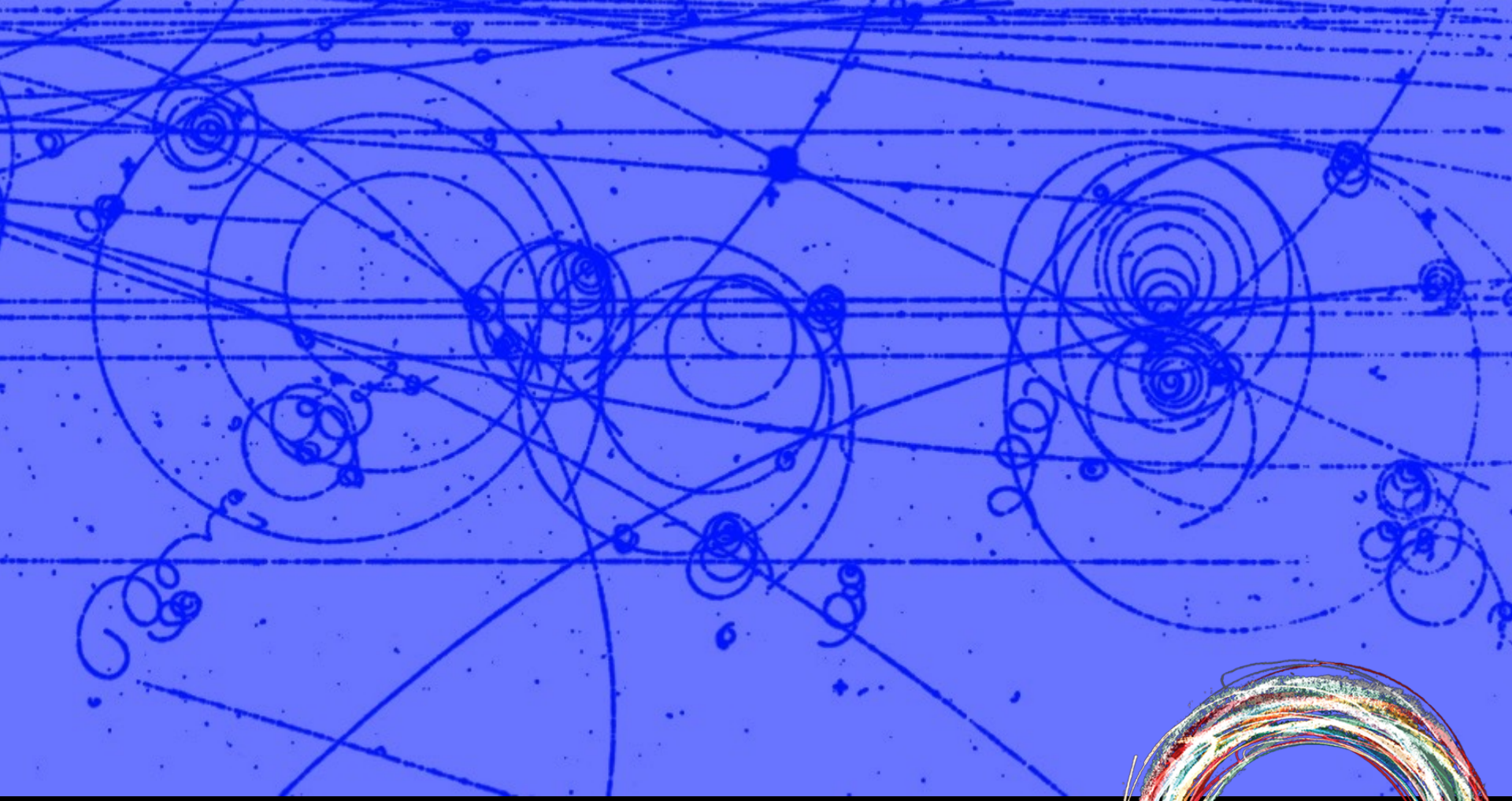
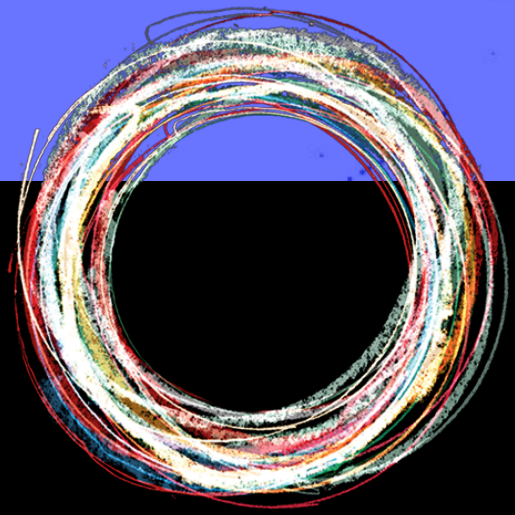


LECTURE 12
QUANTUM ELECTRODYNAMICS



PHY 493/803, 2017

Recap / Up Next

<u>Last time:</u>

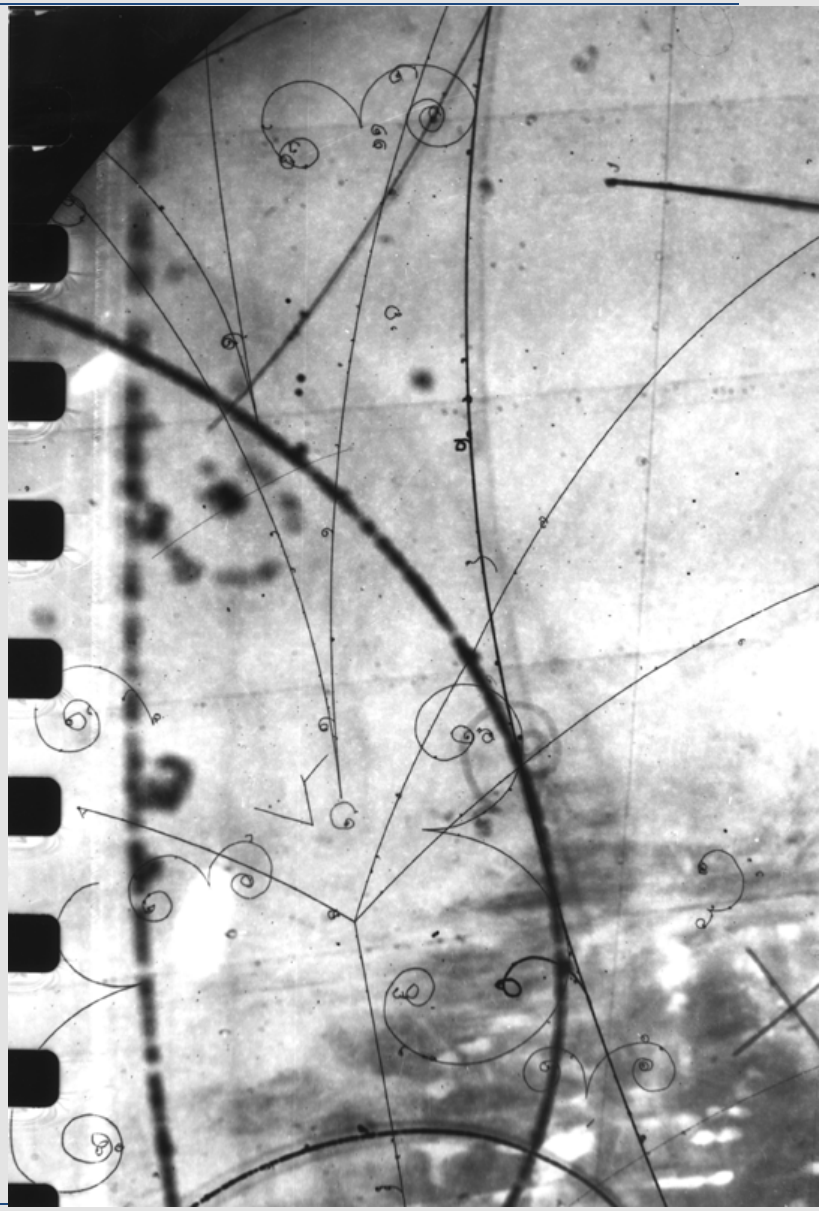
Multicomponent Detectors

- Comprehensive strategies
- Particle Identification
- Triggering

<u>This time:</u>

Quantum Electrodynamics

- The Dirac Equation
- QED Feynman Rules
- Cross Sections
- Renormalization



Upcoming Plans

<u>Chapter 7:</u> Everything

<u>Chapter 8:</u> Secs. 8.1-8.4

<u>Chapter 9:</u> Everything but Sec 9.4

<u>Chapter 10:</u> Useful, but only if you're interested

<u>Chapter 11:</u> Everything

<u>Chapter 12:</u> Only if you're interested

The Schrodinger Equation

Hopefully this is a review for everyone!!

Schrodinger's equation is based on the Hamiltonian, a description of the total energy of a system.

$$(-\frac{\hbar^2}{2m}\nabla^2 + V)\psi = i\hbar\frac{\partial}{\partial t}\psi$$

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V$$

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$$H = -\frac{\hbar^2}{2m}\nabla^2 + V$$

Time-independent solutions

Assume (or know!) that the potential V doesn't depend on time

$$\psi(\vec{r}, t) = \psi(\vec{r})e^{-iEt/\hbar}$$

$$(-\frac{\hbar^2}{2m}\nabla^2 + V)\psi = E\psi$$

$$H\psi = E\psi$$

Deriving Wave Equations

What actually transpired to get us to the Schrodinger Equation?

1) Start with the classical expression for energy conservation

$$\frac{p^2}{2m} + V = E$$

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2) Substitute QM operators

$$\hat{p} = -i\hbar\nabla$$

$$\hat{E} = i\hbar\frac{\partial}{\partial t}$$

$$\Rightarrow \left(-\hbar^2\frac{\nabla^2}{2m} + V\right)\psi = i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi$$

The Schrodinger Equation

1) 1st order in the time domain
2) 2nd order in the spatial domain

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Why are these important?
They imply that solutions to the Schrodinger Eqn satisfy the continuity equation.
IE, probability is conserved!

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1) Multiply SE by conjugate WF:

2) Multiply conjugate SE by WF:

$$\left(-\hbar^2\frac{\nabla^2}{2m} + V\right)\psi\psi^* = i\hbar\frac{\partial\psi}{\partial t}\psi^*$$

$$\psi\left(-\hbar^2\frac{\nabla^2}{2m} + V\right)\psi^* = -i\hbar\frac{\partial\psi^*}{\partial t}\psi$$

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3) Subtract & the result is the continuity equation:

$$\frac{\partial}{\partial t}\rho + \nabla\cdot j = 0$$

$$\rho = |\psi|^2 = \psi^*\,\psi$$

$$j = \frac{\hbar}{2mi}\left(\psi^*(\nabla\psi) - \psi(\nabla\psi^*)\right)$$

The Klein-Gordon Equation

The Schrodinger Equation is a non-relativistic description of the evolution of wave functions

Why can't we just do the same for relativistic particles?

Using the relativistic energy-momentum relationship?

$$E^2 - p^2c^2 = m^2c^4$$

$$\hat{p} = -i\hbar\nabla$$

$$\hat{E} = i\hbar\frac{\partial}{\partial t}$$

Via substitution, we have the Klein-Gordon Equation

$$-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} + \nabla^2\psi = \left(\frac{mc}{\hbar}\right)^2\psi$$

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$$\partial_\mu = \frac{\partial}{\partial x^\mu}$$

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The covariant form of the KG Eqn

$$-\partial_\mu\partial^\mu\psi - m^2 = 0$$

Brief Definitions & Reminders

$$x^\mu = (ct,\ x,\ y,\ z)$$

$$x_\mu = g_{\mu\nu} x^\nu = (ct,\ -x,\ -y,\ -z)$$

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$$\frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t},\ \frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right) \qquad \frac{\partial}{\partial x_\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t},\ -\nabla\right)$$

$$= \left(\frac{1}{c}\frac{\partial}{\partial t},\ \nabla\right)$$

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$$\partial^\mu = \frac{\partial}{\partial x_\mu}$$

$$\partial_\mu \partial^\mu = \partial^\mu \partial_\mu = \frac{1}{c^2}\frac{\partial}{\partial t^2} - \nabla^2 = \Box$$

d'Alembertian = Minkowski Laplace operator

The Klein-Gordon Equation

1) 2nd order in the time domain
2) 2nd order in the spatial domain

Schrodinger also discovered this equation, but discarded it because it didn't obviously satisfy the continuity equation.

Later on Pauli & Weisskopf reformulated a relativistic, quantum continuity equation, so it turns out to be just fine. Oops!

In Schrodinger's defense, the KG Eqn fails to reproduce the Hydrogen spectra. That's because KG only works for spin-0 particles…not spin 1/2 electrons!

$$-\partial_\mu \partial^\mu \psi - m^2 = 0$$

$$-\frac{\partial \psi}{\partial t^2} + \nabla^2 \psi = m^2 \psi$$

$$(\Box + m^2)\psi = 0$$

The Klein-Gordon Equation

The Klein Gordon Equation accepts plane wave solutions

We now use our 4-vectors to describe the solutions

$$\psi(x,t) = \mathrm{e}^{ip_\mu x^\mu} = \mathrm{e}^{iEt + ip\cdot x}$$

$$p_\mu x^\mu = \hbar(k \cdot x - \omega t)$$

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$$-E^2 + p^2 + m^2 = 0 \quad \Rightarrow \quad E = \pm\sqrt{p^2 + m^2}$$

What do we do with the negative energy solutions?

Cannot just throw them away!

More Problems

As we discussed, the KG Eqn does indeed satisfy the continuity equation

We now use our 4-vectors to describe the solutions

Continuity Equation:

$$\frac{\partial}{\partial t}\rho + \nabla \cdot j = 0$$

Probability density from Schrodinger Equation:

$$\rho = |\psi|^2 = \psi^* \psi$$

Probability density from KG Equation:

$$\rho = i\left[\psi^* \frac{\partial \psi}{\partial t} - \left(\frac{\partial \psi^*}{\partial t}\right)\psi\right]$$

This probability term is not constant!

Nor is it even necessarily positive. Hmmm.

Solutions?

The second-order time derivative in the □ of the KG equation is responsible for both the negative-energy plane wave solutions and the misbehaving probability density.

Dirac tried to fix this problem by looking for a relativistic equation that, like the Schrodinger equation, only contained first-order time derivatives

$$\rho = i\left[\psi^* \frac{\partial \psi}{\partial t} - \left(\frac{\partial \psi^*}{\partial t}\right)\psi\right]$$

$$E = \pm\sqrt{p^2 + m^2}$$

Towards a Solution

Dirac noticed that if you make the simplifying assumption that there is zero momentum, then the energy-momentum relation factorizes!

Even better, the factored equations are linear with energy.

$$E^2 - p^2 = m^2$$

Set 3-momentum to zero:

$$(p^0)^2 - m^2 = 0$$

$$(p^0 - m)(p^0 + m) = 0$$

$$\Rightarrow p^0 = \pm m$$

Either linear equation leads to a time-space equation that is first-order in time and satisfies the relativistic energy-momentum relation.

Getting closer….

But what about the general case with non-zero momentum: moving particles!

Not so trivial. Need to solve for a linear energy/momentum relationship.

$$(\vec{\alpha} \cdot \vec{p} + \beta\, m)\psi = i\frac{\partial \psi}{\partial t}$$

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$$(\vec{\alpha}\cdot\vec{p}+\beta\, m)\psi = i\frac{\partial\psi}{\partial t}$$

$$\gamma^0=\beta,\ \gamma^1=\beta\alpha^1,\ \gamma^2=\beta\alpha^2,\cdots$$

This gets us to a general solution, now we just need to understand these gamma factors.

$$i\left(\gamma^0\frac{\partial}{\partial t}+\vec{\gamma}\cdot\nabla\right)\psi = m\psi$$

First Look at Gamma Matrices

Multiplying the Dirac equation by its complex conjugate must give KG:

$$\left(-i\gamma^0\frac{\partial}{\partial t} - i\vec{\gamma}\cdot\nabla - m\right)\left(i\gamma^0\frac{\partial}{\partial t} + i\vec{\gamma}\cdot\nabla - m\right) = 0$$

Klein-Gordon Equation:

$$-\partial_\mu\partial^\mu\psi - m^2 = 0$$

$$-\frac{\partial\psi}{\partial t^2} + \nabla^2\psi = m^2\psi$$

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This leads to a set of conditions on the four coefficients

(1) $(\gamma^0)^2 = 1$ (2) $(\gamma^{1,2,3})^2 = -1$

(3) $\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0$ For $\mu \neq \nu$

The γ^μ are unitary and anti-commute.

How about $\gamma^0=1$ and $\gamma^{1,2,3}=i$?? Not so simple!

Our anti-commutator equation cannot be solved by any set of complex numbers! More on this later.

The Dirac Equation

Having successfully factored the relativistic energy-momentum relation, we can set either factor to zero.

Remember, we're technically still working with the KG equation!

$$(p^\mu p_\mu - m^2) = (\gamma^k p_k + m)(\gamma^\lambda p_\lambda - m) = 0$$

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$$i\left(\gamma^0 \frac{\partial}{\partial t} + \vec{\gamma} \cdot \nabla\right)\psi = m\psi$$

$$i\gamma^\mu \partial_\mu \psi = m\psi$$

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

Slash Notation

When we contract γ^μ with a four-vector q_μ, we can abbreviate this using the Feynman slash notation

$$\gamma^\mu q_\mu = \not{q}$$

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With the slash notation, the Dirac equation becomes

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

$$\not{p} - m = 0$$

$$(i\not{\partial} - m)\psi = 0$$

Back to Gamma Matrices

We had 3 conditions, including an anti-commutation relation.

Can't do this with numbers since they commute (AB=BA always) but we can do it with matrices (which do not, in general, commute).

$$(\gamma^0)^2 = 1 \qquad (\gamma^{1,2,3})^2 = -1 \qquad \{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0$$ For μ≠ν

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Dirac's clever idea was to let γ represent a set of 4x4 matrices

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Back to Gamma Matrices

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

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Back to Gamma Matrices

The "Bjorken & Drell" convention is often used to reduce the notation. Clearly the physics is independent of the representation.

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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We'll use the anti-commutation relation later, so it's worthwhile noting a feature of it:

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0 \quad \text{For } \mu \neq \nu$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

Spin-½ Representations

FLASHBACK!! CHAPTER 4

Now that we're using a vector wavefunction, we must have a different form for spin observable operators

The operator must transform a vector into a vector

The obvious (only?) choice is a matrix

We're now using vectors as spin-½ representations

$$\chi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Spin vectors transform via the 2-D representation of the SU(2) group

The Lie algebra is spanned by 3, 2×2 Hermitian, unitary, complex matrices.

$$\hat{S}_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{S}_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \hat{S}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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Or, more compactly, as the Pauli spin matrices:

$$\hat{\mathbf{S}} = \frac{\hbar}{2} \left(\sigma_x \; \sigma_y \; \sigma_z \right) \longrightarrow \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Dirac Spinors

We have 4x4 matrices operating on our 4-momentum vectors.

Thus our wave functions must be 4-component column vectors!

Bi-spinors, Dirac spinors, or just "spinors"

$$(i\not{\partial} - m)\psi = 0$$

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

It is important to note that while the Dirac spinor ψ is a four component object, is it NOT a four-vector. It's transformation properties are important however, and we will discuss these later on.

Dirac Spinors

We'll frequently want to deal with our spinors as two components

These "look like"…but aren't truly…spin-up and spin-down states

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \qquad \psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$$

$$\psi_A = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \qquad \psi_B = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}$$

The Dirac Equation

Don't let all this shorthand fool you!

We've made our lives simpler by contracting the terms we must write, but at the cost of potential obfuscation.

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The Dirac equation in it's full glory

$$\begin{pmatrix} i\frac{\partial}{\partial t} - m & 0 & i\frac{\partial}{\partial z} & i\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \\ 0 & i\frac{\partial}{\partial t} - m & i\frac{\partial}{\partial x} - \frac{\partial}{\partial y} & -i\frac{\partial}{\partial z} \\ -i\frac{\partial}{\partial z} & -i\frac{\partial}{\partial x} - \frac{\partial}{\partial y} & -i\frac{\partial}{\partial t} - m & 0 \\ -i\frac{\partial}{\partial x} + \frac{\partial}{\partial y} & i\frac{\partial}{\partial z} & 0 & -i\frac{\partial}{\partial t} - m \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \\ \psi^4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Solutions of the Dirac Equation

Solutions to the Dirac equation have two parts:

1) A phase term that describes the characteristic time evolution of the quantum state of energy E and momentum p:

$$\psi \propto \mathrm{e}^{-ip_\mu x^\mu}$$

2) A Dirac spinor term, which will also be a function of the particle's 4-momentum:

$$\psi \propto u(p^\mu)$$

$$\psi = u(p^\mu)\,\mathrm{e}^{-ip_\mu x^\mu}$$

Solutions of the Dirac Equation

Let's make a simplifying assumption as a first approach to the solutions:

Assume the particle is at rest: p=0

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \quad \Rightarrow \quad (i\gamma^0 \partial_0 - m)\psi = 0$$

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$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \quad \Rightarrow \quad (i\gamma^0 \partial_0 - m)\psi = 0$$

$$\begin{pmatrix} i\frac{\partial}{\partial t} - m & 0 & 0 & 0 \\ 0 & i\frac{\partial}{\partial t} - m & 0 & 0 \\ 0 & 0 & -i\frac{\partial}{\partial t} - m & 0 \\ 0 & 0 & 0 & -i\frac{\partial}{\partial t} - m \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \\ \psi^4 \end{pmatrix} = 0$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{\partial \psi_A}{\partial t} \\ \frac{\partial \psi_B}{\partial t} \end{pmatrix} = -im \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$$

Solutions of the Dirac Equation

This leads to the pair of equations, with their associated solutions.

$$\frac{\partial \psi_A}{\partial t} = -im\psi_A$$

$$\psi_A(t) = \psi_A(0)e^{-imt}$$

$$\frac{\partial \psi_B}{\partial t} = +im\psi_B$$

$$\psi_B(t) = \psi_B(0)e^{+imt}$$

Recall that for the Schrodinger equation, the characteristic time dependence of the solutions goes like exp(-iEt).

Evidently, ψ_A is a solution with energy E=+m, as we should expect, but ψ_B seems to have a negative energy E=-m.

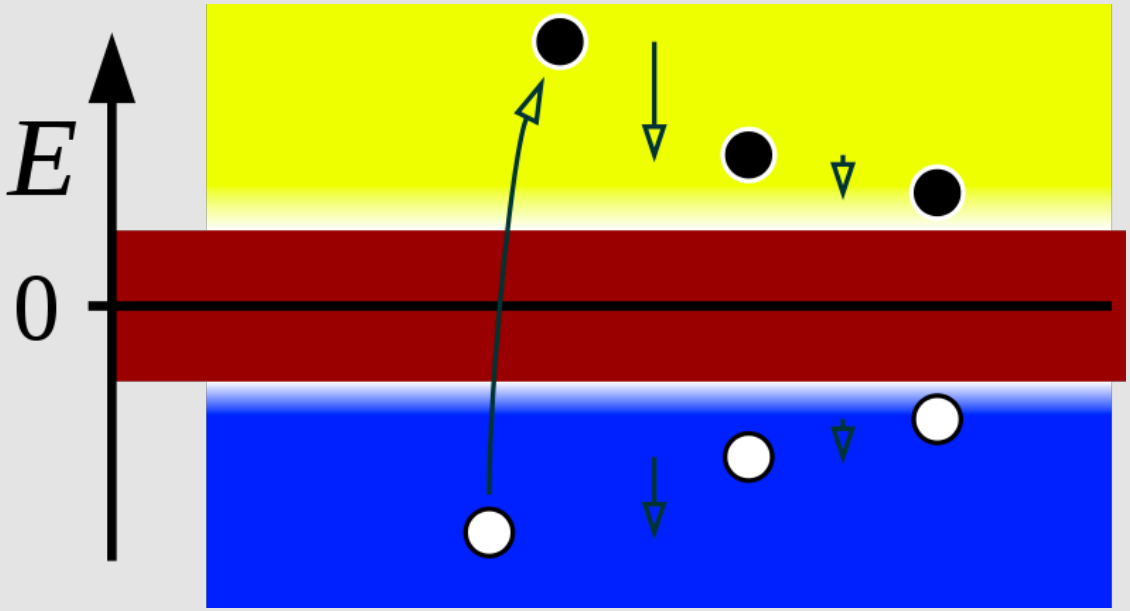
Dirac had hoped that an equation that is first order in time would avoid these negative energy solutions.

Back to the Negative Energy Problem

- Seeing as we seem to be stuck with the negative energy solutions, Dirac's next suggestion was that all possible negative energy states were already filled by a Dirac sea of particles. The Pauli Exclusion Principle would then leave only the positive energy states available.

- The excitation of a sea electron would leave a hole which would behave like a positive energy particle with a positive charge. Eventually, Dirac worked up the courage to predict the existence of the positron.

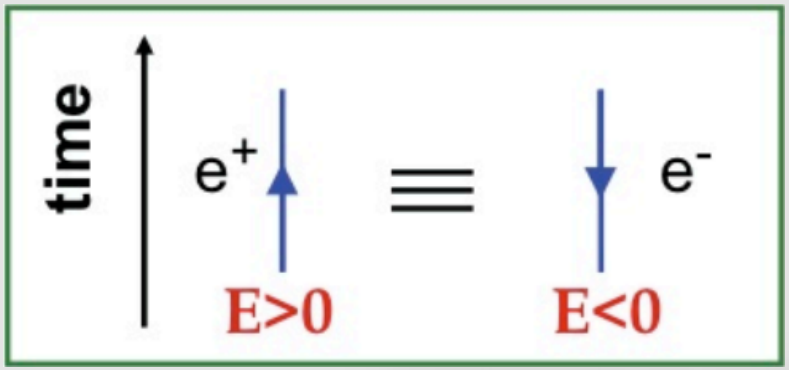
- Experimentalists had secretly been observing evidence for antimatter for years, but had always discarded these unphysical particles. The positron was quickly "discovered".

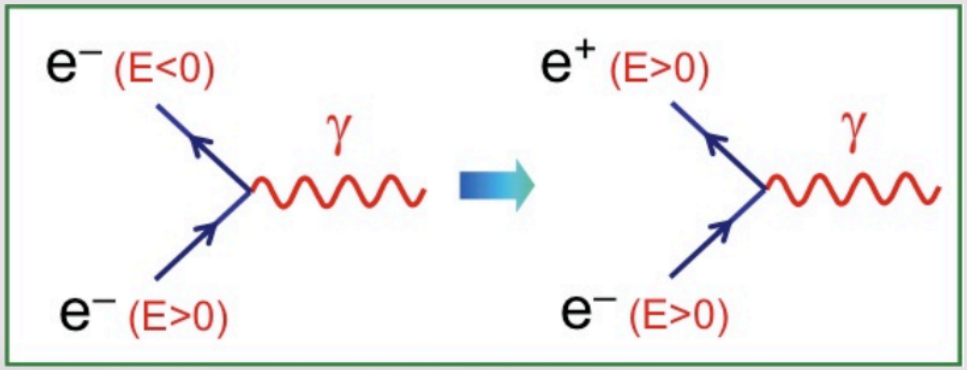


Back to the Negative Energy Problem

- We can't escape negative energy solutions. How should we interpret them?

- Modern Feynman-Stückelberg Interpretation:
  A negative energy solution is a negative energy particle which propagates backwards in time or equivalently a positive energy anti-particle which propagates forwards in time.



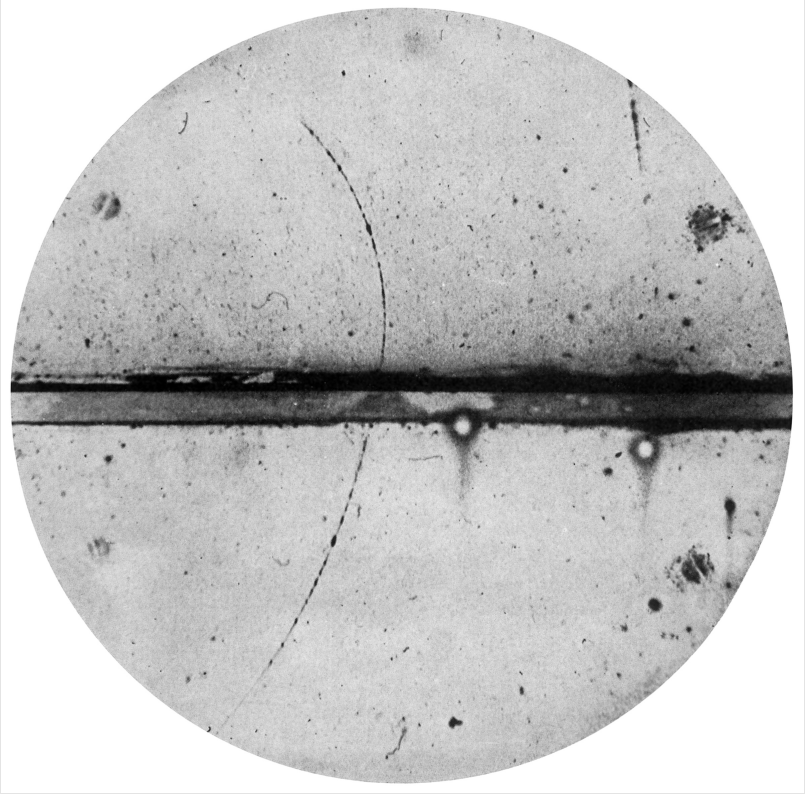


$$e^{-i(-E)(-t)} \rightarrow e^{-i(Et)}$$

This is why in Feynman diagrams the backwards pointing lines represent anti-particles.

Discovery of the Positron

C.D. Anderson, Phys Rev 43 (1933) 491



Left: Carl Anderson & his magnet-surrounded bubble chamber
Right: Positron track in bubble chamber, curving due to magnetic field

Discovery of the Positron

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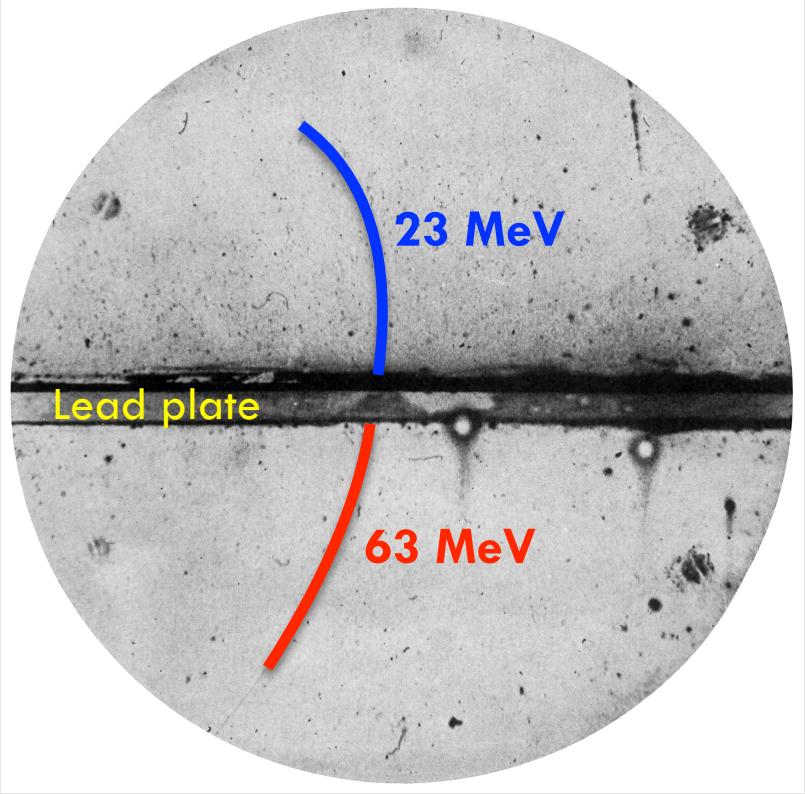

- Positron slows as it traverses the lead plate. Why??
- Track curvature in B-field shows it's a positive particle.
- Cannot be a proton, as a proton would have stopped in the lead.

Solutions of the Dirac Equation

With the interpretation of negative energy states as positive energy antiparticles, we can see how the Dirac equation has four independent solutions for a particle at rest:

$$(e^- \uparrow) \quad \psi^{(1)} = e^{-imt} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \psi^{(2)} = e^{-imt} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad (e^- \downarrow)$$

$$(e^+ \downarrow) \quad \psi^{(3)} = e^{+imt} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \psi^{(4)} = e^{+imt} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad (e^+ \uparrow)$$

We can think of these as spin up/down states of electrons (ψ_A) and positrons (ψ_B)

Solutions of the Dirac Equation

Next we will look for full, plane-wave solutions to the Dirac equation

$$\psi = N\, u(p^\mu)\, \mathrm{e}^{-ip_\mu x^\mu}$$

Solutions of the Dirac Equation

Next we will look for full, plane-wave solutions to the Dirac equation

$$\psi = N\, u(p^\mu)\, \mathrm{e}^{-ip_\mu x^\mu}$$

$u(p^\mu)$ doesn't depend on time and it must satisfy the momentum-space Dirac equation:

$$\begin{aligned}
(i\partial\!\!\!/_\mu - m)u\mathrm{e}^{-ip\cdot x} &= 0 \\
(\,p\!\!\!/_\mu u\mathrm{e}^{-ip\cdot x} - mu\mathrm{e}^{-ip\cdot x}) &= 0 \\
(\gamma^\mu p_\mu - m)u &= 0
\end{aligned}$$

This is purely algebraic (it contains no derivatives) and represents a set of four coupled equations.

Solutions of the Dirac Equation

$$
\begin{aligned}
\not{p} &= \gamma^0 p^0 - \gamma \cdot \mathbf{p} \\
&= E \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \mathbf{p} \cdot \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix} \\
&= \begin{pmatrix} E & -\mathbf{p} \cdot \sigma \\ \mathbf{p} \cdot \sigma & -E \end{pmatrix}
\end{aligned}
$$

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\end{aligned}
$$

$$
\begin{aligned}
(\not{p} - m)u &= \begin{pmatrix} E - m & -\mathbf{p} \cdot \sigma \\ \mathbf{p} \cdot \sigma & -E - m \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} \\
&= \begin{pmatrix} (E - m)u_A - (\mathbf{p} \cdot \sigma)u_B \\ (\mathbf{p} \cdot \sigma)u_A - (E + m)u_B \end{pmatrix}
\end{aligned}
$$

Solutions of the Dirac Equation

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\end{aligned}
$$

The Dirac equation then gives us a pair of coupled equations for U_A and U_B:

$$
u_A = \frac{(\mathbf{p}\cdot\sigma)}{E - m} u_B \qquad u_B = \frac{(\mathbf{p}\cdot\sigma)}{E + m} u_A
$$

These equations can easily be solved by substituting one into the other and noting that

$$
(\mathbf{p}\cdot\sigma)^2 = \mathbf{p}^2 \cdot 1
$$

Solutions of the Dirac Equation

$$u_A = \frac{(\mathbf{p}\cdot\sigma)}{E-m}u_B \qquad u_B = \frac{(\mathbf{p}\cdot\sigma)}{E+m}u_A$$

Substituting the second equation into the first, we have

$$u_A = \frac{\mathbf{p}^2}{E^2-m^2}u_A$$

This requires (E^2 - m^2 = p^2), just as we should expect. The same thing happens with U_B. Either way, we have two solutions for E:

$$E = \pm\sqrt{\mathbf{p}^2+m^2}$$

Solutions of the Dirac Equation

By picking specific forms for U_A or U_B (remember, one is fixed by the other), we can construct a set of four solutions to the Dirac equation for a moving particle.

$$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \Rightarrow \quad u_B = \frac{p \cdot \sigma}{E+m} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{E+m} \begin{pmatrix} p_z \\ p_x + ip_y \end{pmatrix}$$

$$u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \Rightarrow \quad u_B = \frac{p \cdot \sigma}{E+m} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{E+m} \begin{pmatrix} p_x - ip_y \\ -p_z \end{pmatrix}$$

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$$E = +\sqrt{p^2 + m^2} \qquad E = -\sqrt{p^2 + m^2}$$

Solutions of the Dirac Equation

Splitting U_A and U_B each into their 2-part pieces, we have two particle solutions (keep calling these U) and two anti-particle solutions (call these V):

$$u^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \end{pmatrix} \qquad u^{(2)} = N \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

$$(\not{p} - m)u = 0$$

$$E = +\sqrt{p^2 + m^2}$$

$$v^{(1)} = N \begin{pmatrix} \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} \qquad v^{(2)} = -N \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}$$

$$(\not{p} + m)v = 0$$

$$E = -\sqrt{p^2 + m^2}$$

Normalizing Solutions

The normalization constant N is easily calculated once the total wave function is normalized. The standard convention is 2E:

$$u^\dagger u = 2E$$

This convention comes about by converting from the "classical" unit probability calculation to the Lorentz-Invariant formulation

$$\int \psi^\dagger \psi \, dV = 1 \quad \Rightarrow \quad \int \psi^\dagger \psi \, d^4x = 2E$$

With this convention, we trivially find N (with some algebra):

$$N = \sqrt{E + m}$$

Spin Eigenvectors?

We have to be VERY careful when we interpret the spin states of our general, plane-wave.

Our zero momentum solutions were spin-up and spin-down for positrons and electrons.

But with non-zero momentum, things are not so simple anymore!

But we can generalize the Pauli spin matrices to the 4x4 matrices required for Dirac spinors:

$$\mathbf{S} = \frac{\hbar}{2}\Sigma \qquad \Sigma \equiv \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$$

If (and only if) the particles are traveling along the z-axis, the plane-wave solutions U and V will be eigenstates of S_z.

U^1 and V^1 are spin up, while U^2 and V^2 are spin down.

See the homework!

Lorentz Transformations of Spinors

Spinors are not four-vectors, therefore they do not transform via Λ. How do they transform?

$$\psi'(x') = S\,\psi(x)$$

Not the same S as last slide!!!

The transformation is defined as:

$$S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu\,\gamma^\nu$$

$$\frac{\partial_\mu}{\partial x^\mu} = \Lambda^\nu{}_\mu\,\frac{\partial_\mu}{\partial x^\nu}$$

For motion along the x-axis:

$$S = \begin{pmatrix} a_+ & a_-\sigma_1 \\ a_-\sigma_1 & a_+ \end{pmatrix}$$

$$a_\pm = \pm\sqrt{(\gamma \pm 1)/2}$$

$$\gamma = (1 - v^2)^{-1/2}$$

Scalar Quantities from Spinors

Consider

$$\psi^\dagger \psi = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2$$

Under a Lorentz transformation,

$$\begin{aligned} \psi^\dagger \psi \quad &\rightarrow \quad (S\psi)^\dagger (S\psi) \\ &\rightarrow \quad \psi^\dagger (S^\dagger S)\psi \end{aligned}$$

Since $S^\dagger S \neq 1$ (check for yourself using the explicit representation of S on the previous page), $\psi^\dagger \psi$ is not a Lorentz scalar.

The Adjoint Spinor: Towards Fermion Currents

16 Lorentz invariant quantities can be defined from spinors.
Each describes a different kind of fermion currents (fermion lines of Feynman diagrams)

Just as four-vector contractions need a few well-placed minus signs (i.e., $g^{\mu\nu}$) in order to make a scalar, we can add a couple of minus signs to a spinor by defining the ***adjoint spinor***:

$$\bar{\psi} \equiv \psi^\dagger \gamma^0 = \begin{pmatrix} \psi_1^* & \psi_2^* & -\psi_3^* & -\psi_4^* \end{pmatrix}$$

Since $S^\dagger \gamma^0 S = \gamma^0$ (again, check this yourself),

$$\bar{\psi}\psi = |\psi_1|^2 + |\psi_2|^2 - |\psi_3|^2 - |\psi_4|^2$$

is a Lorentz scalar

γ^5: The Black Sheep of the Family

Define an additional γ-matrix by

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$$

In the Bjorken and Drell representation:

$$\gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

<u>Note:</u>
$(\gamma^5)^2=1$ and anti-commutes with every other γ:

$$\{\gamma^\mu, \gamma^5\} = 0 \qquad \Rightarrow \qquad \gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu$$

Two Scalars?

We have already seen how $\bar{\psi}\psi$ is a Lorentz scalar

Since $S^\dagger \gamma^0 \gamma^5 S = \gamma^0 \gamma^5$ (check this too),

$$\bar{\psi}\gamma^5\psi$$

is also a Lorentz scalar.

This gives us two Lorentz scalars. What's the difference??

Parity?

Under a parity transformation we have:

$$\psi \to \gamma^0 \psi$$

Thus, since:

$$\begin{aligned}
\bar{\psi}\psi &\to (P\psi)^\dagger \gamma^0 (P\psi) \\
&\to \psi^\dagger (\gamma^0)^\dagger \gamma^0 \gamma^0 \psi \\
&\to \psi^\dagger (\gamma^0)^\dagger \psi \\
&\to \bar{\psi}\psi
\end{aligned}$$

$$\begin{aligned}
\bar{\psi}\gamma^5\psi &\to (P\psi)^\dagger \gamma^0 \gamma^5 (P\psi) \\
&\to \psi^\dagger (\gamma^0)^\dagger \gamma^0 \gamma^5 \gamma^0 \psi \\
&\to -\psi^\dagger (\gamma^0)^\dagger \gamma^5 \psi \\
&\to -\bar{\psi}\gamma^5\psi
\end{aligned}$$

True scalar: $\bar{\psi}\psi$

Pseudo-scalar: $\bar{\psi}\gamma^5\psi$

Bilinear Covariants

There are 16 possible products of the form $\psi^*_i\psi_j$. These 16 products can be grouped together into bilinear covariants:

$\bar{\psi}\psi$	Scalar	1 component
$\bar{\psi}\gamma^5\psi$	Pseudoscalar	1 component
$\bar{\psi}\gamma^\mu\psi$	Vector	4 components
$\bar{\psi}\gamma^\mu\gamma^5\psi$	Pseudovector	4 components
$\bar{\psi}\sigma^{\mu\nu}\psi$	Antisymmetric tensor	6 components

Note that:

$$\sigma^{\mu\nu} \equiv \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right]$$

Bilinear Covariants: Why??

We now have a simple basis set { 1, γ^μ, γ^5, $\gamma^\mu\gamma^5$, $\sigma^{\mu\nu}$ } for any 4x4 matrix, therefore we can always simplify more complicated combinations of γ matrices.

The tensorial and parity character of each bilinear is evident. This makes it easy to see why the QED interaction Lagrangian

$$-eA_\mu\bar{\psi}\gamma^\mu\psi$$

leads to a parity-conserving electromagnetic force mediated by a vector (spin-1) boson.

To describe the parity-violating weak interaction, we could (and will!) mix vector and axial interactions.

$$(\bar{\psi}\gamma^\mu\psi) \quad \pm \quad (\bar{\psi}\gamma^\mu\gamma^5\psi)$$

EM & The Photon

Warning: The next few slides make comparisons between the classical EM formulation and the QM formulation. Do not get confused by trying to connect directly between them. We'll discuss more in class, but always remember that the classical formulation is merely our observation when the QM effects are small.

EM & The Photon

We start discussing the photon in the context of Maxwell's equations.

"homogeneous" equations

"inhomogeneous" equations

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial B}{\partial t} = 0$$

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho$$

$$\vec{\nabla} \times \vec{B} - \frac{1}{c}\frac{\partial E}{\partial t} = \frac{4\pi}{c}\vec{J}.$$

EM & The Photon

The relativistic formulation of Maxwell's equations takes advantage of our 4-vector and tensor notation:

Field Strength Tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$\begin{aligned} \vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial B}{\partial t} &= 0 \\ \vec{\nabla} \cdot \vec{E} &= 4\pi\rho \\ \vec{\nabla} \times \vec{B} - \frac{1}{c}\frac{\partial E}{\partial t} &= \frac{4\pi}{c}\vec{J}. \end{aligned}$$

Density 4-Vector
or EM 4-Current

$$J^{\mu} = (c\rho, \vec{J})$$

EM & The Photon

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$$\begin{aligned} \vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial B}{\partial t} &= 0 \\ \vec{\nabla} \cdot \vec{E} &= 4\pi\rho \\ \vec{\nabla} \times \vec{B} - \frac{1}{c}\frac{\partial E}{\partial t} &= \frac{4\pi}{c}\vec{J}. \end{aligned}$$

Density 4-Vector
or EM 4-Current

$$J^\mu = (c\rho, \vec{J})$$

Inhomogeneous Equations

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu$$

EM & The Photon

The homogenous equations can also be put in the relativistic format, but we have to make some observations first.

The result is that we see the E and B fields arise from a scalar and a vector potential.

$$
\begin{aligned}
\textbf{(1)} \quad \vec{\nabla} \cdot \vec{B} &= 0 \\
\textbf{(2)} \quad \vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial B}{\partial t} &= 0
\end{aligned}
$$

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(1)⇒(2)

$$\nabla \times \left(E + \frac{1}{c}\frac{\partial A}{\partial t} \right) = 0$$

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(1)⇒(2)

$$\nabla \times \left(E + \frac{1}{c}\frac{\partial A}{\partial t} \right) = 0 \quad \Longrightarrow \quad E = -\nabla V - \frac{1}{c}\frac{\partial A}{\partial t}$$

EM & The Photon

By defining a 4-vector potential, we can reformulate the classical EM theory in terms of the evolution of fields that satisfy a wave equation.

$$\begin{aligned} \vec{\nabla}\cdot\vec{B} &= 0 \\ \vec{\nabla}\times\vec{E}+\frac{1}{c}\frac{\partial B}{\partial t} &= 0 \\ \vec{\nabla}\cdot\vec{E} &= 4\pi\rho \\ \vec{\nabla}\times\vec{B}-\frac{1}{c}\frac{\partial E}{\partial t} &= \frac{4\pi}{c}\vec{J}. \end{aligned}$$

4-Vector Potential

$$A^\mu = (V, \vec{A})$$

$$\Downarrow$$

Re-write the Field Strength Tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

The inhomogeneous equations take on an interesting form. Note the return of the d'Alembertian from the KG equation.

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= \partial_\mu\partial^\mu A^\nu - \partial^\nu(\partial_\mu A^\mu) = \frac{4\pi}{c}J^\nu \\ &= \Box A^\nu - \partial^\nu(\partial_\mu A^\mu) \end{aligned}$$

EM & The Photon

The only problem with this formulation is that the 4-vector potential is not unique for a given set of E and B fields.

If we want to use the potentials to describe something physical, we'll have to confront this uniqueness issue.

Consider a gauge transformation of the 4-vector potential:

$$A'_\mu = A_\mu + \partial_\mu \lambda$$

The field strength tensor is unchanged by this transformation, leaving Maxwell's equations unchanged.

$$\partial^\mu A^{\nu\prime} - \partial^\nu A^{\mu\prime} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

A not completely arbitrary choice to reduce the ambiguity is the Lorentz condition:

$$\partial_\mu A^\mu = 0 \implies \Box A^\mu = \frac{4\pi}{c} J^\mu$$

EM & The Photon

Even with the Lorentz condition, we can make further gauge transformations of the form $A^\mu \rightarrow A^\mu + \partial^\mu \lambda$

without disturbing Maxwell's equations provided that: $\Box \lambda = 0$

To solve this second issue, we impose one more constraint:

In empty space (no charge or current!) there is zero scalar potential (V)

$$A^0 = 0$$ and thus: $$\nabla \cdot A = 0$$

The free-space 4-vector potentials satisfy the KG equation for a massless particle

We can thus find associated plane-wave solutions

$$\Box A^\mu = 0 \quad \Longrightarrow \quad A^\mu(x) = a\, \mathrm{e}^{-ip^\mu x_\mu}\, \epsilon^\mu(p^\mu)$$

polarization vector

EM & The Photon

Plugging these 4-vector field solutions back into the massless-KG equation yield some natural constraints:

$$A^\mu(x) = a\,\mathrm{e}^{-ip^\mu x_\mu}\,\epsilon^\mu(p^\mu)$$

Zero mass constraint

$$p_\mu p^\mu = 0$$

Lorentz condition

$$p^\mu \epsilon_\mu = 0$$

Coulomb gauge

$$\epsilon^0 = 0$$

Lorentz + Coulomb

$$\vec{\epsilon}\cdot\vec{p} = 0$$

The combination of the Lorentz condition and Coulomb gauge leads us to understand that the polarization of the 4-potential is perpendicular to the momentum: transverse!

Example: motion along the z-axis allows:

$$\epsilon^{(1)} = (1,0,0)$$

$$\epsilon^{(2)} = (0,1,0)$$

Coulomb gauge ate one polarization DOF, but why aren't there three??

Because these are massless solutions to the KG equation, we cannot polarize along the direction of motion!

EM & The Photon

So where is the photon??

The free-space 4-vector potentials satisfy the KG equation for a massless particle

We can thus find associated plane-wave solutions

$$\Box A^\mu = 0 \quad \Longrightarrow \quad A^\mu(x) = a\,\mathrm{e}^{-ip^\mu x_\mu}\,\epsilon^\mu(p^\mu)$$

In Quantum Electrodynamics (QED), we recognize that the plane-wave solution for a massless particle (from the KG equation) matches exactly with the solutions of Maxwell's equations for the 4-vector potential.

We have a duality between particle and wave descriptions of EM.

All that remains is to demonstrate that we can build a consistent QM description of the photon-fermion interactions!

Summary Up to this Point

	Electrons	Positrons
Spinors satisfy the Dirac Equation:	$(\gamma^\mu p_\mu - m)u = 0$	$(\gamma^\mu p_\mu + m)v = 0$
Adjoint spinors satisfy:	$\bar{u} = u^\dagger \gamma^0$	$\bar{v} = v^\dagger \gamma^0$
Adjoints satisfy the Dirac Eqn:	$\bar{u}(\gamma^\mu p_\mu - m) = 0$	$\bar{v}(\gamma^\mu p_\mu + m) = 0$
Orthogonality:	$\bar{u}^{(1)} u^{(2)} = 0$	$\bar{v}^{(1)} v^{(2)} = 0$
Normalization:	$\bar{u}u = 2m$	$\bar{v}v = -2m$
Completeness:	$\sum_{s=1,2} u^{(s)} \bar{u}^{(s)} = (\gamma^\mu p_\mu + m)$	$\sum_{s=1,2} v^{(s)} \bar{v}^{(s)} = (\gamma^\mu p_\mu - m)$

Summary Up to this Point

Photons

Lorentz condition:

$$p^\mu \epsilon_\mu = 0$$

Orthogonality:

$$\epsilon^{\mu *}_{(1)} \epsilon_{\mu(2)} = 0$$

Normalization:

$$\epsilon^{\mu *} \epsilon_\mu = -1$$

Coulomb gauge:

$$\epsilon^0 = 0, \quad \vec{\epsilon} \cdot \vec{p} = 0$$

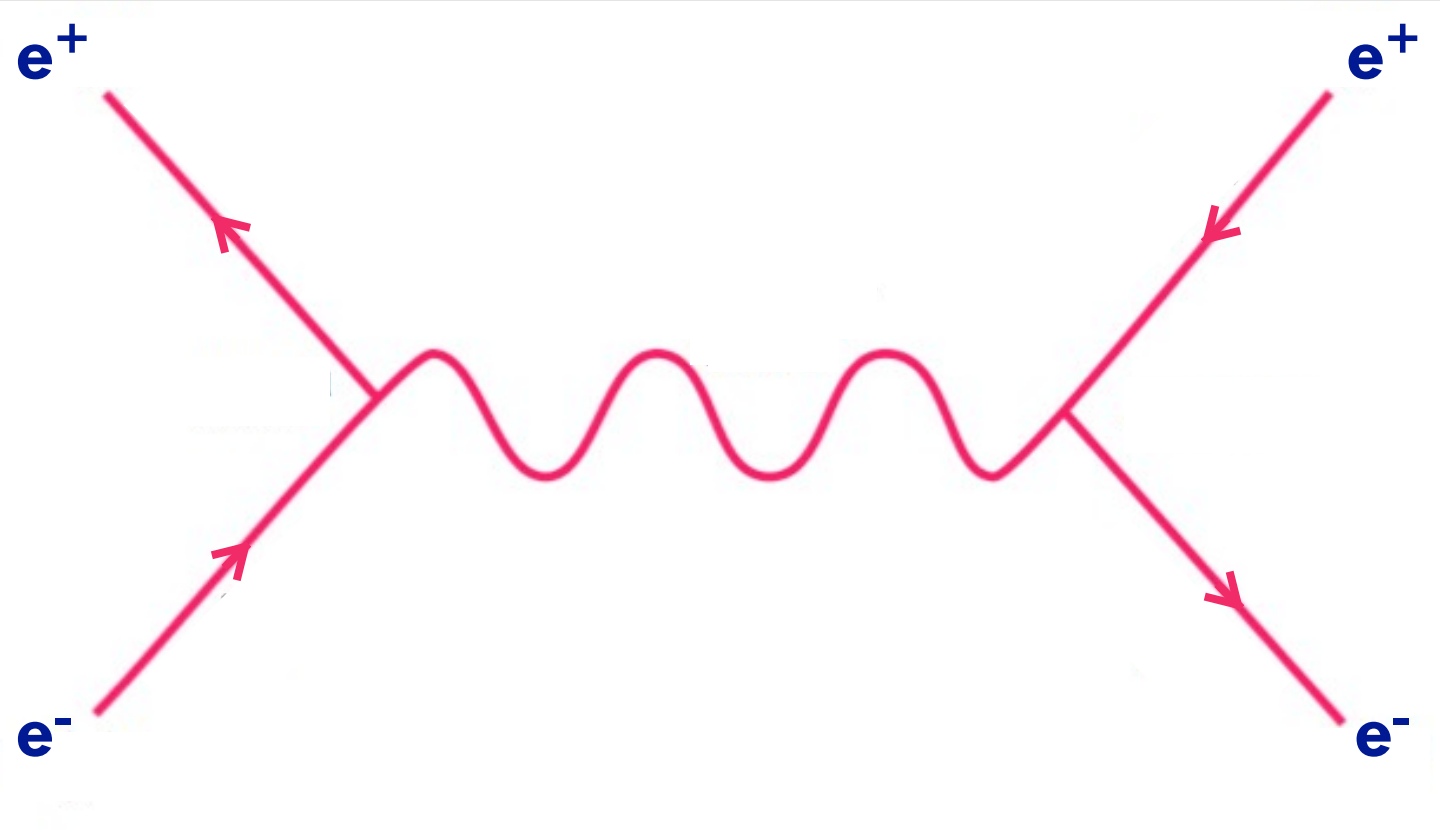
Completeness:

$$\sum_{s=1,2} \epsilon_i^{(s)} \epsilon_j^{(s)*} = \delta_{ij} - \hat{p}_i \hat{p}_j$$

Feynman Rules for QED

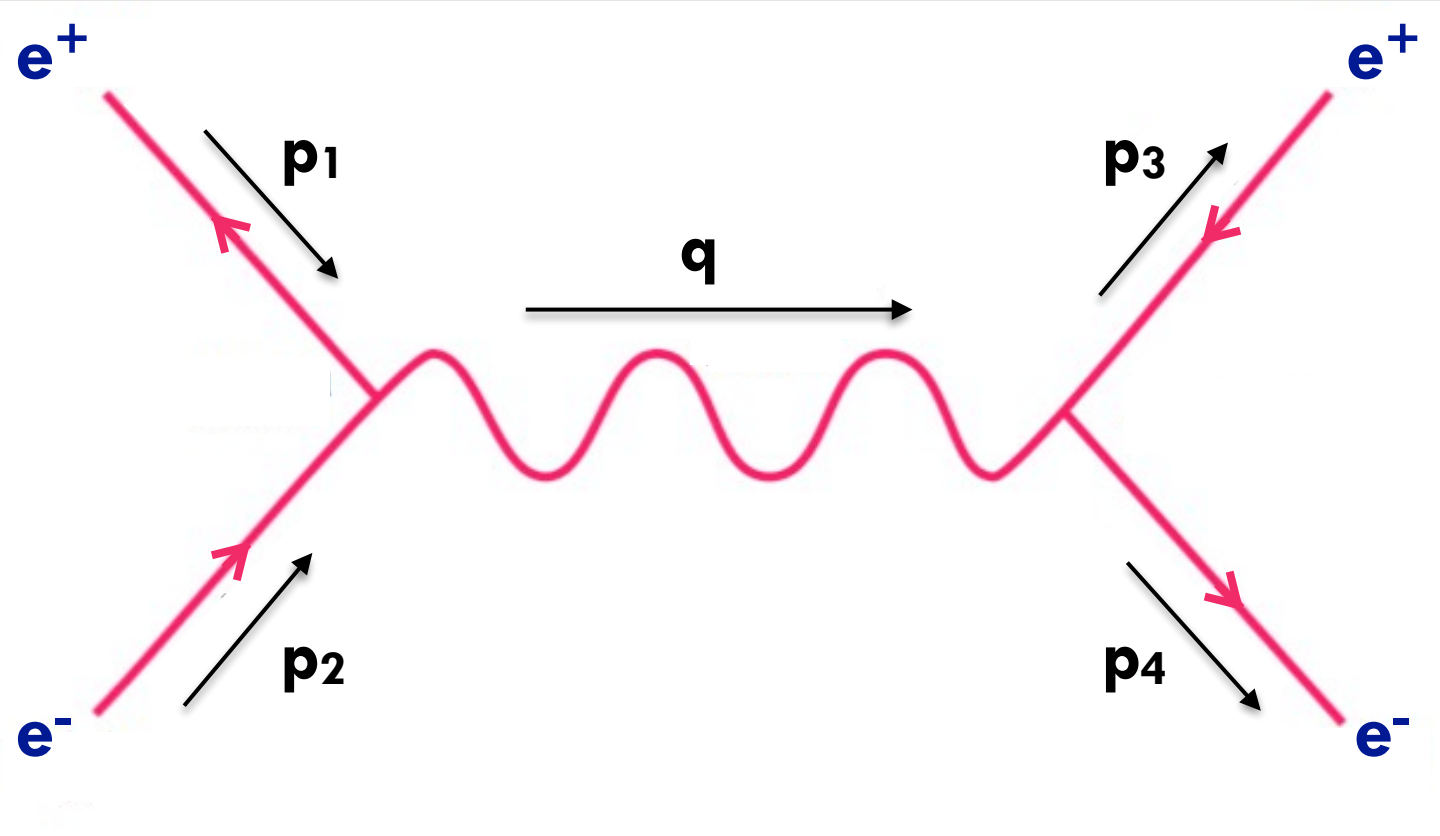
Recall: The Feynman rules provide the recipe for constructing an amplitude $\mathscr{M}$ from a Feynman diagram.

<u>Step 1:</u> For a particular process of interest, draw a Feynman diagram with the minimum number of vertices. There may be more than one.



Feynman Rules for QED

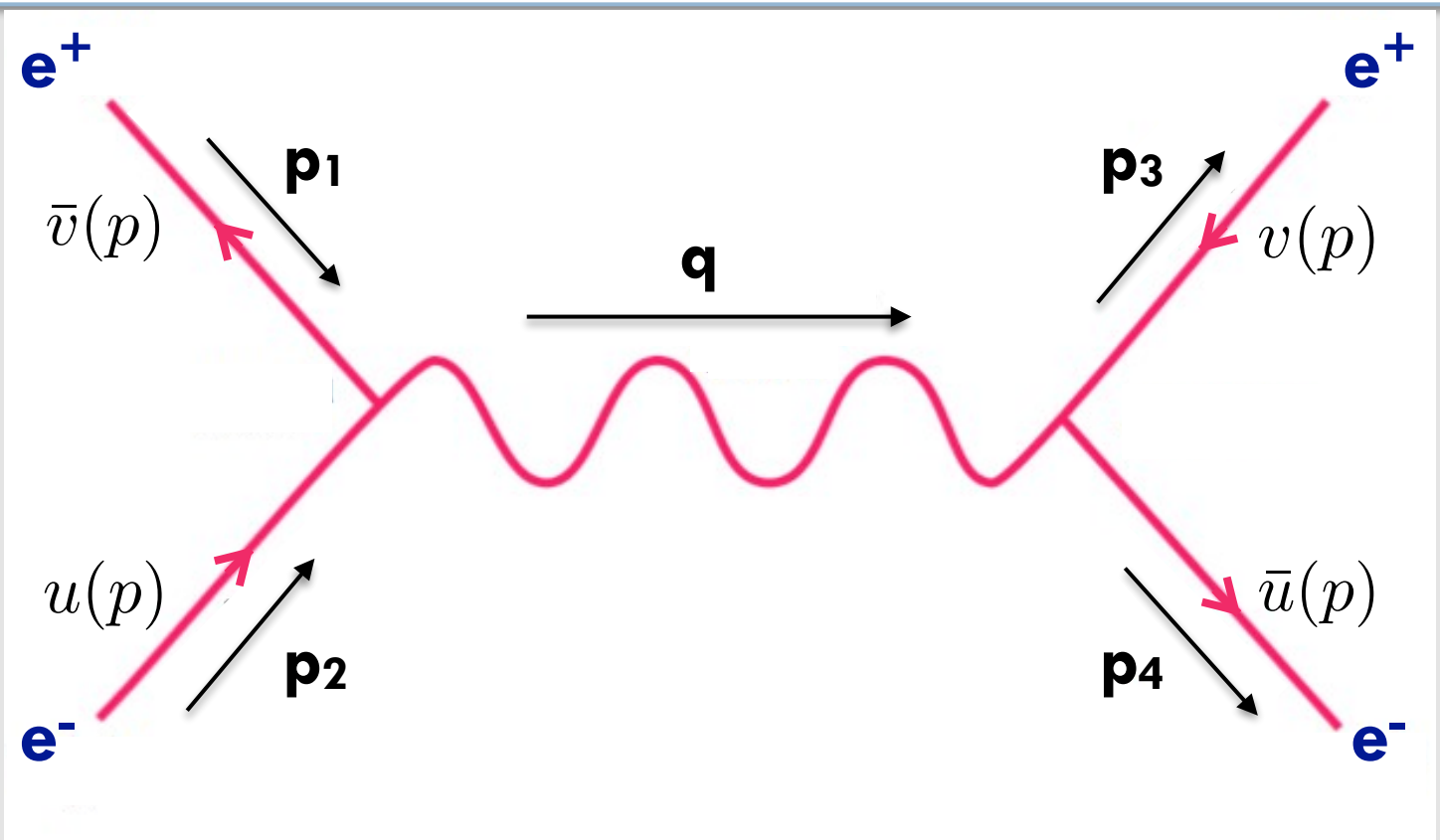
Step 2:
For each Feynman diagram, label the four-momentum of each line, enforcing four-momentum conservation at every vertex.
Note that arrows are only present on fermion lines and they represent particle flow, not momentum.



Feynman Rules for QED

Step 3: For each external line, include a factor for the particle wave function:

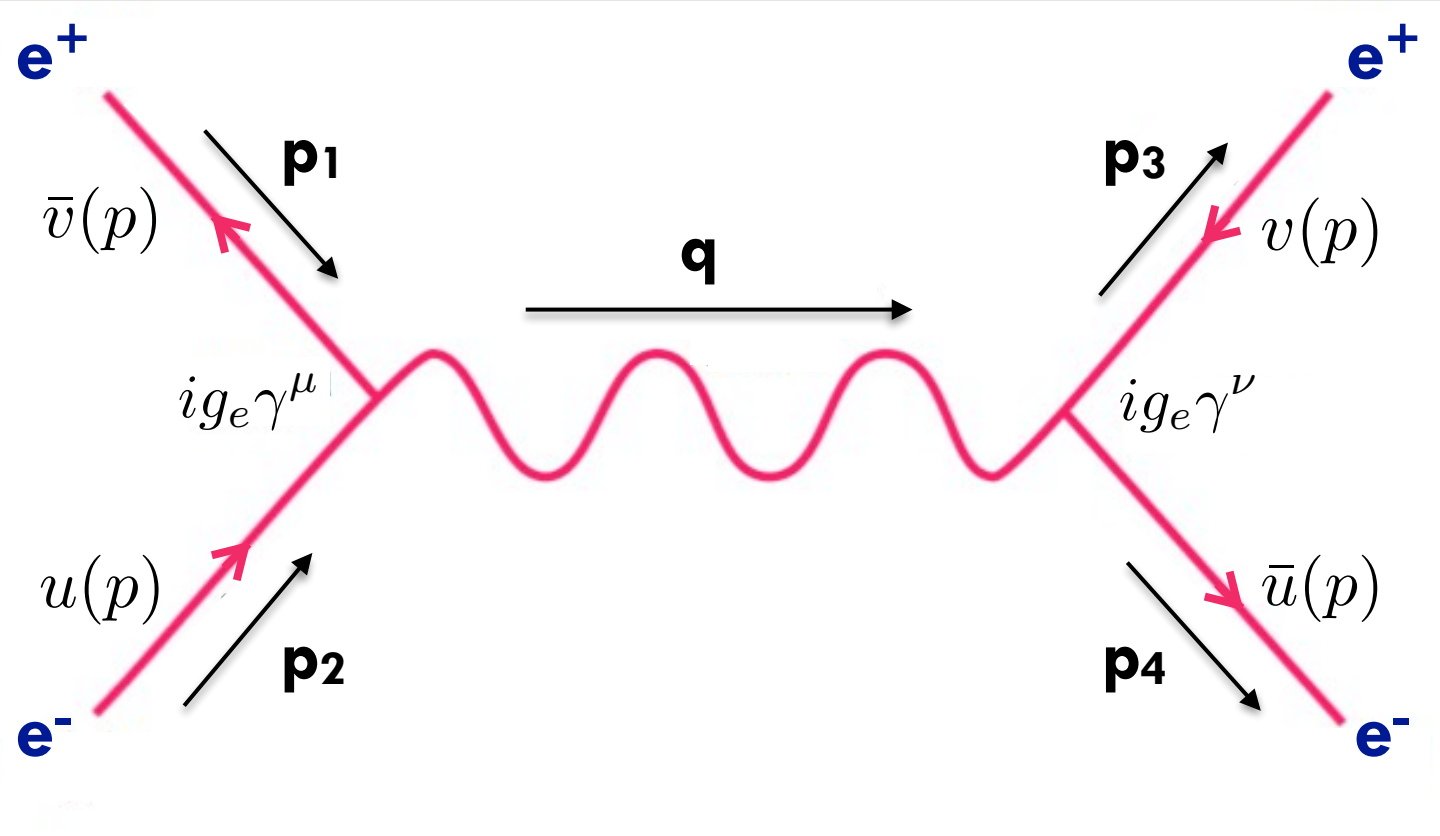
spin 1/2	incoming particle	$u(p)$
	outgoing particle	$\bar{u}(p)$
	incoming antiparticle	$\bar{v}(p)$
	outgoing antiparticle	$v(p)$
spin 1	incoming photon	$\varepsilon^{\mu}(p)$
	outgoing photon	$\varepsilon^{\mu}(p)^{*}$



Feynman Rules for QED

Step 4:
Every QED vertex contributes a factor of $ig_e\gamma^\mu$

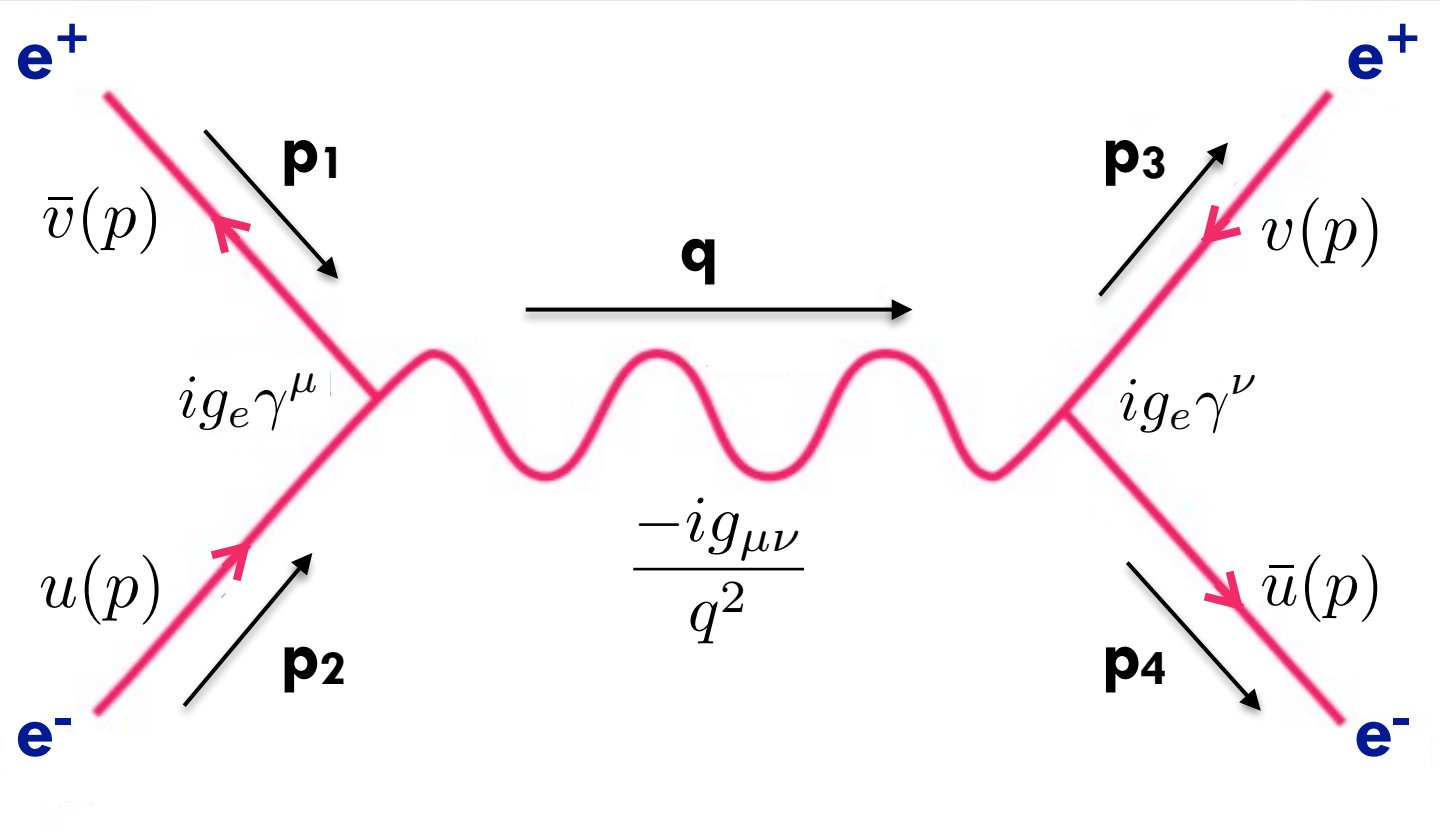
where ge is a dimensionless coupling constant and is related to the fine-structure constant by $\alpha = \frac{g_e^2}{4\pi}$



Feynman Rules for QED

Step 5:
Each internal line contributes a factor as follows:

Photons: $\dfrac{-ig_{\mu\nu}}{q^2}$ **Fermions:** $\dfrac{i(\gamma^\mu q_\mu + m)}{q^2 - m^2}$



Feynman Rules for QED

Step 6:
Each vertex gets a delta function over the 4-momenta into/out of the vertex. Take care to get the 4-momentum signs right!!

$$(2\pi)^4\,\delta^4(k_1 + k_2 + k_3)$$

Step 7:
Each internal momentum gets a phase space integral factor.

$$\frac{d^4q}{(2\pi)^4}$$

Step 8:
After integrating, the result will include a delta function reflecting total energy/momentum conservation. Cancel this factor and multiply by i. The result is the matrix element.

Feynman Rules for QED

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After integrating, the result will include a delta function reflecting total energy/momentum conservation. Cancel this factor and multiply by i. The result is the matrix element.

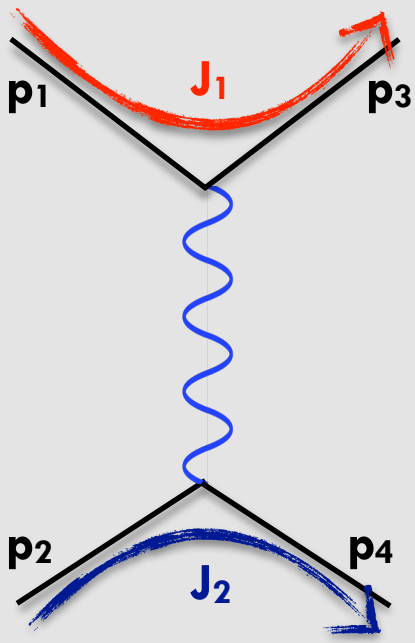
Step 9:
Include a minus sign between diagrams that differ only in the interchange of two incoming (or outgoing) electrons (or positrons), or of an incoming electron with an outgoing positron (or vice versa).

Anti-symmetrization

The anti-symmetrization issue is hiding a more important aspect of QFT.

What we're really doing on some level is tracing the "current" in question.

This can be electric charge, probability, weak hypercharge, etc.

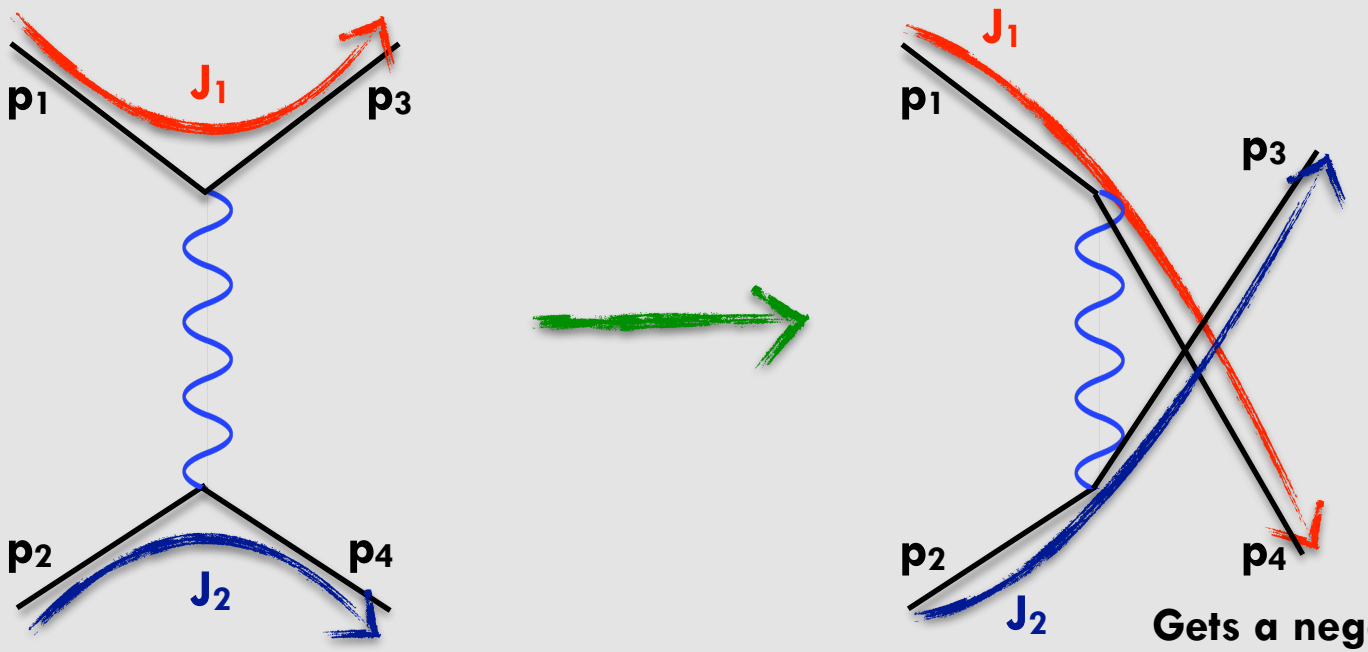


Anti-symmetrization

What matters is the "current"

Black = electron **Red=positron**

This case: electron-electron scattering



Gets a negative sign in the matrix element sum!

The exchange of the final state electrons interchanges momentum definitions.

J_1 goes from (p_3-p_1) to (p_4-p_1)

J_2 goes from (p_4-p_2) to (p_3-p_2)

Anti-symmetrization

What matters is the "current"

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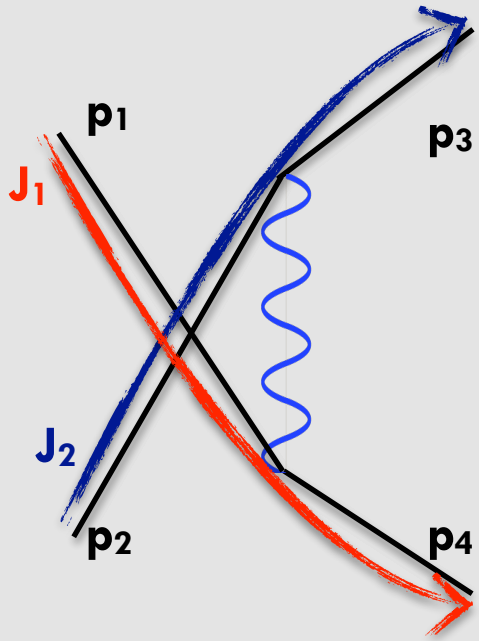
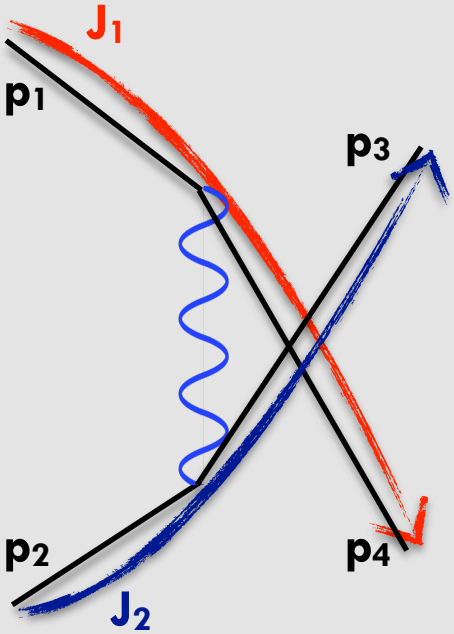


???

We don't do both diagrams because they arrive at the same current definitions! The convention is the one on the right.



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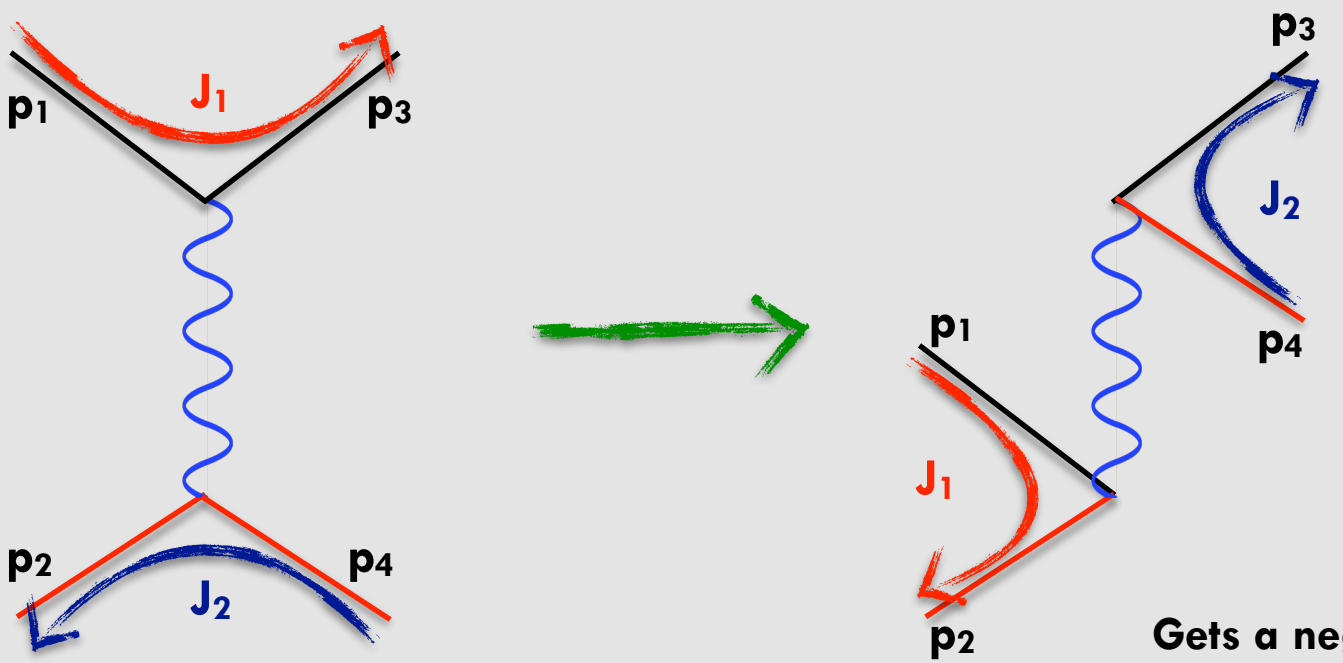
Anti-symmetrization

What matters is the "current"

Black = electron **Red=positron**

This case: electron-positron scattering



Gets a negative sign in the matrix element sum!

The exchange of the initial state electron with final state positron interchanges:

J_1 goes from (p_3-p_1) to (p_2-p_1)

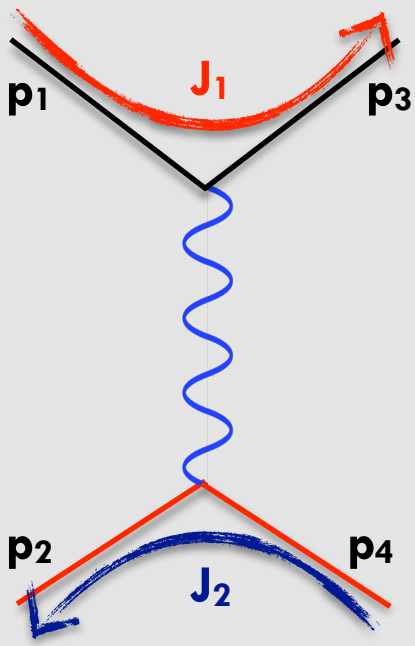
J_2 goes from (p_4-p_2) to (p_4-p_3)

Anti-symmetrization

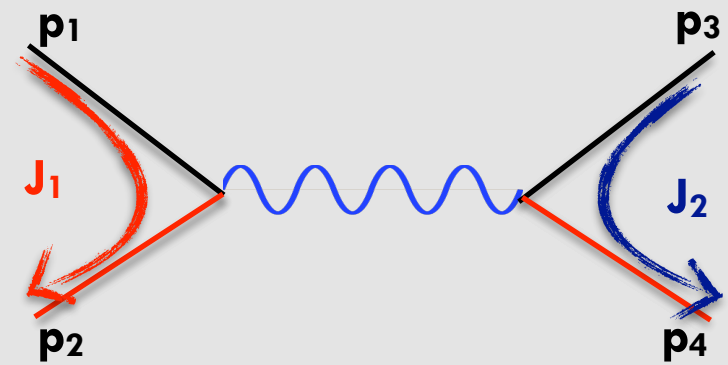
What matters is the "current"

Black = electron **Red=positron**

This case: electron-positron scattering



Rearranging gets you here!



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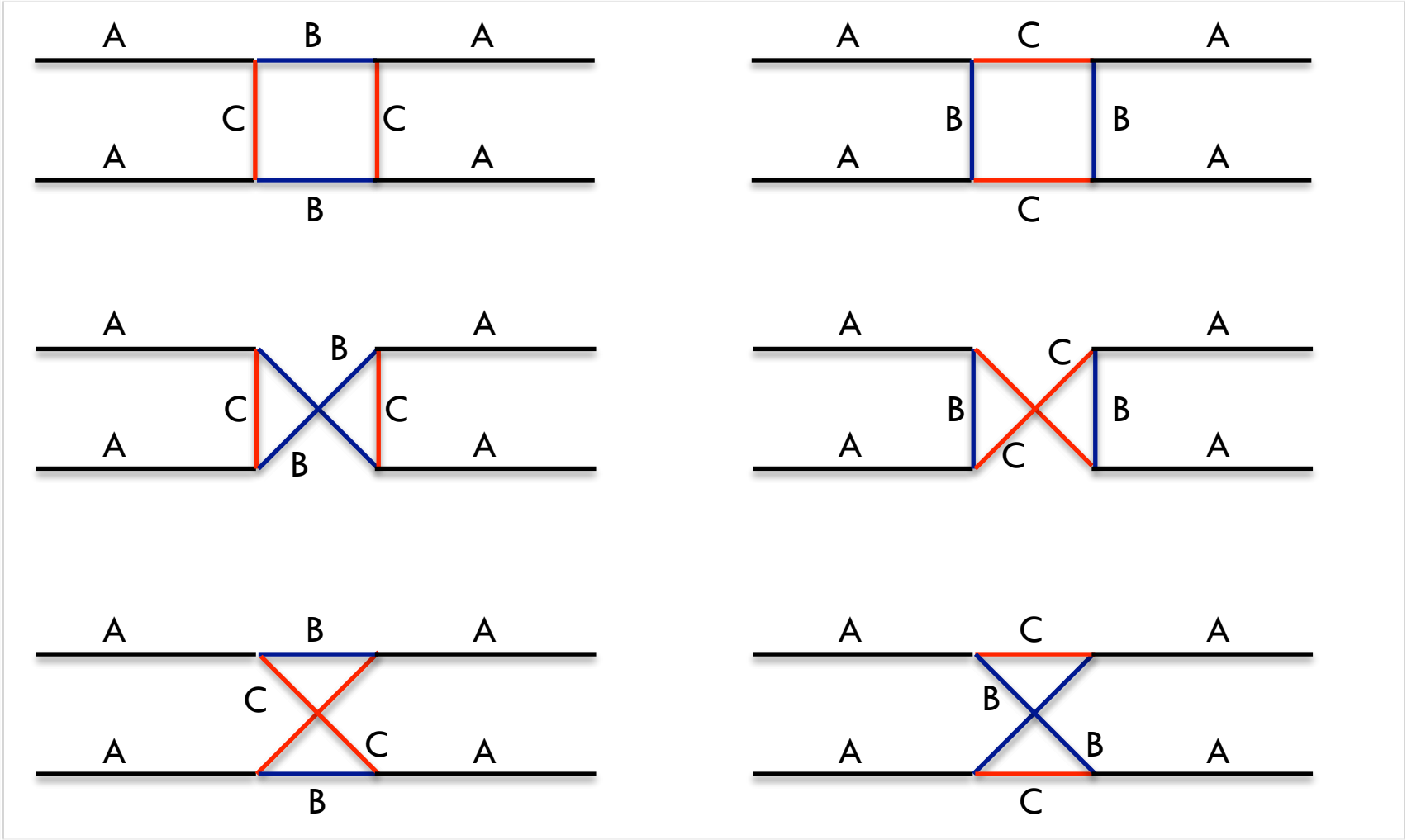
The exchange of the initial state electron with final state positron interchanges:

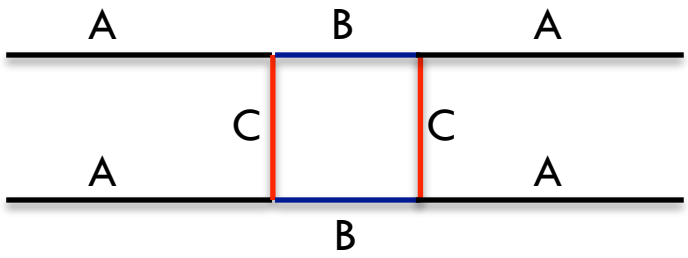
J_1 goes from (p_3-p_1) to (p_2-p_1)

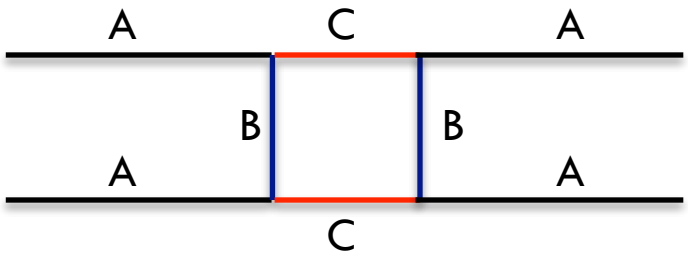
J_2 goes from (p_4-p_2) to (p_4-p_3)

From Exam 2, Problem 5

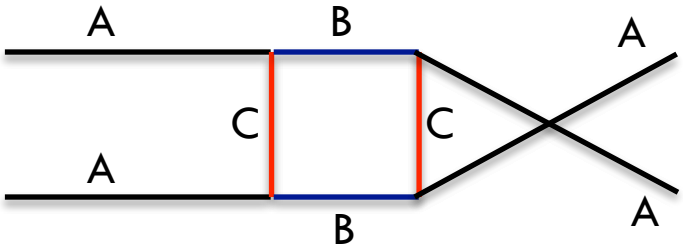


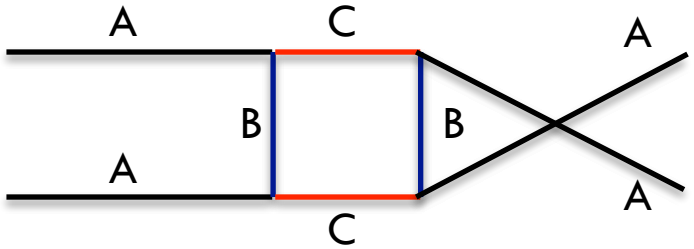
From Exam 2, Problem 5



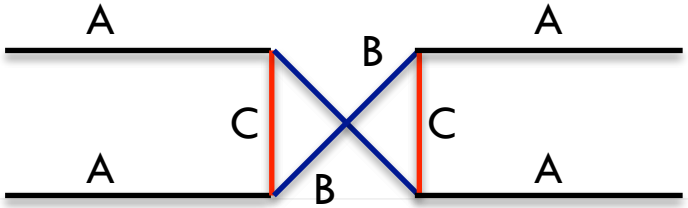


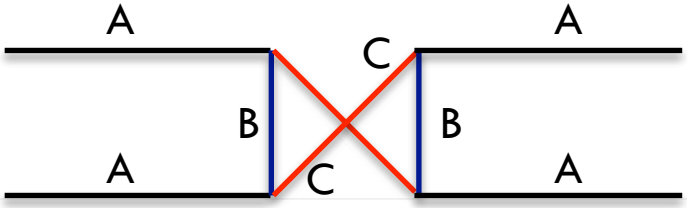
Why not this one??





Because it's the same currents as this one!





Current "Sandwiches"

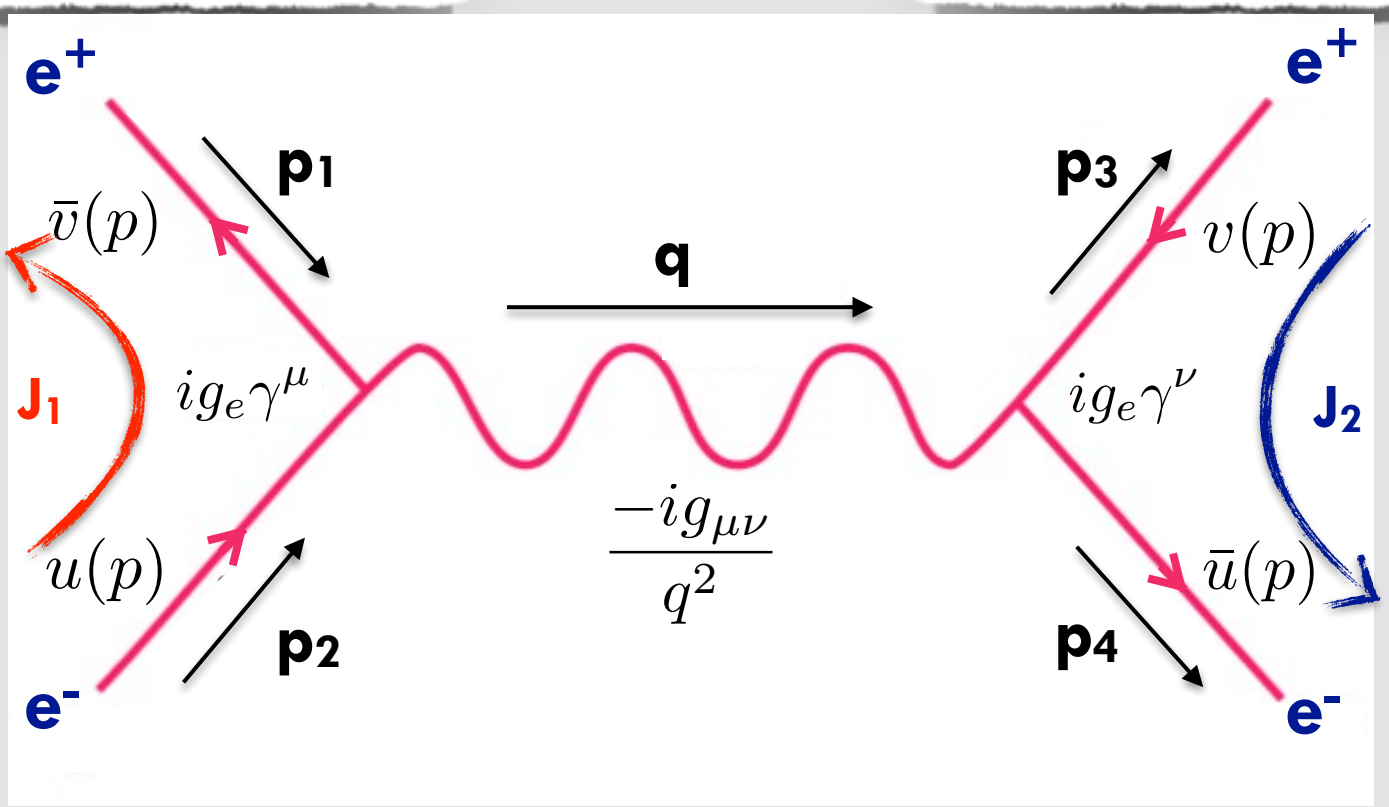
When building QED matrix elements, it's easiest to think of following currents and building current "sandwiches".

Follow the particle current, sandwich the vertex factor in the middle!

Adjoint spinors on the left, spinors on the right.

J1: $\bar{v}(p_1)(ig_e\gamma^\mu)u(p_2)$

J2: $\bar{u}(p_4)(ig_e\gamma^\mu)v(p_3)$

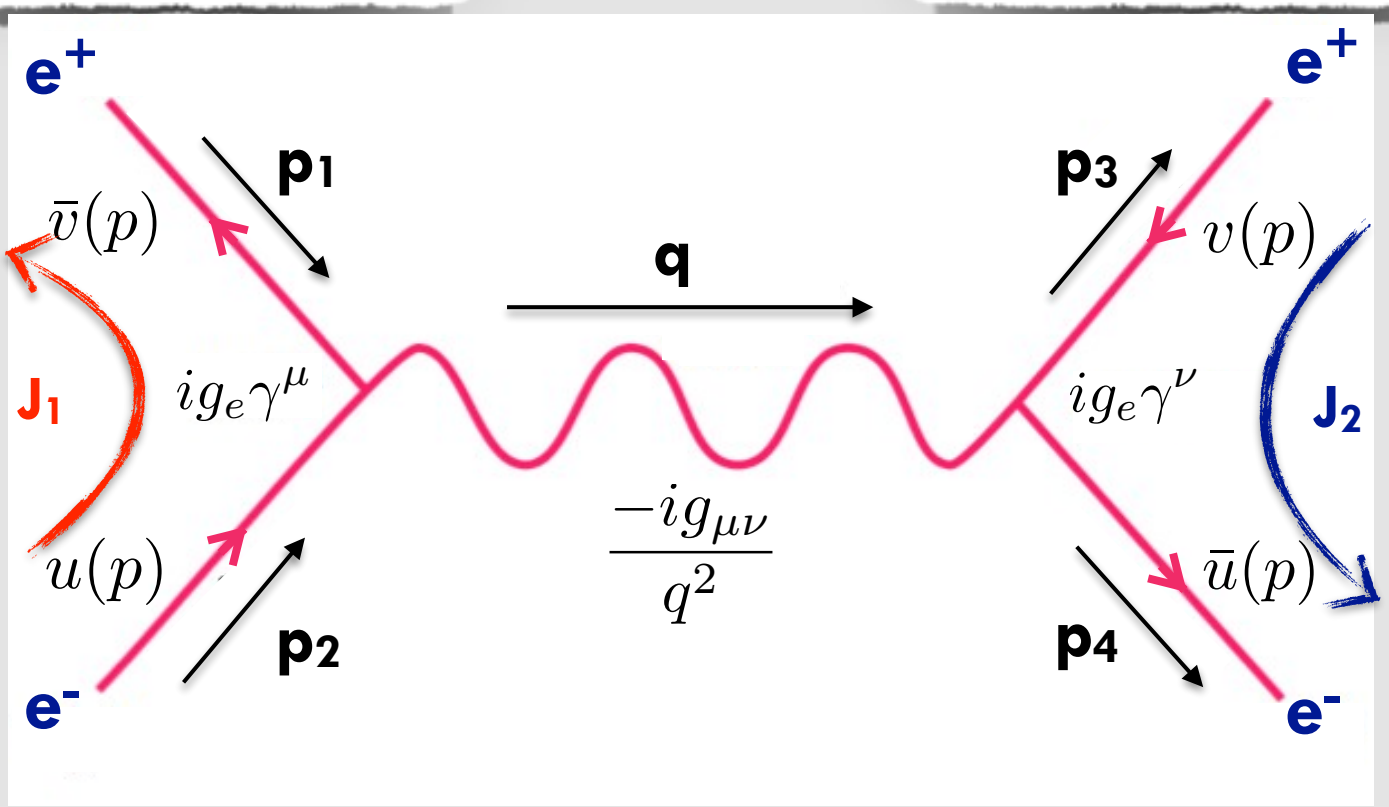


The Matrix Element

$$\mathcal{M} = (2\pi)^8 \int (J_1) \left(\frac{-ig_{\mu\nu}}{q^2} \right) (J_2)\, \delta^4(p_1 + p_2 - q)\, \delta^4(q - p_3 - p_4)\, \frac{d^4q}{(2\pi)^4}$$

J₁: $\bar{v}(p_1)(ig_e\gamma^\mu)u(p_2)$

J₂: $\bar{u}(p_4)(ig_e\gamma^\mu)v(p_3)$



The Matrix Element

$$\mathcal{M} = (2\pi)^8 \int (J_1) \left(\frac{-ig_{\mu\nu}}{q^2} \right) (J_2)\, \delta^4(p_1 + p_2 - q)\, \delta^4(q - p_3 - p_4)\, \frac{d^4q}{(2\pi)^4}$$

1) Substitute the forms of the currents:

$$= i(2\pi)^4\, g_e^2 \int [\bar{v}(p_1)\gamma^\mu u(p_2)] \left(\frac{g_{\mu\nu}}{q^2} \right) [\bar{u}(p_4)\gamma^\mu v(p_3)]\, \delta^4(p_1 + p_2 - q)\, \delta^4(q - p_3 - p_4)\, d^4q$$

2) Do the integration over the propagator momentum: $q \rightarrow p_1 + p_2$

$$= \frac{i(2\pi)^4\, g_e^2}{(p_1 + p_2)^2} [\bar{v}(p_1)\gamma^\mu u(p_2)]\, [\bar{u}(p_4)\gamma_\mu v(p_3)]\, \delta^4(p_1 + p_2 - p_3 - p_4)$$

3) Cancel the remaining delta function (and its 2π factor!), multiply by i

$$= -\frac{g_e^2}{(p_1 + p_2)^2} [\bar{v}(p_1)\gamma^\mu u(p_2)]\, [\bar{u}(p_4)\gamma_\mu v(p_3)]$$

Examples

Examples worked in class:

1. Electron-muon scattering
2. Compton scattering
3. Pair Annihilation

Dealing with Spin

A typical QED amplitude might look something like

$$-\frac{g_e^2}{(p_1+p_2)^2}\left[\bar{v}(p_1)\gamma^\mu u(p_2)\right]\left[\bar{u}(p_4)\gamma_\mu v(p_3)\right]$$

The Feynman rules won't take us any further, but to get a number for $\mathscr{M}$ we will need to substitute explicit forms for the wavefunctions of the external particles

If all external particles have a known polarization, this might be a reasonable way to calculate things. More often, though, we are interested in unpolarized particles.

If we do not care about the polarizations of the particles then we need to

1. Average over the polarizations of the initial-state particles
2. Sum over the polarizations of the final-state particles in the squared amplitude $|\mathscr{M}|^2$.

We call this the spin-averaged amplitude and we denote it by $\left\langle |\mathcal{M}|^2 \right\rangle$

Note that the averaging over initial state polarizations involves summing over all polarizations and then dividing by the number of independent polarizations, so the spin-averaging involves a sum over the polarizations of ***all*** external particles.

Casimir's Trick

The procedure of calculating spin-averaged amplitudes in terms of traces is known as Casimir's Trick.

$$\sum_{\text{all spins}} \left[\bar{u}_a \Gamma_1 u_b\right] \left[\bar{u}_a \Gamma_2 u_b\right]^* = \mathrm{Tr}\left[\Gamma_1 (\not{p}_b + m_b) \bar{\Gamma}_2 (\not{p}_a + m_a)\right]$$

If antiparticle spinors (v) are present in the spin sum, we use the corresponding completeness relation

$$\sum_{s_i=1,2} v_i^{s_i} \bar{v}_i^{s_i} = (\not{p}_i - m_i)$$

Dealing with Spin

Let's simplify things even further and suppose that we have:

$$\mathcal{M} \sim [\bar{u}_1 \Gamma u_2]$$

Then we have:

$$\begin{aligned}
|\mathcal{M}|^2 &\sim [\bar{u}_1 \Gamma u_2]\,[\bar{u}_1 \Gamma u_2]^* \\
&\sim [\bar{u}_1 \Gamma u_2]\left[u_1^\dagger \gamma^0 \Gamma u_2\right]^\dagger \\
&\sim [\bar{u}_1 \Gamma u_2]\left[u_2^\dagger \Gamma^\dagger \gamma^{0\dagger} u_1\right] \\
&\sim [\bar{u}_1 \Gamma u_2]\left[u_2^\dagger \gamma^0 \gamma^0 \Gamma^\dagger \gamma^0 u_1\right] \\
&\sim [\bar{u}_1 \Gamma u_2]\left[\bar{u}_2 \bar{\Gamma} u_1\right]
\end{aligned}$$

$\bar{u} = u^\dagger \gamma^0$

$(\gamma^0)^2 = 1$

$\gamma^0 \Gamma^\dagger \gamma^0 = \bar{\Gamma}$

Dealing with Spin

We have worked up to:

$$|\mathcal{M}|^2 \sim [\bar{u}_1 \Gamma u_2]\,[\bar{u}_2 \Gamma u_1]$$

We can simplify by applying the completeness relation to the 2nd particle (u_2):

$$\sum_{s_i=1,2} u_i^{s_i} \bar{u}_i^{s_i} = (\not{p}_i + m_i)$$

Then we get:

$$\begin{aligned} \sum_{s_2} |\mathcal{M}|^2 &\sim [\bar{u}_1 \Gamma (\not{p}_2 + m_2) \bar{\Gamma} u_1] \\ &\sim [\bar{u}_1 Q u_1] \end{aligned}$$

Dealing with Spin

We have worked up to:

$$\sum_{s_2} |\mathcal{M}|^2 \sim \left[\bar{u}_1 \Gamma (\not{p}_2 + m_2) \bar{\Gamma} u_1\right]$$

$$\sim \left[\bar{u}_1 Q u_1\right]$$

The RHS is just a number, but we can rewrite the matrix multiplication with summations over indices and simplify:

$$\begin{aligned}
\left[\bar{u}_1 Q u_1\right] &= (\bar{u}_1)_i \, Q_{ij} \, (u_1)_j \\
&= Q_{ij} \, (u_1 \bar{u}_1)_{ji} \\
&= \left[Q \, (u_1 \bar{u}_1)\right]_{ii} \\
&= \mathrm{Tr}\left[Q (u_1 \bar{u}_1)\right]
\end{aligned}$$

A slight of hand!

$$\bar{u}_i u_j = (u^\dagger \gamma^0)_i u_j = \left\{ \begin{pmatrix} u_1^* & u_2^* & u_3^* & u_4^* \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \right\}_i u_j$$

$$u\bar{u} = uu^\dagger \gamma^0 = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \begin{pmatrix} u_1^* & u_2^* & u_3^* & u_4^* \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\bar{u}_i u_j = \begin{matrix} u_i^* u_j & (i = 1, 2) \\ -u_i^* u_j & (i = 3, 4) \end{matrix} \qquad (u\bar{u})_{ij} = \begin{matrix} u_i u_j^* & (j = 1, 2) \\ -u_i u_j^* & (j = 3, 4) \end{matrix}$$

Dealing with Spin

We have worked up to:

$$
\begin{aligned}
[\bar{u}_1 Q u_1] &= (\bar{u}_1)_i \, Q_{ij} \, (u_1)_j \\
&= Q_{ij} \, (u_1 \bar{u}_1)_{ji} \\
&= [Q \, (u_1 \bar{u}_1)]_{ii} \\
&= \mathrm{Tr}\,[Q(u_1 \bar{u}_1)]
\end{aligned}
$$

Next, we apply the completeness relation once again, so that we get

$$\sum_{s_1} |\mathcal{M}|^2 \sim \mathrm{Tr}\,[Q(\not{p}_1 + m_1)]$$

Thus in total we have:

$$\left\langle |\mathcal{M}|^2 \right\rangle \sim F \cdot \mathrm{Tr}\left[\Gamma(\not{p}_2 + m_2)\bar{\Gamma}(\not{p}_1 + m_1)\right]$$

F= 1/4 (2 initial state fermions)
1/2 (1 initial state fermion)
1 (2 initial state photons)

Casimir's Trick

The procedure of calculating spin-averaged amplitudes in terms of traces is known as Casimir's Trick.

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If antiparticle spinors (v) are present in the spin sum, we use the corresponding completeness relation

$$\sum_{s_i=1,2} v_i^{s_i} \bar{v}_i^{s_i} = (\not{p}_i - m_i)$$

Trace Theorems

Because of Casimir's Trick, we're going to find ourselves calculating a lot of traces involving γ-matrices. Some general identities about traces:

1. $Tr(A+B) = Tr(A) + Tr(B)$

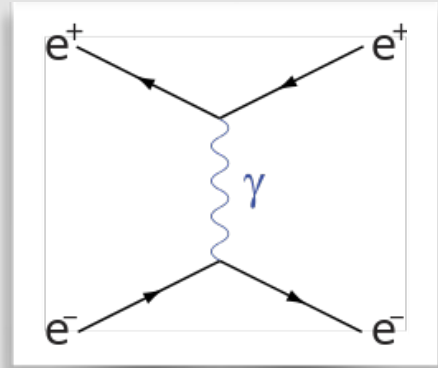
2. $Tr(\alpha A) = \alpha Tr(A)$

10. The trace of the product of an odd number of γ matrices is 0

12. $Tr(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu}$

13. $Tr(\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma) = 4(g^{\mu\nu} g^{\lambda\sigma} - g^{\mu\lambda} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\lambda})$

Trace Example



Consider Bhabha scattering. We'll get the following trace:

$$T = \mathrm{Tr}\left[\gamma^\mu(\not{p}_1 + m)\gamma^\nu(\not{p}_3 + m)\right]$$

We can expand this out to create 4 terms, but 2 of these terms (the ones linear in m) will involve 3 γ-matrices, and are therefore zero. Thus, we have:

$$\begin{aligned} T &= \mathrm{Tr}(\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3) + m^2 \mathrm{Tr}(\gamma^\mu \gamma^\nu) \\ &= 4\left(p_1^\mu p_3^\nu + p_3^\mu p_1^\nu - (p_1 \cdot p_3) g^{\mu\nu}\right) + 4m^2 g^{\mu\nu} \end{aligned}$$

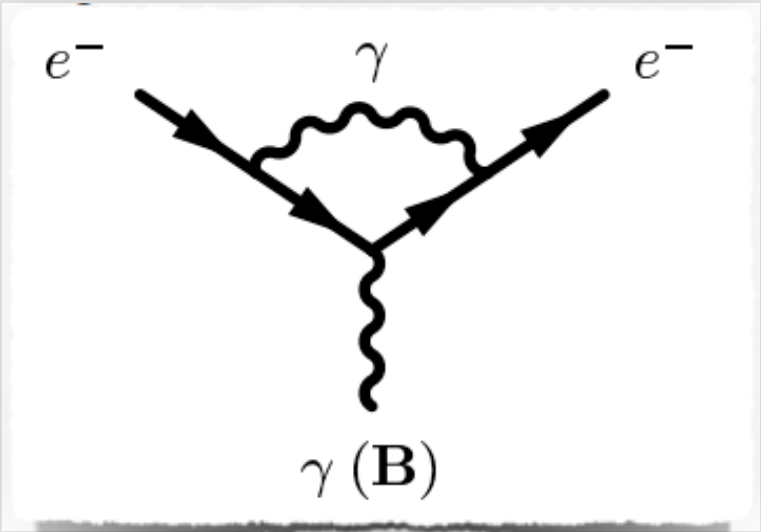
Calculations

Examples worked in class:

1. Electron-muon scattering
2. Compton Scattering

Higher-Order QED Diagrams

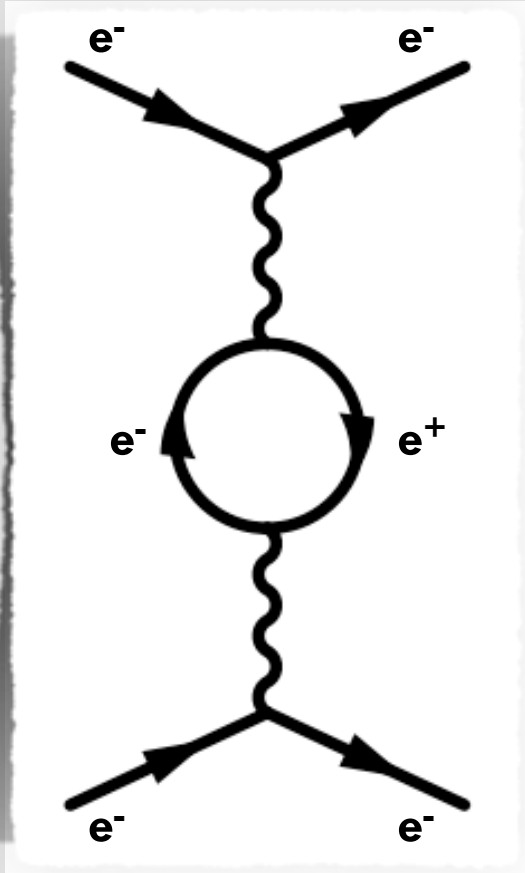
The most famous higher-order process in QED is the anomalous magnetic moment of the electron (or muon), arising from the diagram



In 1948, Schwinger showed that this modifies the electron g-factor from 2 to $(2+\alpha/\pi)$. It is currently known to α^4, corresponding to an uncertainty in g_e of about10^{-12}.

Higher-Order QED Diagrams

Recall from Chapter 5, that the Lamb Shift arises from vacuum polarization effects in QED:



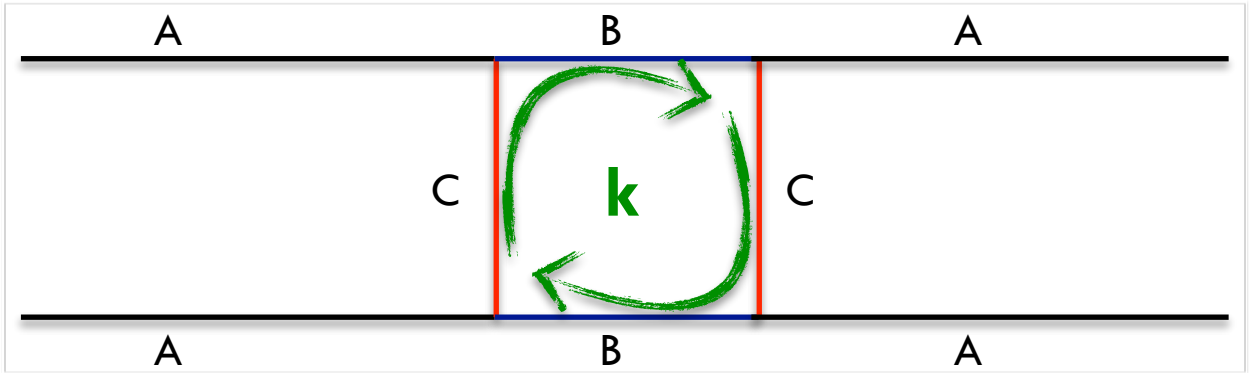
Intuitively, we expect the electromagnetic force to strengthen at high energies (short distances), as two particles will see each other's unscreened charges more than at low energies.
Quantitatively, the leading-order effect due to virtual e^+e^- pairs leads to an change in the effective coupling strength:

$$\alpha(|q^2|) = \frac{\alpha(0)}{1 - \left(\frac{\alpha(0)}{3\pi}\right) \ln\left(\frac{|q^2|}{m^2}\right)}$$

Other types of virtual pairs modify this expression as various energy thresholds are passed.

Renormalization

"Box" diagrams also contribute to the total matrix element



Two extra vertices ⇒ the contribution is suppressed by a factor of $\alpha = 1/137$

- The four momentum must be conserved at each vertex.
- However, four momentum q flowing round the loop can be anything!

- In calculating M integrate over all possible allowed momentum configurations: $\int f(k)\, d^4k \sim \ln(k)$ leads to a divergent integral!
- This is solved by renormalisation in which the infinities are "miraculously swept up into redefinitions of mass and charge"

Renormalization

Impose a "cutoff" mass M, do not allow the loop four momentum to be larger than M. Use $M^2 \gg q^2$, the momentum transferred between initial and final state.

- This can be interpreted as a limit on the shortest range of the interaction
- Or interpreted as possible substructure in point-like fermions

- Physical amplitudes should not depend on choice of M
  - Find that $\ln(M^2)$ terms appear in the M
  - Absorb $\ln(M^2)$ into redefining fermion masses and vertex couplings
- Masses $m(q^2)$ and couplings $\alpha(q^2)$ are now functions of q^2
  - e.g. Renormalisation of electric charge (considering only effects from one type of fermion):

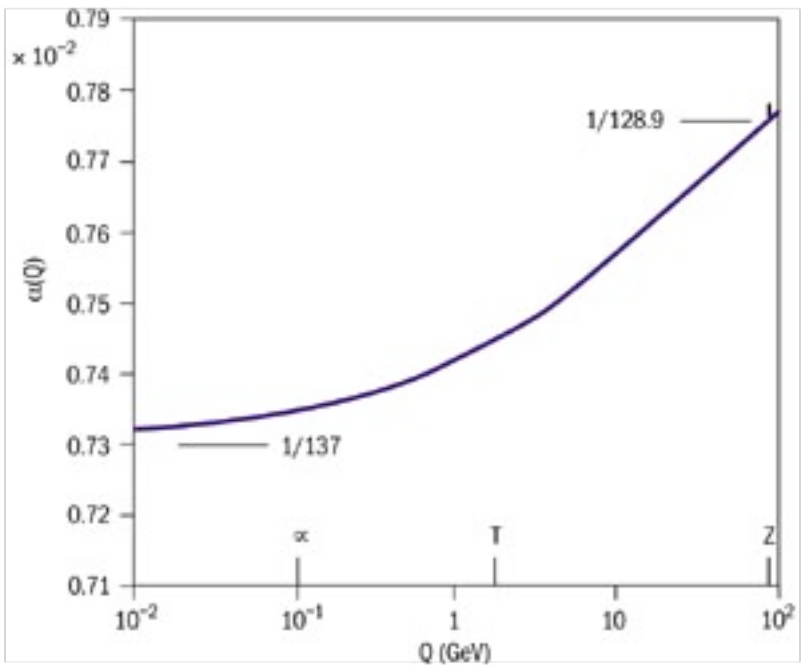
$$\alpha(|q^2|) = \frac{\alpha(0)}{1 - \left(\frac{\alpha(0)}{3\pi}\right) \ln\left(\frac{|q^2|}{m^2}\right)}$$

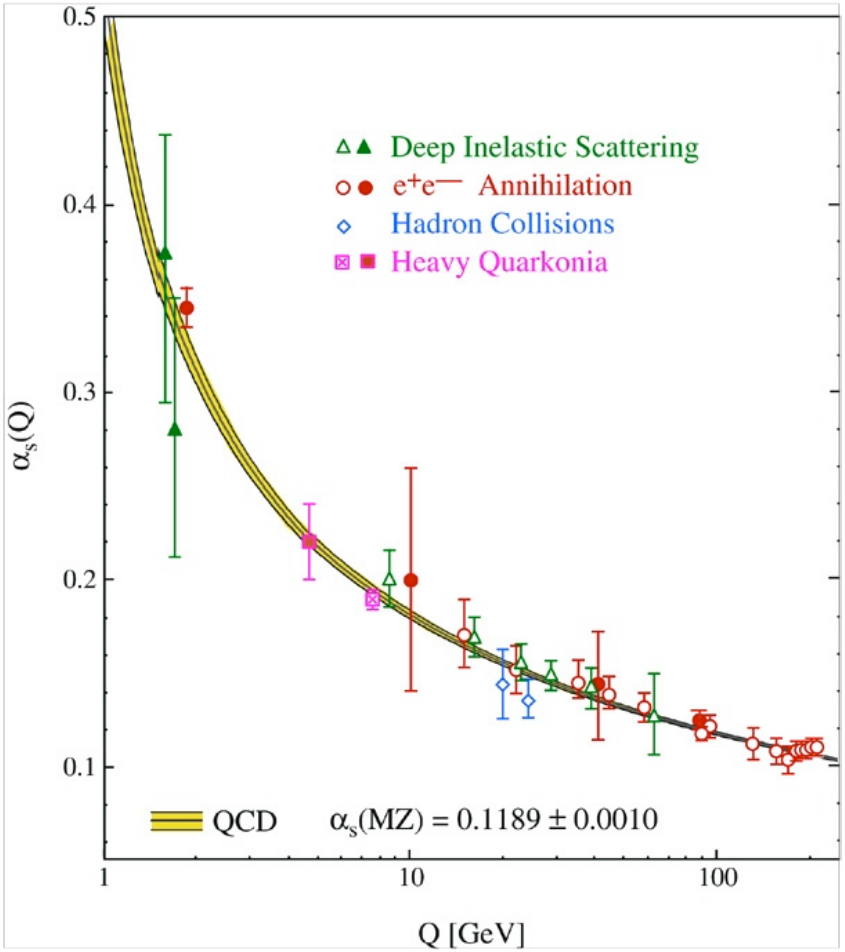
Can be interpreted as a "screening" correction due to the production of electron/positron pairs in a region round the primary vertex

- The new $\alpha(q^2)$ represents the effective charge we actually measure!

Running Coupling Constants

Recall, from Ch 5, while the QED coupling constant increases for higher energies, the QCD coupling constant gets smaller!





Recap / Up Next

<u>This time:</u>

Quantum Electrodynamics

- The Dirac Equation
- QED Feynman Rules
- Cross Sections
- Renormalization

<u>Next time:</u>

Quark Dynamics

- QED for quarks
- Quantum Chromodynamics
- Color
- Asymptotic Freedom