Calculating the $e + \mu \to e + \mu$ scattering cross section using Schoonschip

The Feynman diagram for $e + \mu \to e + \mu$ is

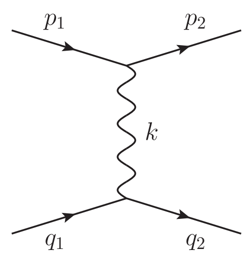

Figure 1: The upper solid line represents an electron, the lower solid line represents a muon and the wiggly line represents a photon.

This diagram can be expressed in terms of a complex amplitude that takes the form

$$\mathcal{M} = \frac{e^2\, \bar{u}(q_2)\gamma_\mu u(q_1)\bar{u}(p_2)\gamma_\mu u(p_1)}{k^2}\,, \tag{1}$$

where $k = p_2 - p_1 = q_1 - q_2$. $u(p_1)$ and $u(q_1)$ are the initial electron and muon wave functions, $\bar{u}(p_2)$ and $\bar{u}(q_2)$ are the final electron and muon wave functions and $k^2 = (p_2 - p_1)^2$ is the photon propagator. The repeated μ suffix implies a summation over $1, 2, 3, 4$. The four vector is $p_\mu = (\vec{p}, ip_0)$. For a free particle $p_0 = E(\vec{p}) = \sqrt{\vec{p}^2 + m^2}$, so $p_\mu p_\mu = p{\cdot}p = p^2 = -m^2$, where m is the mass of the particle.

To calculate the cross section it is necessary to obtain an expression for the transition rate $\mathcal{M}\mathcal{M}^*$ in terms of the energy and scattering angle by using explicit forms of p_1, p_2, q_1, q_2 in a convenient reference frame. When reducing $\mathcal{M}\mathcal{M}^*$ to a real number, one can usually sums over the final electron and muon spins and averages over the initial electron and muon spins. As with Compton scattering, Schoonschip will do this.

In this example, we choose the center of mass frame where

$$\begin{aligned}
p_1 &= (0, 0, p, iE_e) && (2)\\
q_1 &= (0, 0, -p, iE_\mu) && (3)\\
p_2 &= (p' \sin(\theta), 0, p' \cos(\theta), iE'_e) && (4)\\
q_2 &= (-p' \sin(\theta), 0, -p' \cos(\theta), iE'_\mu) && (5)\\
k &= p_2 - p_1 && (6)
\end{aligned}$$

From conservation of energy and momentum, $p_1 + q_1 = p_2 + q_2$, it follows that $E_e + E_\mu = E'_e + E'_\mu = \sqrt{s}$, where s is the square of the center of mass energy, $s = -(p_1 + q_1)^2$. Then, the Schoonschip code is

```
A m,M,cos,sin,cos2,sin2,Ee,Em
V p1,p2,q1,q2,k
I mu,nu,i1,i2
F T=u
Z R=T(mu)*Conjg(T(nu))
Id,T(mu~)=Ubg(i1,M,q2)*G(i1,mu)*Ug(i1,M,q1)*
     Ubg(i2,m,p2)*G(i2,mu)*Ug(i2,m,p1)*kDk^-2
Id,Spin,i1,i2
Id,Trick,Trace,i1,i2
B kDk^-4
P out
*yep
C p1=(0,0,p,iEe)
C q1=(0,0,-p,iEm)
C p2=(pp*sin,0,pp*cos,iE'e)
C q2=(-pp*sin,0,-pp*cos,iE'm)
C sin2=sin^2(theta/2) cos2=cos^2(theta/2)
Id,p1Dq1=-p^2-Ee*Em
Id,p1Dq2=-p^2*cos-Ee*Em
Id,q1Dq2=-2*p^2*cos2^2-M^2
Id,p1Dp2=-2*p^2*sin2^2-m^2
Id,p2Dq1=-p^2*cos-Ee*Em
Id,p2Dq2=-p^2-Ee*Em
C Id,kDk^-4=(p^4*sin2^4)^-1
Id,Addfac,1/16
B kDk^-4
P out
*yep
C Neglect masses
Id,Ee=p
Al,Em=p
Id,m=0
Al,M=0
Id,Multi,cos=2*cos2^2-1
Id,Addfa,kDk^4*p^-4
P out
*yep
Id,Addfa,al^2*s^-1*16^-1*sin2^-4
B al,sin2^-4,s^-1
P out
*end
```

and the output is

```
Schoonschip, 68000 version of June 27, 1991. Public version.
Date: Thu Jun  7 2001 18:01:29.  Memory: start 00020008, length 476860.
```

```
Command line: ee2mumu.e ee2mumu.txt

A m,M,cos,sin,cos2,sin2,Ee,Em
V p1,p2,q1,q2,k
I mu,nu,i1,i2
F T=u
Z R=T(mu)*Conjg(T(nu))
L 2 Id,T(mu~)=Ubg(i1,M,q2)*G(i1,mu)*Ug(i1,M,q1)*
     Ubg(i2,m,p2)*G(i2,mu)*Ug(i2,m,p1)*kDk^-2
L 3 Id,Spin,i1,i2
L 4 Id,Trick,Trace,i1,i2
B kDk^-4
> P out
*yep

R = + kDk^-4
  * ( 64*m^2*M^2 + 32*m^2*q1Dq2 + 32*M^2*p1Dp2 + 32*p1Dq1*p2Dq2
  + 32*p1Dq2*p2Dq1 ) + 0.

C p1=(0,0,p,iEe)
C q1=(0,0,-p,iEm)
C p2=(p*sin,0,p*cos,iEe)
C q2=(-p*sin,0,-p*cos,iEm)
C sin2=sin^2(theta/2) cos2=cos^2(theta/2)
L 1 Id,p1Dq1=-p^2-Ee*Em
L 2 Id,p1Dq2=-p^2*cos-Ee*Em
L 3 Id,q1Dq2=-2*p^2*cos2^2-M^2
L 4 Id,p1Dp2=-2*p^2*sin2^2-m^2
L 5 Id,p2Dq1=-p^2*cos-Ee*Em
L 6 Id,p2Dq2=-p^2-Ee*Em
C Id,kDk^-4=(p^4*sin2^4)^-1
L 7 Id,Addfac,1/16
B kDk^-4
> P out
*yep

R =
 + kDk^-4
  * ( - 4*m^2*cos2^2*p^2 - 4*M^2*sin2^2*p^2 + 4*cos*Ee*Em*p^2
  + 2*cos^2*p^4 + 4*Ee*Em*p^2 + 4*Ee^2*Em^2 + 2*p^4 ) + 0.

C Neglect masses
L 1 Id,Ee=p
L 1 Al,Em=p
L 2 Id,m=0
L 2 Al,M=0
```

```
L 3 Id,Multi,cos=2*cos2^2-1
L 4 Id,Addfa,kDk^4*p^-4
> P out
*yep

R =  + 8 + 8*cos2^4 + 0.

L 1 Id,Addfa,al^2*s^-1*16^-1*sin2^-4
B al,sin2^-4,s^-1
> P out
*end

R = + al^2*sin2^-4
  * ( 1/2*cos2^4*s^-1 + 1/2*s^-1 ) + 0.


End run. Time 0 sec.
```

The result after the last *yep, $R = 8(1+\cos^4(\theta/2))$, comes from squaring the matrix element and dropping the factor kDk^{-4}, which is included in the last Addfa. Adding the flux factor and phase space[1], the differential cross section $d\sigma/d\Omega$ is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s}\frac{(1+\cos^4(\theta/2))}{\sin^4(\theta/2)}\,. \tag{7}$$

Appendix: $d\sigma/d\Omega$ in the center of mass.

Including the energy-momentum conserving delta function, the standard normalization factors, the average over initial spins, the incoming flux and the phase space factor, the expression for $d\sigma$ is

$$d\sigma = \frac{1}{4}\frac{1}{|\vec{v}_e - \vec{v}_\mu|}\frac{1}{4E_e(\vec{p}_1)E_\mu(\vec{q}_1)}\delta^{(4)}(p_1+q_1-p_2-q_2)\frac{e^4}{(2\pi)^2}\frac{d^3\vec{p}_2\,d^3\vec{q}_2}{4E_e(\vec{p}_2)E_\mu(\vec{q}_2)}|\mathcal{M}|^2\,, \tag{8}$$

where $E_e(\vec{p_1}) = \sqrt{\vec{p}_1^{\,2}+m_e^2}$ and $E_\mu(\vec{q}_1) = \sqrt{\vec{q}_1^{\,2}+m_\mu^2}$. When integrating over $d^3\vec{p}_2\,d^3\vec{q}_2$, we can use the constraints of the delta function in $|\mathcal{M}|^2$,. From the initial values of p_1 and q_1, the delta function can be written as

$$\delta^{(4)}(p_1+q_1-p_2-q_2) = \delta^{(3)}(\vec{p}_2+\vec{q}_2)\delta(\sqrt{s}-E_e(\vec{p}_2)-E_\mu(\vec{q}_2))\,. \tag{9}$$

Apart from terms the only depend on p_1 and q_1, the $d^3\vec{q}_2$ integration then leaves

$$\int d^3\vec{p}_2\frac{\delta(\sqrt{s}-E_e(\vec{p}_2)-E_\mu(\vec{p}_2))}{E_e(\vec{p}_2)E_\mu(\vec{p}_2)}\,. \tag{10}$$

Since $E_e(\vec{p}_2)$ and $E_\mu(\vec{p}_2)$ depend on $|\vec{p}_2|$, we can write $d^3\vec{p}_2$ as

$$d^3\vec{p}_2 = d|\vec{p}_2||\vec{p}_2|^2\,d\Omega\,, \tag{11}$$

[1] see Appendix

where $d\Omega$ denotes $\sin(\theta) d\theta\, d\phi$. The $d|\vec{p}_|$ integration has the form

$$\int \frac{dp\, p^2 \delta(\sqrt{s} - E_e(p) - E_\mu(p))}{E_e(p) E_\mu(p)}. \tag{12}$$

Since we are integrating over p, the delta function has the form $\delta(F(p))$ and we need to find all those values of p such that $F(p) = 0$. If $\hat{p}$ is one of these points and p is close to $\hat{p}$, then $\delta(F(p)) = \delta(F'(\hat{p})(p - \hat{p})) = \delta(p - \hat{p})/|F'(\hat{p})|$. In our case, $|F'(p)|$ is

$$|F'(p_3)| = \frac{p_3}{E_e(p_3)} + \frac{p_3}{E_\mu(p_3)} = \frac{p_3(E_e(p_3) + E_m(p_3))}{E_e(p_3) E_m(p_3)} = \frac{p_3 \sqrt{s}}{E_e(p_3) E_\mu(p_3)}. \tag{13}$$

The point where $\sqrt{s} - E_e(p) - E_\mu(p) = 0$ is

$$E_e(p_3) + E_\mu(p_3) = \sqrt{s}\,. \tag{14}$$

Multiplying by $E_e(p_3) - E_\mu(p_3)$ gives

$$E_e(p_3) - E_\mu(p_3) = \frac{m_e^2 - m_\mu^2}{\sqrt{s}}\,. \tag{15}$$

Then,

$$E_e(p_3) = \frac{s + m_e^2 - m_\mu^2}{2\sqrt{s}}\,. \tag{16}$$

Using $E_e^2(p_3) = p_3^2 + m_e^2$, p_3 is

$$p_3 = \frac{\sqrt{s^2 + m_e^4 + m_\mu^4 - 2s\, m_e^2 - 2s\, m_\mu^2 - 2m_e^2\, m_\mu^2}}{2\sqrt{s}} \equiv \frac{\lambda(s, m_e^2, m_\mu^2)}{2\sqrt{s}}\,. \tag{17}$$

Hence, the phase space integral is

$$\frac{\lambda(s, m_e^2, m_\mu^2)}{2s} d\Omega \tag{18}$$

The flux factor is

$$|\vec{v}_e - \vec{v}_\mu| = \frac{p_1}{E_e(\vec{p_1})} + \frac{p_1}{E_\mu(\vec{p_1})} = \frac{p_1(E_e(\vec{p_1}) + E_\mu(\vec{p_1}))}{E_e(\vec{p_1}) E_\mu(\vec{p_1})} = \frac{p_1 \sqrt{s}}{E_e(\vec{p_1}) E_\mu(\vec{p_1})} \tag{19}$$

Then,

$$|\vec{v}_e - \vec{v}_\mu| E_e(\vec{p_1}) E_\mu(\vec{p_1}) = p_1 \sqrt{s} = \frac{\lambda(s, m_e^2, m_\mu^2)}{2}\,. \tag{20}$$

If one uses $s = m_e^2 + m_\mu^2 - 2p_1 \cdot q_1$, it is possible to show that

$$\frac{\lambda(s, m_e^2, m_\mu^2)}{2} = \sqrt{(p_1 \cdot q_1)^2 - m_e^2\, m_\mu^2}. \tag{21}$$