = n_H$. Denote the ionized fraction by $x = n_+/n_H = n_i/n_H$, so that the left-hand side of equation (10.7) is $n_H x^2/(1-x)$. In the hydrogen atom ground state, the electron spin and proton spin are either aligned or anti-aligned. These states are very nearly degenerate, so that $g_0 = 2$. Both the proton and electron have spin 1/2; there are really only two available states, however, because of the freedom in choosing our coordinate system. As a result, $g_+ g_- = 2$ as well.

Inserting these factors into equation (10.7), and using $k_B = 8.6173 \times 10^{-5}$ eV/K, we obtain

$$\frac{x^2}{1-x} = \frac{2.41 \times 10^{21}\,\mathrm{cm}^{-3}}{n_H}\left(\frac{T}{10^4\,\mathrm{K}}\right)^{3/2}\exp\left(-\frac{15.78 \times 10^4\,\mathrm{K}}{T}\right). \tag{10.8}$$

[1] The quantity Λ is called the *fugacity*.

This equation defines relationship between density and temperature at which $x = 1/2$. At fixed density, the transition from neutral to fully ionized is very rapid.

10.2 Line Widths

We saw in Section 5.3 that there is always an intrinsic width to any absorption or emission feature in a spectrum. This intrinsic width is very small, however, and in practice the width of lines are set by random Doppler shifts from thermal motion of the gas (or small-scale turbulent eddies) and collisions.

Suppose we model our oscillator as being started and stopped by impacts; in between impacts it just radiates as $e^{i\omega_0 t}$. To get the spectrum, we take the Fourier transform,

$$F(\omega, t) = \int_0^t \mathrm{d}t' \;\exp[i(\omega_0 - \omega)t'],$$

where t is some time between impacts. Now if the impacts are distributed randomly and are uncorrelated, then the distribution of wait times follows a Poisson distribution,

$$W(t)\,\mathrm{d}t = e^{-t/\tau}\,\mathrm{d}t/\tau,$$

where τ is the average time between collisions. Using this to compute the energy spectrum, we obtain

$$E(\omega) = \frac{1}{2\pi\tau}\int_0^\infty \mathrm{d}t\; F(\omega, t)F^*(\omega, t)W(t) = \frac{1}{\pi\tau}\frac{1}{(\omega_0 - \omega)^2 + (1/\tau)^2};$$

the line profile is again Lorentzian, with a full-width at half-maximum (FWHM) $2/\tau$.

We might be inclined to treat the atoms as hard spheres, but this gives a large τ, or equivalently a narrow line width. We are therefore led to consider longer-range interactions for setting the intrinsic line width. Table 10.1 lists such interactions. The picture is similar to our considerations of collisions in §7.1. For a given impact parameter, the interaction perturbs the energy levels; by integrating over a distribution of impact parameters one gets the intrinsic damping. Of course, we should really use a quantum mechanical calculation. We can scale our cross-section to the classical result (eq. [5.23]), however, by writing

$$\sigma_\nu = \left(\frac{\pi e^2}{m_e c}\right) f\varphi_\nu, \tag{10.9}$$

where φ_ν is the line profile (dimension $\sim$ Hz^{-1}) and f is a dimensionless cross-section called the **oscillator strength**.

Table 10.1: Interactions in stellar atmospheres. From Mihalas [1978].

perturbation	form	source	affects
linear Stark	$C_2 r^{-2}$	e^-, p, ions	H (Hα, Hβ, ...)
quadratic Stark	$C_4 r^{-4}$	e^-	non-hydrogenic ions
van der Waals	$C_6 r^{-6}$	atoms, H	most atomic lines

EXERCISE 10.1— H I gas distributed between clouds within our in the galaxy has [Spitzer, 1978] typical densities $n \approx 0.1\,\mathrm{cm}^{-3}$ and temperatures $T \approx 6\,000\mathrm{K}$. We wish to model the widths of lines for transitions in atoms in this cloud. We showed that the line profile is Lorentzian, with a FWHM $2/\tau$, where τ is the mean time between collisions. Suppose we model the atoms has hard spheres. Use your best order-of-magnitude estimates to answer the following questions.

1. If the atoms can be treated as hard spheres, what is the appropriate radius and cross-section?
2. Given that, what is a typical mean free path and collision time in the H I gas considered here?
3. Does the resulting line width seem reasonable? Is the hard sphere model a good approximation?
4. Now suppose we consider the solar photosphere, where the density is $n \sim 10^{16}\,\mathrm{cm}^{-3}$. What is a typical line width in this case? For transitions at optical frequencies, does the hard sphere approximation make sense in this regime?

10.3 The Curve of Growth

A classical technique in the analysis of stellar spectra is to construct the *curve of growth*, which relates the equivalent width of a line W_ν to the opacity in the line. This discussion follows Mihalas[2].

[2] D. Mihalas. *Stellar Atmospheres*. W. H. Freeman, 2d edition, 1978

Let's first get the opacity in the line. Write the cross-section for the transition $i \to j$ as

$$\sigma_\nu = \left(\frac{\pi e^2}{m_e c}\right) f_{ij}\varphi_\nu,$$

where the first term is the classical oscillator cross-section, f_{ij} is the oscillator strength and contains the quantum mechanical details of the interaction, and φ_ν is the line profile. Now recall that the opacity is given by $\kappa_\nu = n_i\sigma_\nu/\rho$, where n_i denotes the number density of available atoms in state i available to absorb a photon. Furthermore, we need to allow for *stimulated emission* from state j to state i. With this added, the opacity is[3]

[3] I'm writing the line opacity as χ_ν to distinguish it from the *continuum opacity*.

$$\rho\chi_\nu = \left(\frac{\pi e^2}{m_e c}\right) f_{ij}\varphi_\nu n_i \left[1 - \frac{g_i}{g_j}\frac{n_j}{n_i}\right]. \tag{10.10}$$

If we are in LTE, then the relative population of n_i and n_j follow a Boltzmann distribution,

$$1 - \frac{g_i}{g_j}\frac{n_j}{n_i} = 1 - \exp\left(-\frac{h\nu}{kT}\right).$$

This ensures we have a positive opacity. If our population were inverted, i. e., more atoms in the upper state j, then the opacity would be negative and we would have a *laser*.

Now for the line profile. In addition to damping, there is also Doppler broadening from thermal (or convective) motion. Let the line profile[4] be Lorentzian,

$$\varphi = \frac{\Gamma/(4\pi)}{(\nu-\nu_0)^2 + (\Gamma/[4\pi])^2}.$$

[4] Here we'll switch to ν, rather than ω.

In a Maxwellian distribution, the probability of having a line-of-sight velocity in $(u, u + \mathrm{d}u)$ is

$$\mathcal{P}(u)\,\mathrm{d}u = \frac{1}{\sqrt{\pi}u_0}\exp\left(-\frac{u^2}{u_0^2}\right),$$

where $u_0 = (2kT/m)^{1/2} = 12.85\,\mathrm{km\,s^{-1}}\left(T/10^4\,\mathrm{K}\right)$ (for H) is the mean thermal velocity. The atom absorbs at a it shifted frequency $\nu(1 - u/c)$, so the mean cross section is

$$\sigma_\nu = \int_{-\infty}^{\infty}\sigma\left[\nu\left(1-\frac{u}{c}\right)\right]\mathcal{P}(u)\,\mathrm{d}u. \qquad (10.11)$$

After some algebraic manipulations, we have the cross-section

$$\begin{aligned}\sigma_\nu &= \left(\frac{\sqrt{\pi}e^2}{m_e c}\right) f_{ij}\frac{1}{\Delta\nu_D}\left\{\frac{a}{\pi}\int_{-\infty}^{\infty}\frac{\exp(-y^2)\,\mathrm{d}y}{(v-y)^2+a^2}\right\} \\ &\equiv \frac{1}{\Delta\nu_D}H(a,v) \qquad (10.12)\end{aligned}$$

where $\Delta\nu_D \equiv \nu u_0/c$ is the doppler width, $a = \Gamma/(4\pi\Delta\nu_D)$ is the ratio of the damping width Γ to the doppler width, and $v = \Delta\nu/\Delta\nu_D$ is the difference in frequency from the line center in units of the doppler width. The function $H(a, v)$ is called the *Voigt* function.

Let's combine the line opacity with the continuum opacity and solve the equation of transfer. For simplicity, we are going to assume pure absorption in both the continuum and the line. Under these conditions, the source function is[5] $S_\nu = B_\nu$, the Planck function. For a plane-parallel atmosphere, the equation of transfer is then

[5] See the notes on the Eddington atmosphere.

$$\mu\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = I_\nu - B_\nu \qquad (10.13)$$

where μ is the cosine of the angle of the ray with vertical. Solving equation (10.13) for the emergent intensity at $\tau_\nu = 0$ gives

$$I_\nu(\mu) = \frac{1}{\mu}\int_0^{\infty} B_\nu[T(\tau_\nu)]\exp(-\tau_\nu/\mu)\,\mathrm{d}\tau_\nu. \qquad (10.14)$$

The opacity is given by

$$\kappa_\nu = \kappa_\nu^C + \chi_\nu, \tag{10.15}$$

where κ_ν^C is the continuum opacity and $\chi_\nu = \chi_0\varphi_\nu$ is the line opacity, with

$$\chi_0 = \frac{1}{\rho}\left(\frac{\pi e^2}{m_e c}\right) f_{ij} n_i \left(1 - e^{h\nu_\ell/kT}\right)$$

being the line opacity at the line center ν_ℓ.

As a further simplification, we can usually ignore the variation with ν in κ_ν^C over the width of the line. As a more suspect approximation (although it is not so bad in practice), let's assume that $\beta_\nu \equiv \chi_\nu/\kappa_C$ is independent of τ_ν. With this assumption we can write $\mathrm{d}\tau_\nu = (1+\beta_\nu)\mathrm{d}\tau$, where $\tau = -\rho\kappa^C\,\mathrm{d}z$. Finally, let's assume that in the line forming region, the temperature does not vary too much, so that we can expand B_ν to first order in τ,

$$B_\nu[T(\tau)] \approx B_0 + B_1\tau,$$

where B_0 and B_1 are constants. Inserting these approximations into equation (10.14), multiplying by the direction cosine μ and integrating over outward bound rays gives us the flux,

$$\begin{aligned} F_\nu &= 2\pi \int_0^1 \int_0^\infty [B_0 + B_1\tau] \exp\left[-\frac{\tau}{\mu}(1+\beta_\nu)\right](1+\beta_\nu)\ \mathrm{d}\tau\,\mathrm{d}\mu \\ &= \pi\left[B_0 + \frac{2}{3}\frac{B_1}{1+\beta_\nu}\right]. \end{aligned} \tag{10.16}$$

Far from the line-center, $\beta_\nu \to 0$, implying that the continuum flux is

$$F_\nu^C = \pi\left[B_0 + \frac{2B_1}{3}\right].$$

Hence the depth of the line is

$$A_\nu \equiv 1 - \frac{F_\nu}{F_\nu^C} = A_0\frac{\beta_\nu}{1+\beta_\nu}, \tag{10.17}$$

where

$$A_0 \equiv \frac{2B_1/3}{B_0 + 2B_1/3} \tag{10.18}$$

is the depth of an infinitely opaque ($\beta_\nu \to \infty$) line.

EXERCISE 10.2— Explain why an infinitely opaque line (Eq. [10.18]) is not completely black, i.e., why $A_0 \neq 1$.

Now that we have the depth of the line A_ν we can compute the *equivalent width*,

$$W_\nu \equiv \int_0^\infty A_\nu\,\mathrm{d}\nu = A_0\int_0^\infty \frac{\beta_\nu}{1+\beta_\nu}\,\mathrm{d}\nu. \tag{10.19}$$

Let's change variables from ν to $v = \Delta\nu/\Delta\nu_D = (\nu - \nu_\ell)/\Delta\nu_D$. Since $H(a, v)$ is symmetrical about the line center, we will just integrate over $\Delta\nu > 0$, giving

$$W_\nu = 2A_0 \Delta\nu_D \int_0^\infty \frac{\beta_0 H(a, v)}{1 + \beta_0 H(a, v)}\, \mathrm{d}v, \tag{10.20}$$

with $\beta_0 = \chi_0/(\kappa^C \Delta\nu_D)$.

It's useful to understand the behavior of W_ν in various limits. First, at small line optical depth ($\beta_0 \ll 1$) only the core of the line will be visible (Fig. 10.1, blue curves). In the core of the line, $H(a, v) \approx \exp(-v^2)$ so we insert this into equation (10.20) and expand the denominator to give

$$\begin{aligned} W_\nu^\star \equiv \frac{W\nu}{2A_0 \Delta\nu_D} &= \int_0^\infty \sum_{k=1}^{\infty} (-1)^{k-1} \beta_0^k e^{-kv^2}\, \mathrm{d}v \\ &= \frac{1}{2}\sqrt{\pi}\beta_0 \left[1 - \frac{\beta_0}{\sqrt{2}} + \frac{\beta_0^2}{\sqrt{3}} - \dots\right]. \end{aligned} \tag{10.21}$$

Here $W_\nu^\star$ is the *reduced equivalent width*. Notice that since $\beta_0 \propto 1/\Delta\nu_D$ (cf. eq. [10.12]), the equivalent width W_ν is independent of $\Delta\nu_D$ in this *linear regime*. Physically, in the limit of small optical depth, each atom in state i is able to absorb photons, and the flux removed is just proportional to the number of atoms n_i.

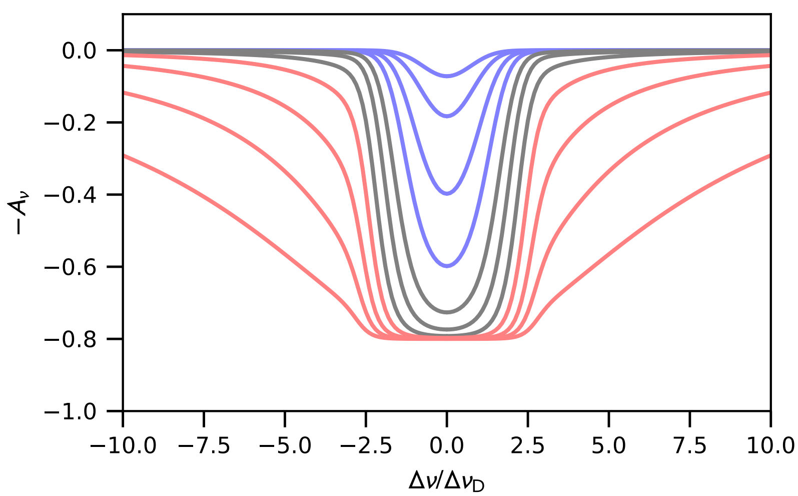

Figure 10.1: Profiles for lines of various optical depths.

As we increase β_0 eventually the core of the line saturates—no more absorption in the core is possible (Fig. 10.1, grey curves). As a result, the equivalent width should be nearly constant until there are so many absorbers that the damping wings contribute to the removal of flux. In the *saturation regime*, the Voigt function is still given by e^{-v^2}, but we can no longer assume $\beta_0 \ll 1$, so our expansion in equation (10.21) won't work. Let's go back to our integral, eq. (10.20), change variables to $z = v^2$, and define $\alpha = \ln \beta_0$ to find

$$W_\nu^\star = \frac{1}{2}\int_0^\infty \frac{z^{-1/2}}{e^{z-\alpha}+1}\,\mathrm{d}z.$$

This may not look like an improvement, but you might notice that it bears a resemblance to a Fermi-Dirac integral, which are used in computing the equation of state of degenerate electrons. You can find a description of how to integrate it in a graduate-level textbook on statistical mechanics. In this saturation regime,

$$W_\nu^\star \approx \sqrt{\ln \beta_0}\left[1 - \frac{\pi^2}{24(\ln \beta_0)^2} - \frac{7\pi^4}{384(\ln \beta_0)^4} - \ldots\right]. \qquad (10.22)$$

Note that the amount of flux removed is basically $2A_0\Delta\nu_D$: the line is maximally dark across the gaussian core.

Finally, if we continue to increase the line opacity, there will finally be so many absorbers that there will be significant flux removed from the wings (Fig. 10.1, red curves). Now the form of the Voigt profile is $H(a,v) \approx (a/\sqrt{\pi})v^{-2}$, so our integral (eq. [10.20]) in this *damping regime* becomes

$$\begin{aligned} W_\nu^\star &= \int_0^\infty \left(1 + \frac{\sqrt{\pi}v^2}{\beta_0 a}\right)^{-1} \mathrm{d}v \\ &= \frac{1}{2}\left(\pi a \beta_0\right)^{1/2}. \end{aligned} \qquad (10.23)$$

Note that since $a\beta_0 \propto \Delta\nu_D^{-2}$, W_ν is again independent of the doppler width in this regime.

Now that we have this "curve of growth", $W_\nu^\star(\beta_0)$, why is it useful? Since it only involves the equivalent width, it is possible to construct the curve of growth empirically without a high-resolution spectrum. Next, let's put some of the factors back into the quantities in the curve of growth. First, for a set of lines, the population of the excited state depends on the Boltzmann factor $\exp(-E/kT)$. Second, we can expand out the Doppler width in both $W_\lambda^\star$ and β_0,

$$\log\left(\frac{W_\lambda}{\Delta\lambda_D}\right) = \log\left(\frac{W_\lambda}{\lambda}\right) - \log\left(\frac{u_0}{c}\right) \qquad (10.24)$$

$$\log \beta_0 = \log(g_i f_{ij}\lambda) - \frac{E}{kT} + \log(N/\kappa^C) + \log C \qquad (10.25)$$

where C contains all of the constants and the continuum opacity. The temperature T is picked as a free parameter, and is picked to minimize scatter about a single curve that is assumed to fit all of the lines. What is measured then is $\log(W_\lambda/\lambda)$ and $\log(g_i f_{ij}\lambda)$; by comparing them to theoretical curves one gets an estimate of $\log(u_0/c)$, the mean velocity of atoms (may be thermal or turbulent). Since the continuum opacity κ^C usually depends on the density of H, one gets from equation (10.25) an estimate of the abundance of the line-producing element to H.

A

Units

The choice of dimensions and units for physical quantities is arbitrary; they are chosen for our convenience[1]. Here we shall give three examples of how one chooses quantities based on the phenomena being considered.

[1] Raymond T. Birge. On the establishment of fundamental and derived units, with special reference to electric units. Part I. *Am. J. Phys.*, 3:102, 1935a; and Raymond T. Birge. On the establishment of fundamental and derived units, with special reference to electric units. Part II. *Am. J. Phys.*, 3:171, 1935b

For nuclear phenomena, it is convenient to set the speed of light $c = 1$. The dimension of c is $[c] \sim LT^{-1}$; in this case, then, we can choose either a unit of length or a unit of time. For example, if we choose 1 m to be our unit of length, then our unit of time is $1\,\mathrm{m}/c$. In nuclear physics, it is convenient to pick the femtometer, also known as the fermi[2], for the unit of length; the "size" of a nucleon is of order 1 fm.

[2] $1\,\mathrm{fm} = 10^{-13}\,\mathrm{cm}$

To define units that connect the macroscopic world to our microscopic calculations, we turn to the world of accelerator physics. The electric potential is defined as the energy per unit charge and has a unit of a *volt* (V). If we accelerate a single electron through a 1 V electrostatic potential, then the energy gained by the electron is $1\,\mathrm{eV} = 1.602 \times 10^{-19}\,\mathrm{J} = 1.602 \times 10^{-12}\,\mathrm{erg}$. The electron volt, and powers thereof, are convenient scales: the electronic binding energy of a hydrogen atom is 13.6 eV; the rest mass of an electron is 0.511 MeV; and nuclear energy levels are spaced by keV to MeV, with the rest mass of a proton being close to 1 GeV.

In nuclear physics a convenient choice is the MeV for energy. In this system of units, $\hbar c = 197\,\mathrm{MeV\,fm}$, and the unit of charge is[3] $e^2 = [e^2/(\hbar c)]\hbar c = 1.44\,\mathrm{MeV\,fm}$. Since the "size" of a nucleon is of order 1 fm, this immediately tells you the scale of the electrostatic potential between two protons in the nucleus. The temperature scale in these units is $1\,\mathrm{MeV}/k = 1.16 \times 10^{10}\,\mathrm{K}$.

[3] Recall that $e^2/(\hbar c) = 1/137$ is the fine-structure constant; since it is dimensionless, the unit of e^2 is [energy] $\times$ [length].

For high-energy physics, we can go further and set both $\hbar$ and c to unity. The dimension of $\hbar$ is $[\hbar] \sim ML^2T^{-1} \sim ET$. Time therefore has dimensions E^{-1}, and since $c = 1$, length also has dimensions E^{-1}. Our sole dimension is energy, which we can measure in units of MeV, for example. In this system, $e^2 = 1/137$ is dimensionless and the unit of length is $1\,\mathrm{MeV}^{-1} = \hbar c/(1\,\mathrm{MeV}) = 197\,\mathrm{fm} = 1.97 \times 10^{-11}\,\mathrm{cm}$.

If instead we were investigating topics involving stellar-mass black holes, we could choose $c = G = 1$. The dimension of c is $[c] \sim LT^{-1}$ and the dimension of G is $[G] \sim L^3T^{-2}M^{-1}$, so our units are specified once we choose a unit of mass. If we pick our unit of mass to be $1\,M_\odot$ (a convenient choice for astrophysics) then our unit of length becomes $GM_\odot/c^2 = 1.5\,\mathrm{km}$ and our unit of time becomes $GM_\odot/c^3 = 4.9\,\mu\mathrm{s}$.

FINALLY, IF WE REALLY WANT TO HAVE NO ARBITRARILY CHOSEN UNITS, we can set $\hbar = c = G = 1$, which gives the *Planck scale*. The unit of mass is $m_\mathrm{P} = (\hbar c/G)^{1/2} = 2.18 \times 10^{-5}\,\mathrm{g}$; the unit of length is $(\hbar G c^{-3})^{1/2} = 1.62 \times 10^{-33}\,\mathrm{cm}$ and the unit of time is $(\hbar G c^{-5})^{1/2} = 5.39 \times 10^{-44}\,\mathrm{s}$.

EXERCISE A.1— For atomic problems, we are non-relativistic, so setting $c = 1$ is not the most convenient choice. Instead, we might choose to set $e^2 = \hbar = m_e = 1$. If we do this what are units of length, time, and energy?

TO ILLUSTRATE HOW TO CONVERT UNITS, WE SHALL START WITH A SIMPLE EXAMPLE. Suppose we measure the length of a rod with both a meter stick and a yardstick. The length of the rod, when measured with the meter stick is l_m m; when measured with the yardstick, $l_\mathrm{yd}$yd. When written in this way, both l_yd and l_m are pure numbers, and clearly l_m and l_yd are different numbers! It is the same rod, however, so

$$l_\mathrm{m} \times 1\,\mathrm{m} \equiv l_\mathrm{yd} \times 1\,\mathrm{yd}.$$

The lengths 1 m and 1.0936 yd are equivalent, so if we divide both sides by this length, we obtain

$$l_\mathrm{yd} = 1.0936 \times l_\mathrm{m};$$

or, put differently,

$$l_\mathrm{yd} \times 1\,\mathrm{yd} = \frac{1.0936\,\mathrm{yd}}{1\,\mathrm{m}} \times 1\,\mathrm{m} \times l_\mathrm{m}.$$

Let's now apply this algorithm to find how to convert from charge in SI ($q_\mathrm{SI} \times 1\,\mathrm{C}$) to charge in gaussian CGS ($q_\mathrm{CGS} \times 1\,\mathrm{statcoul}$). The potential energy of two identical charges q separated by a distance d is, in SI and gaussian CGS respectively[4],

[4] why do we use this relation?

$$\begin{aligned} \Phi_\mathrm{SI} &= \left[\frac{1}{4\pi\epsilon_0}\right]_\mathrm{SI} \frac{q_\mathrm{SI}^2}{d_\mathrm{SI}} \\ \Phi_\mathrm{CGS} &= \frac{q_\mathrm{CGS}^2}{d_\mathrm{CGS}}. \end{aligned}$$

Hence,

$$
\begin{aligned}
q_{\rm CGS} \times 1\,\text{statcoul} &= [(\Phi_{\rm CGS} \times 1\,\text{erg}) \times (d_{\rm CGS} \times 1\,\text{cm})]^{1/2} \\
&= \left[\frac{10^7\,\text{erg}}{1\,\text{J}}(\Phi_{\rm SI}\,\text{J}) \times \frac{100\,\text{cm}}{1\,\text{m}}(d_{\rm SI}\,\text{m})\right]^{1/2} \\
&= \left[10^9 \left(\frac{1}{4\pi\epsilon_0}\right)_{\rm SI}\right]^{1/2} q_{\rm SI} \times (\text{erg cm})^{1/2} \\
&= [10 c_{\rm SI}]\, q_{\rm SI} \times (\text{erg cm})^{1/2}\,.
\end{aligned}
$$

Here $c_{\rm SI} = 2.99792458 \times 10^8$ is the numerical value of the speed of light in meters per second and we used $(4\pi\epsilon_0)^{-1} = 10^{-7}c^2$.

For example, the charge of an electron in SI is $e = 1.602 \times 10^{-19}$ C; in gaussian CGS, the charge is $e = (2.99792458 \times 10^9) \times (1.602 \times 10^{-19}) = 4.803 \times 10^{-10}$ statcoul. In practice, the easiest way to remember the electron charge is to recall that the fine structure constant is $\alpha = e^2/(\hbar c) \approx 1/137$ and therefore $e = \sqrt{\hbar c/137}$. Indeed, this latter relation is useful in making the transition to "natural" units, in which $c = \hbar = 1$.

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