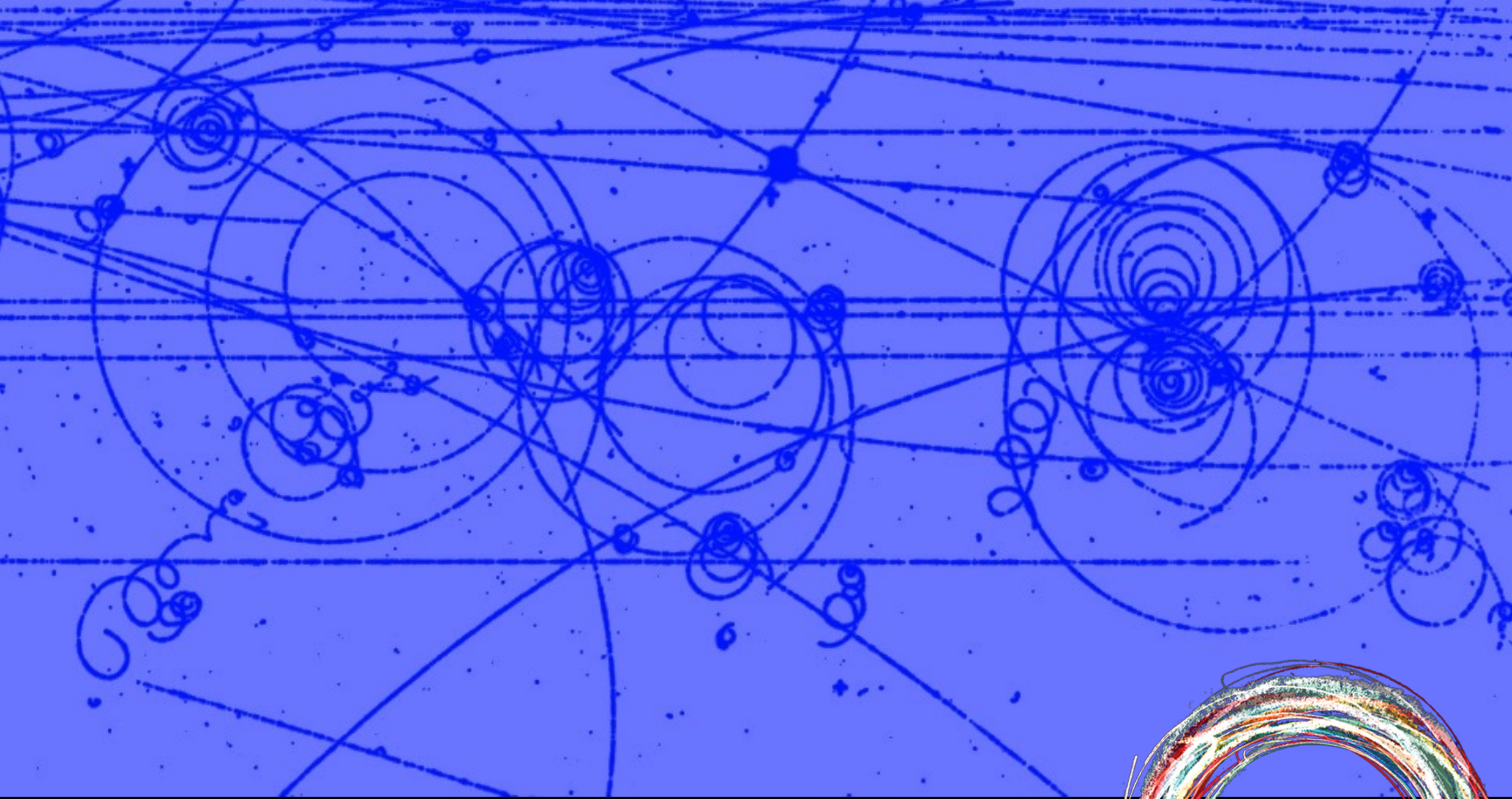
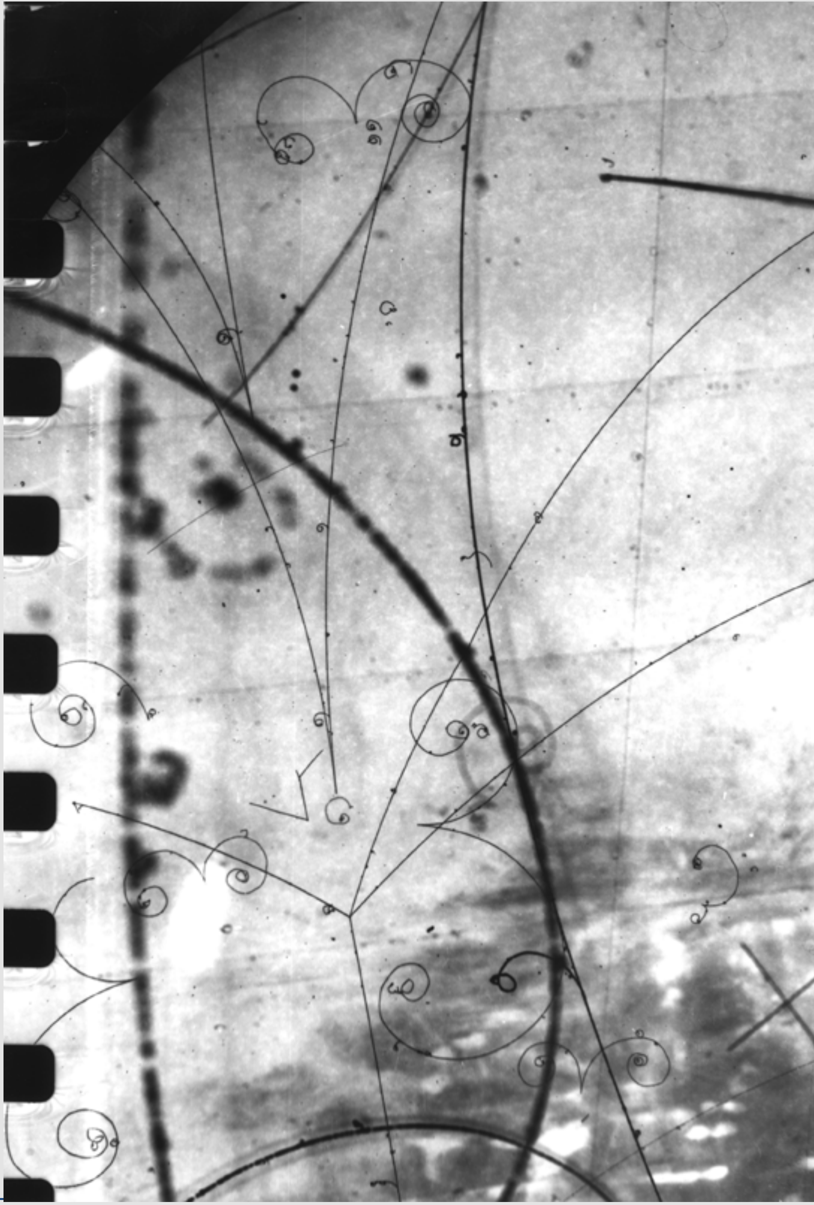


LECTURE 13
QUANTUM CHROMODYNAMICS

PHY 493/803, 2017

Recap / Up Next



Last time:

Quantum Electrodynamics

- The Dirac Equation
- QED Feynman Rules
- Cross Sections
- Renormalization

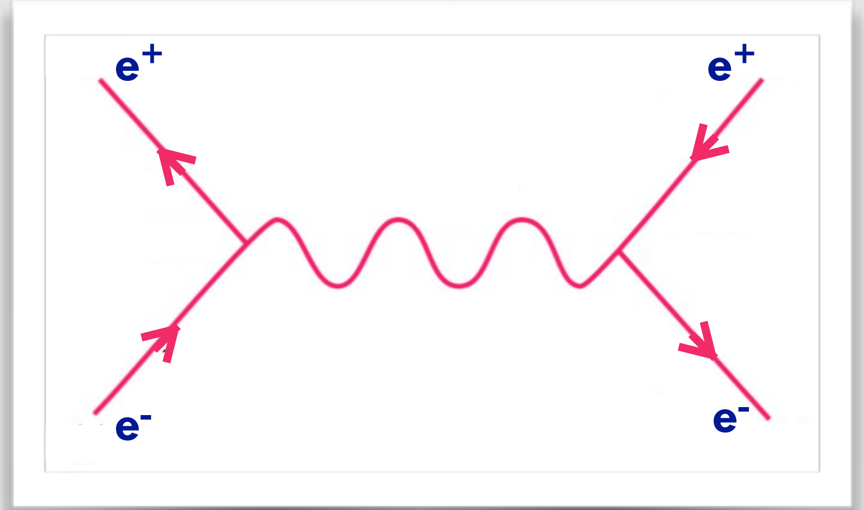
This time:

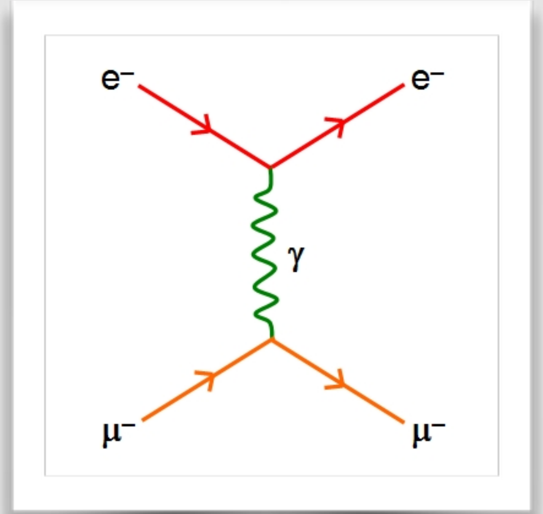
Quantum Chromodynamics

- Quarks in QED
- QCD Feynman Rules
- Asymptotic Freedom

Quarks in QED

We saw previously how quantum electrodynamics describes the relationship between charged particles and photons.



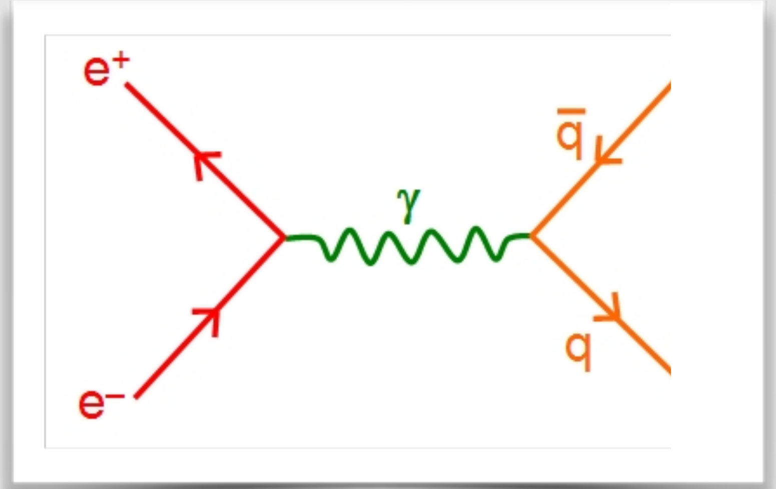


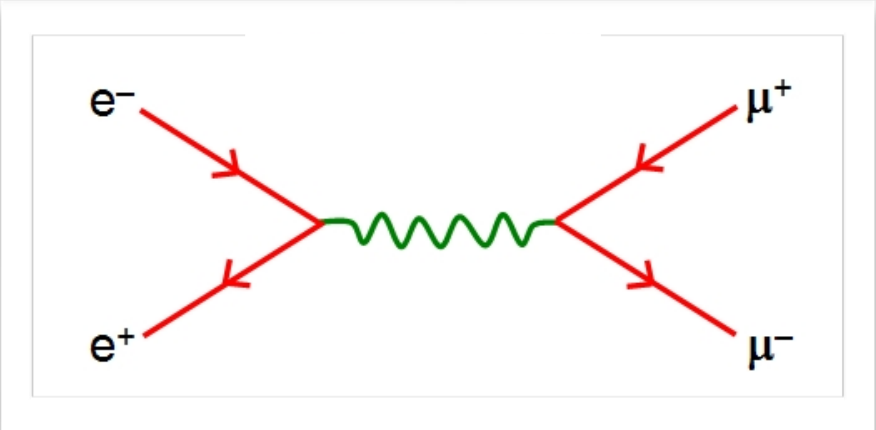
The QED couplings (vertex factors) only depend on the charge involved, not the flavor (or type) of the particle itself.

Cross sections and decay rates obviously depend on particle mass, but that's the only difference between electron and muon currents.

Quarks in QED

Thus we should expect that QED interactions with quarks should have an identical behavior as with electrons or muons!



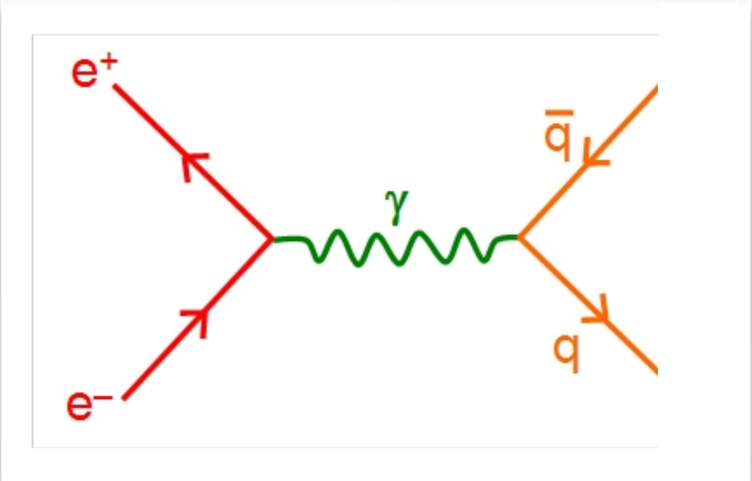


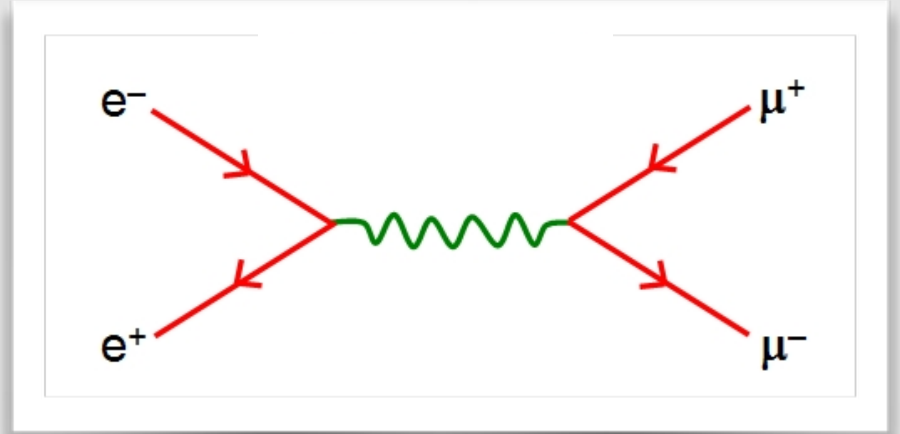
For diagrams that are identical aside from the particle type (and charge!), we should expect that the matrix element will be highly similar.

This was almost trivial for electrons vs muons

For quark vs charged lepton, it's also simple but there are some extra considerations!

QED s-Channel Matrix Element

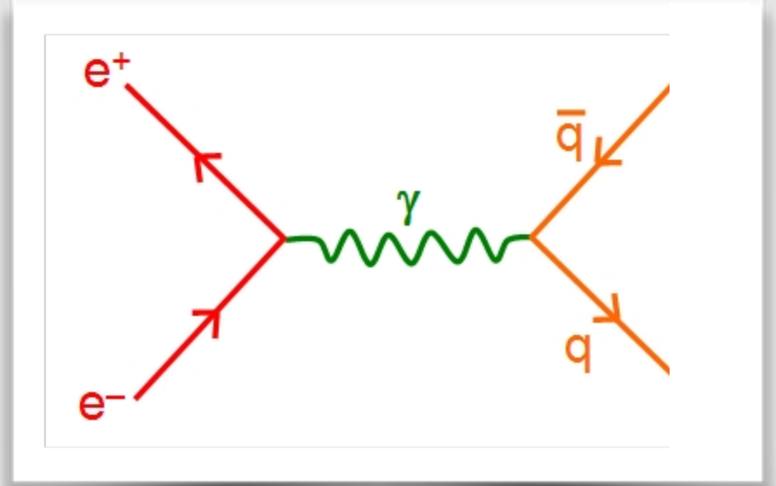


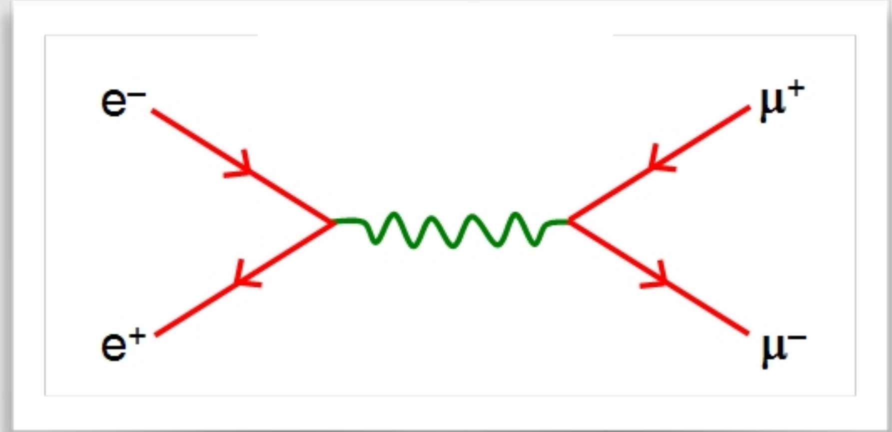


$$\mathcal{M} = (2\pi)^8 \int (J_1) \left(\frac{-ig_{\mu\nu}}{q^2}\right) (J_2)\, \delta^4(p_1 + p_2 - q)\, \delta^4(q - p_3 - p_4)\, \frac{d^4q}{(2\pi)^4}$$

$$\mathcal{M} = \frac{-g^2}{(p_1 + p_2)^2} [\bar{v}(p_1)\gamma^\mu u(p_2)][\bar{u}(p_4)\gamma_\mu v(p_3)]$$

QED s-Channel Matrix Element

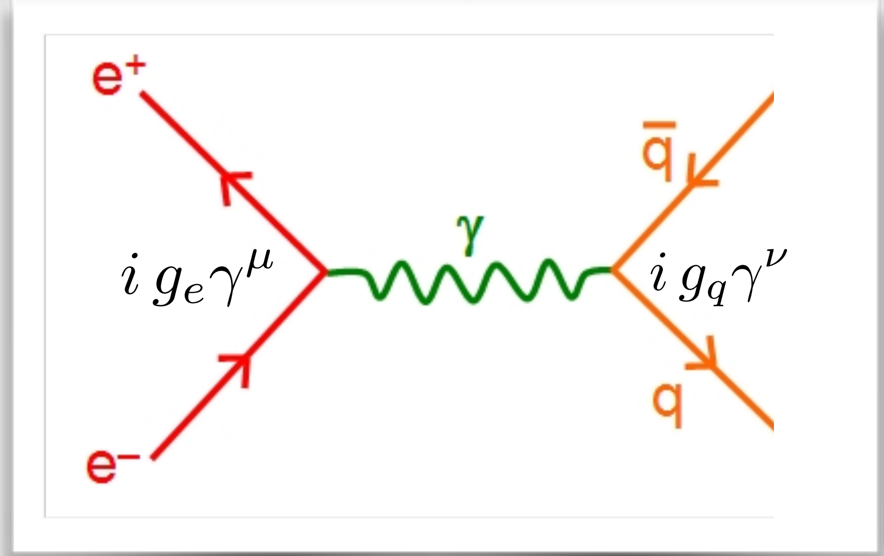


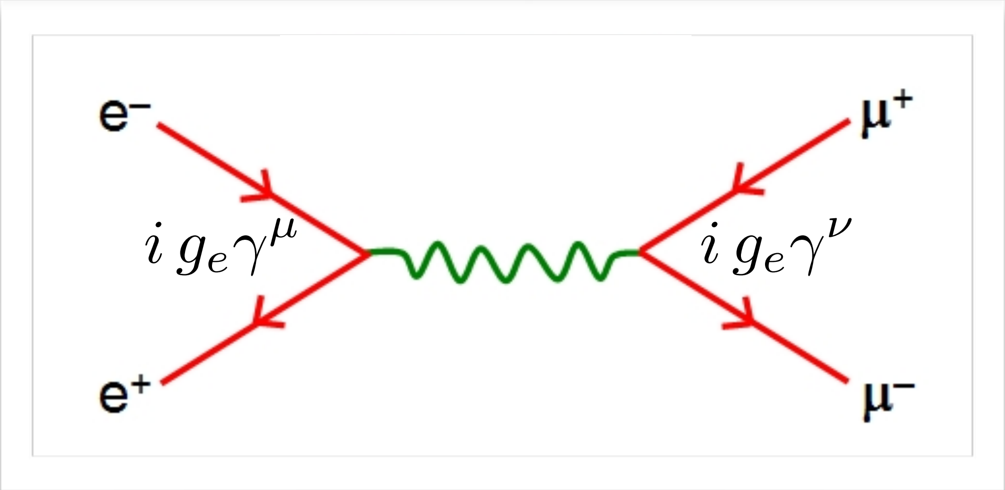


$$\sigma(e^+e^- \to \mu^+\mu^-) = \frac{\alpha^2\pi}{3E^2} = \frac{4\pi\alpha^2}{3s}$$

$$\sigma(e^+e^- \to q\bar{q}) = ??$$

Comparing Vertex Factors





Because the photon couples to charge, the vertex factors only differ by the magnitude of the charge involved!

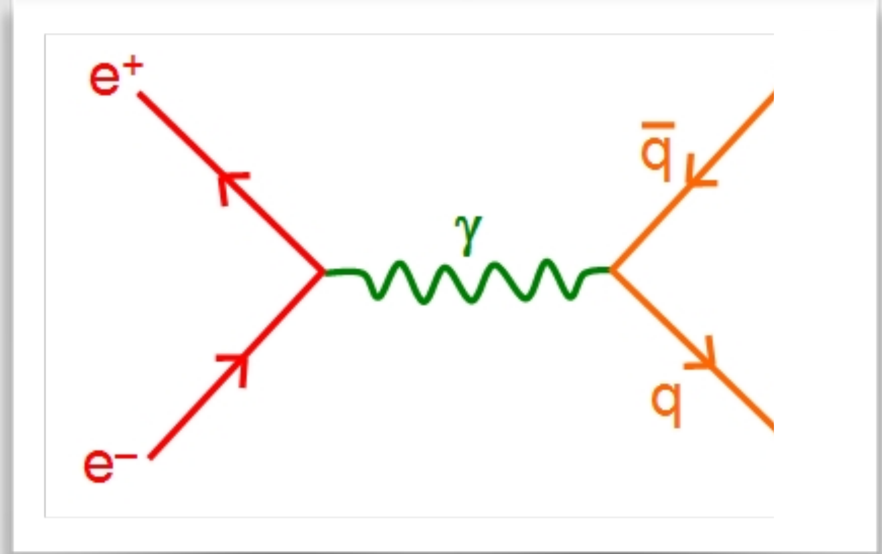
$$g_e = e\sqrt{4\pi/\hbar c} = \sqrt{4\pi\alpha}$$
$$g_q = Q_q\, e\sqrt{4\pi/\hbar c} = Q_q\sqrt{4\pi\alpha}$$

What is the final state?

We also have to consider what our final state is!

Which quarks? u, d, c, s, t, b?

It matters! The charge of up-type quarks is +2/3e and down-type is -1/3e



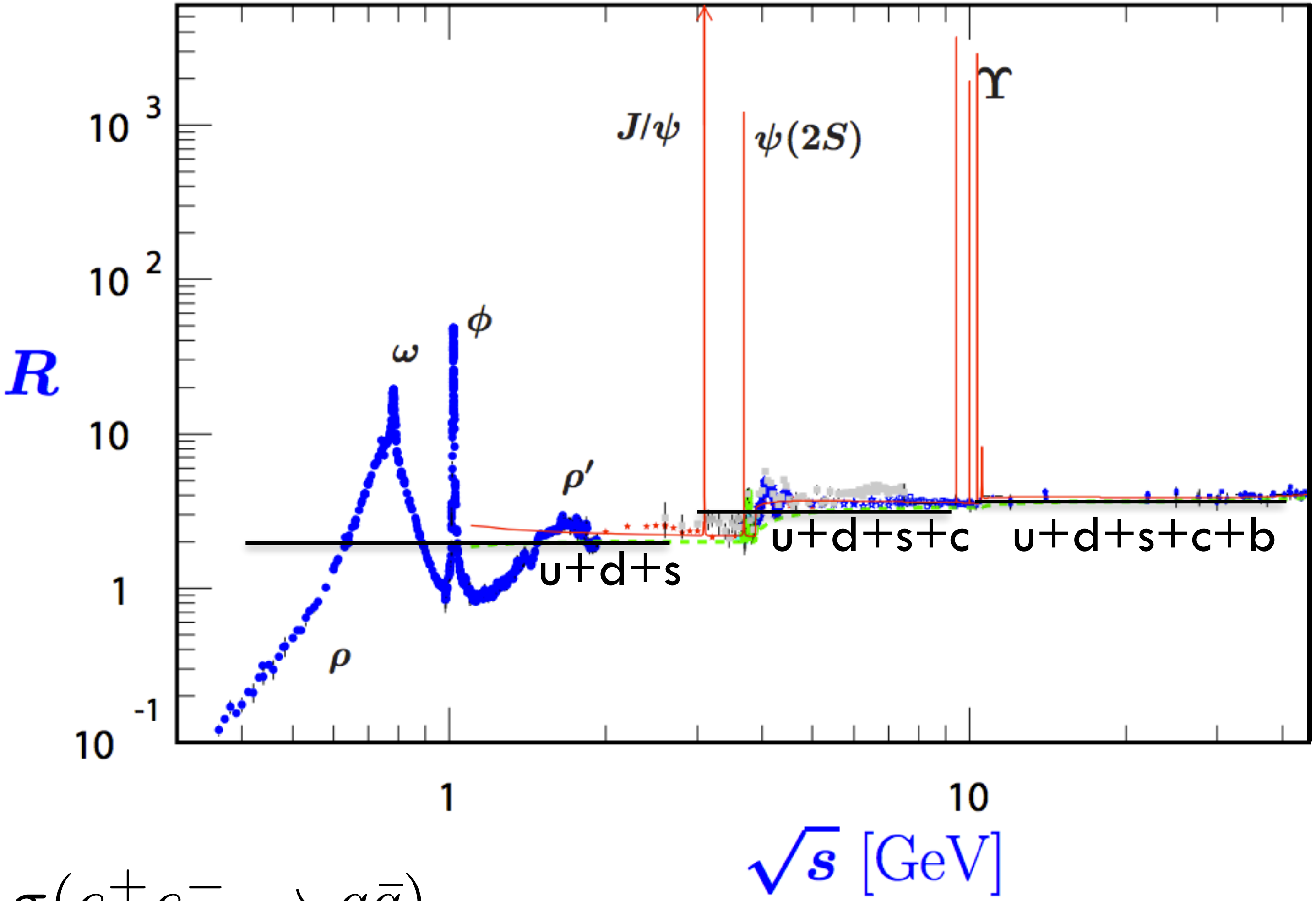
Which quarks are involved?

All of them? u, d, c, s, t, b? How do we decide?

Conservation of energy tells us that for a given quark flavor:

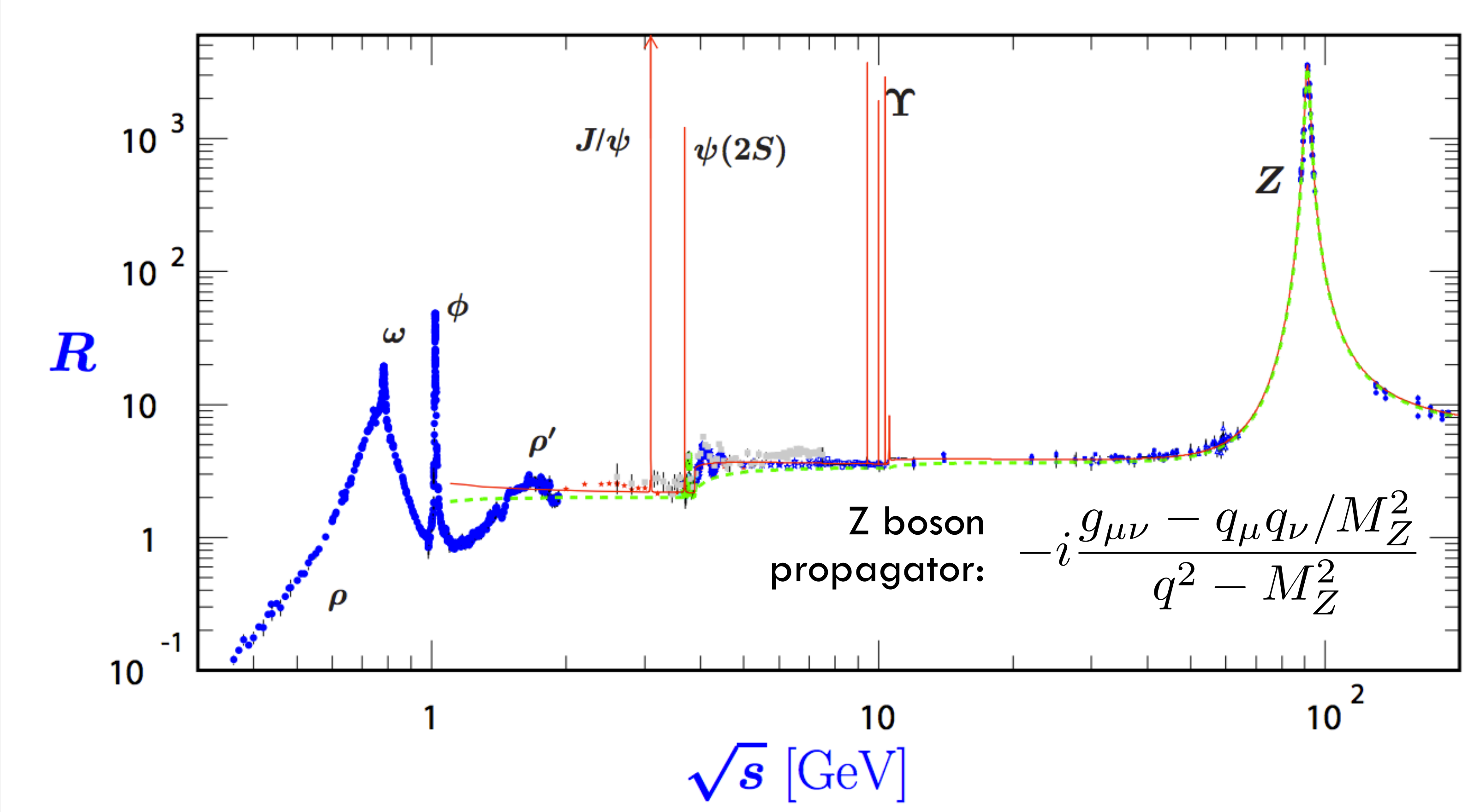
$$\sqrt{s} \geq 2m_q$$

What is the final state?



$$R = \frac{\sigma(e^+e^- \to q\bar{q})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

What is the final state?



Z boson propagator: $-i\dfrac{g_{\mu\nu} - q_\mu q_\nu/M_Z^2}{q^2 - M_Z^2}$

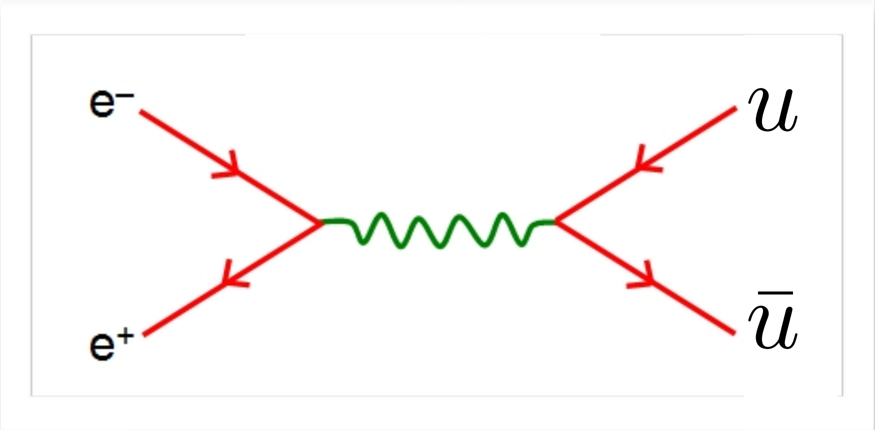
- For E≫m the denominator (for the Z contribution) looks like E^2-$(M_Z)^2$ which blows up at E=M_Z.
- We will see that this “pole” does not cause problems since M_Z has a width associated with it.
- The photon propagator goes like ~q^{-2} so it’s contribution is small at very high energies.
- At very very high energies (that is, for q^2≫$(M_Z)^2$) the contributions from the photon and the Z become similar.

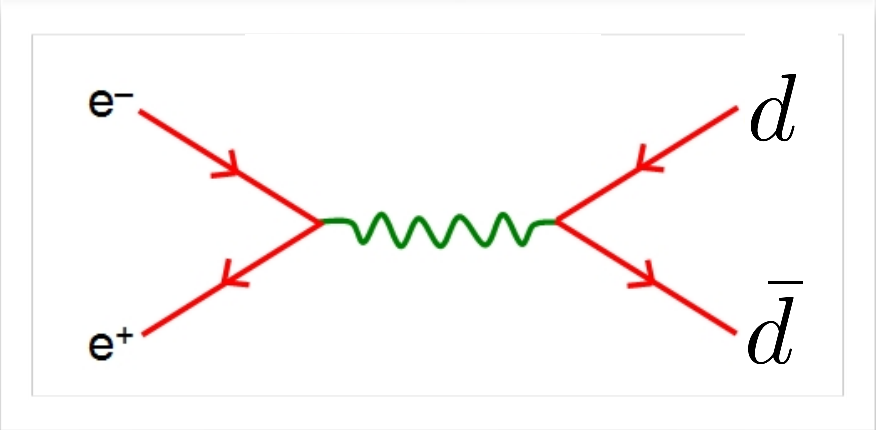
How do we sum over quark flavors?

For multiple Feynman diagrams that contribute to the same final state, we would normally add them up with the appropriate symmetrization factors.

For example, s-channel and t-channel electron/positron scattering

$$\mathcal{M}_{\rm tot} = \mathcal{M}_1 \pm \mathcal{M}_2 \pm \cdots$$





For final states that aren't identical particles, we do not sum matrix elements.

Instead, we have to sum observables!

Eg, cross sections

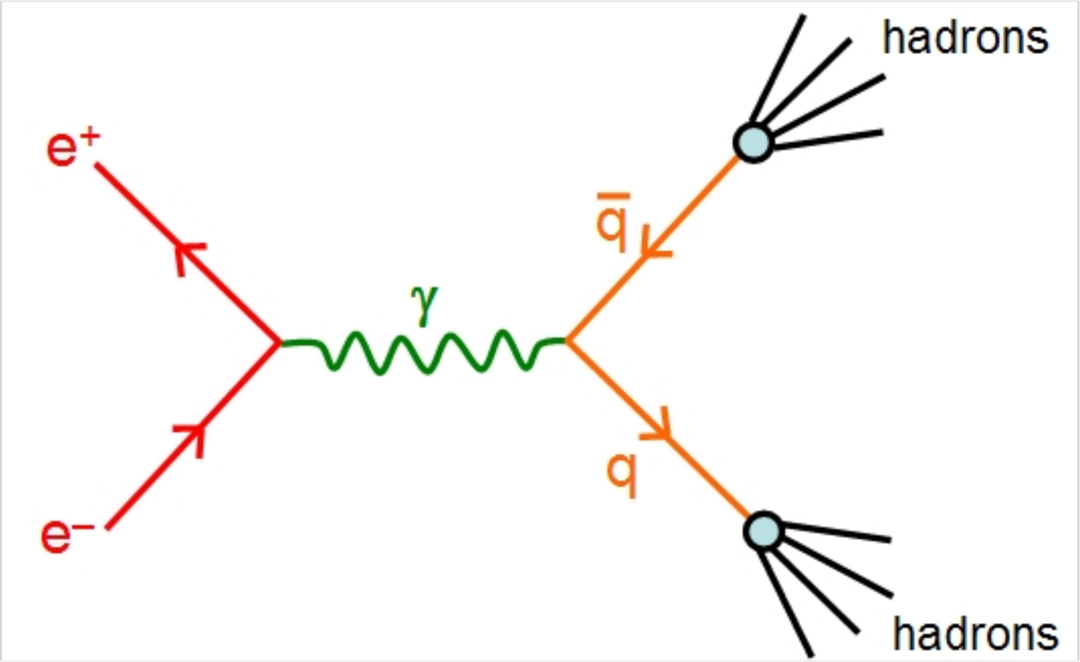
You still need to do individual matrix elements correctly, though!

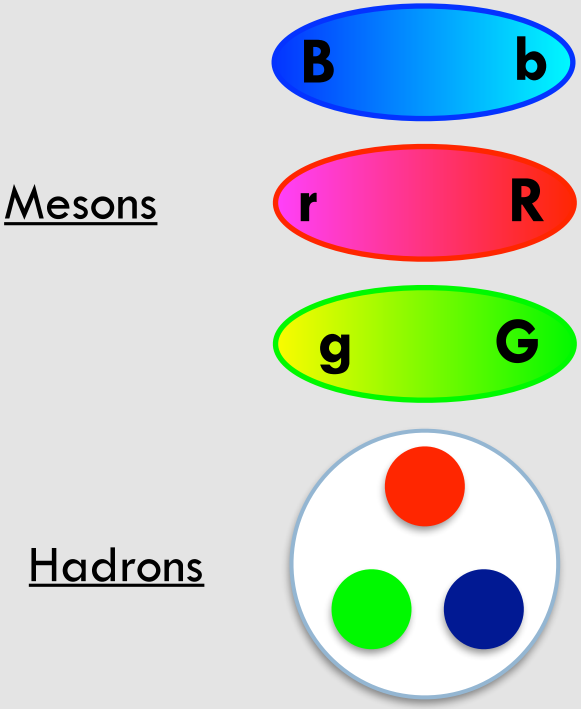
$$\sigma_{\rm tot} = \sigma_1 + \sigma_2 + \cdots$$

Why are we summing?

Why are we summing over quark flavors anyway? Why not just count each separately?

Short answer: because of quark confinement!

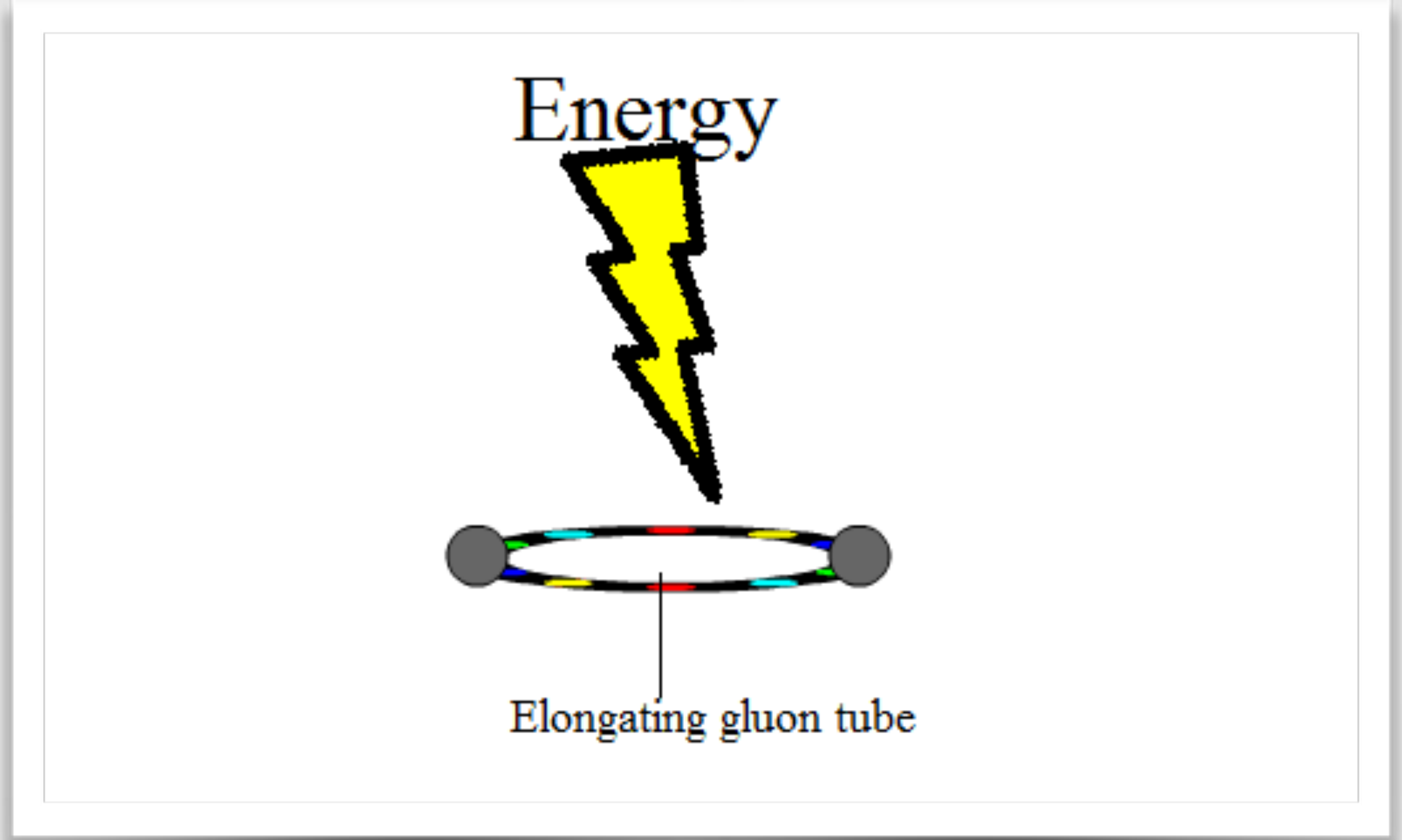




Individual quarks cannot exist in unbound states

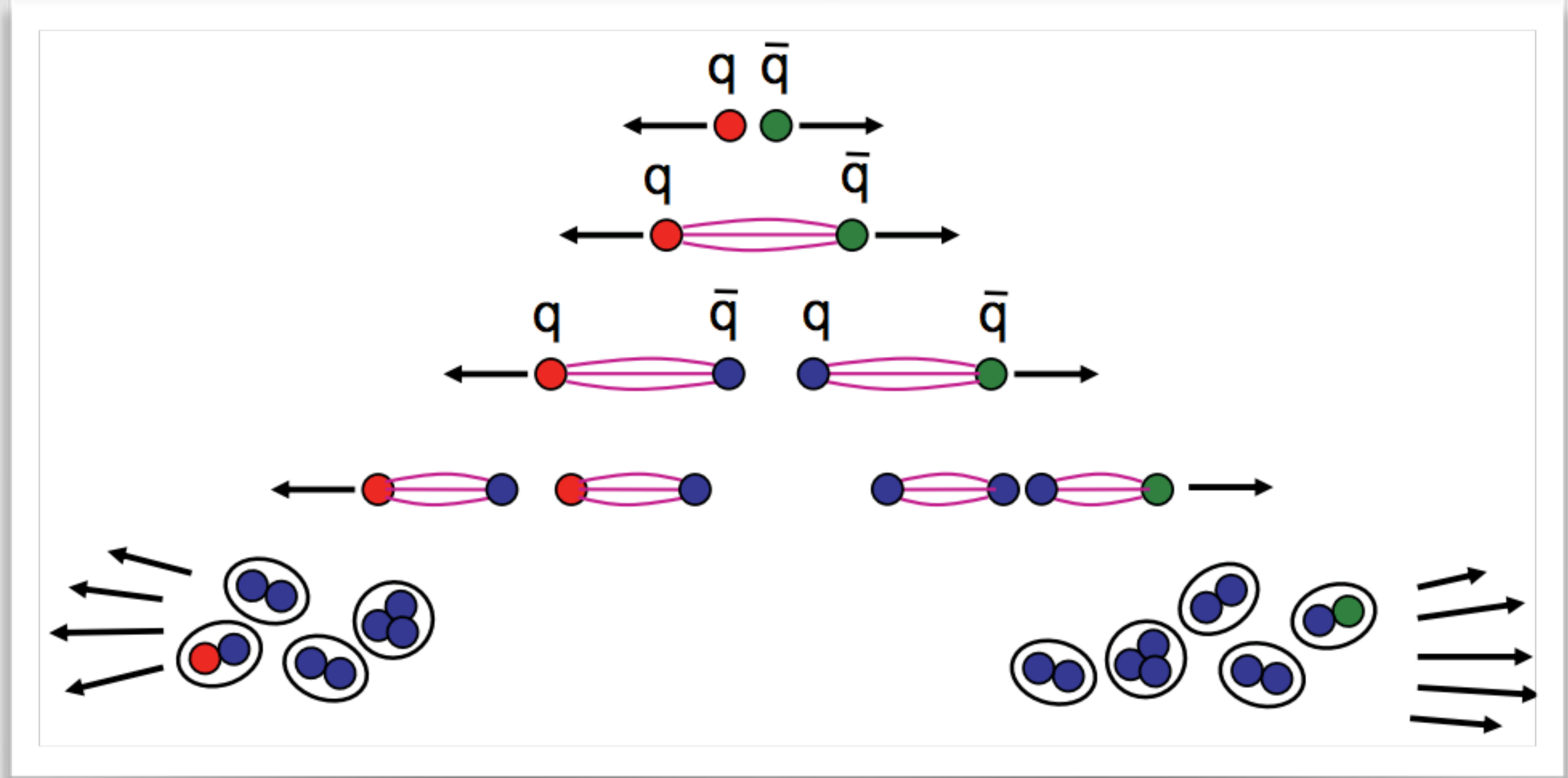
Because bare color is not allowed (don't forget color!)

Color Confinement



Color Confinement

Hadronic Jets



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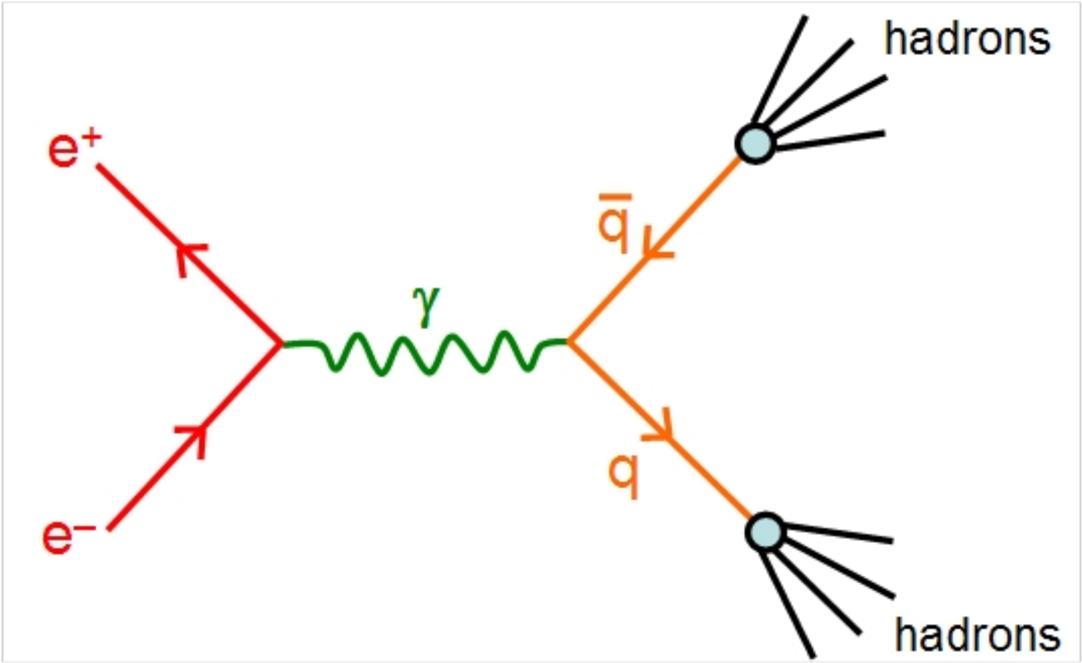
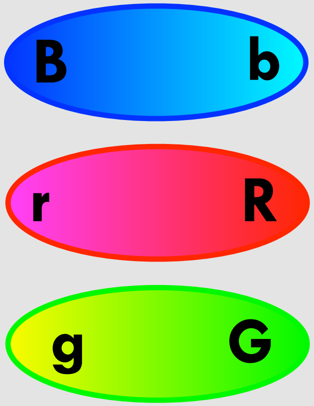


Mesons



Hadrons

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Don’t forget to sum over colors

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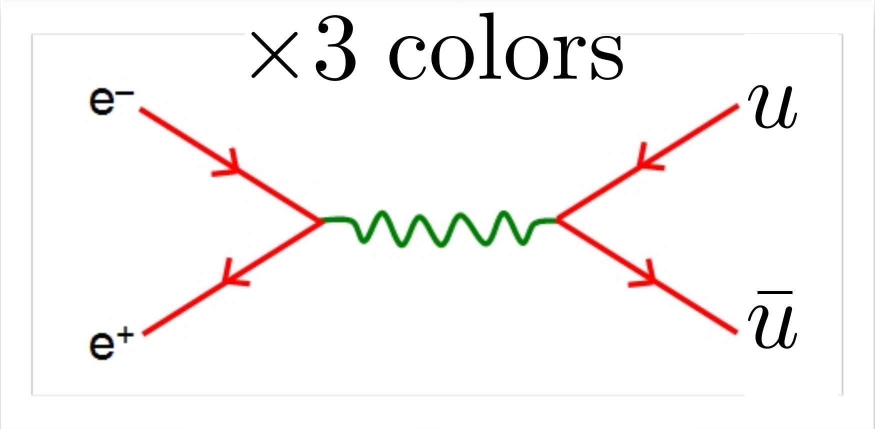
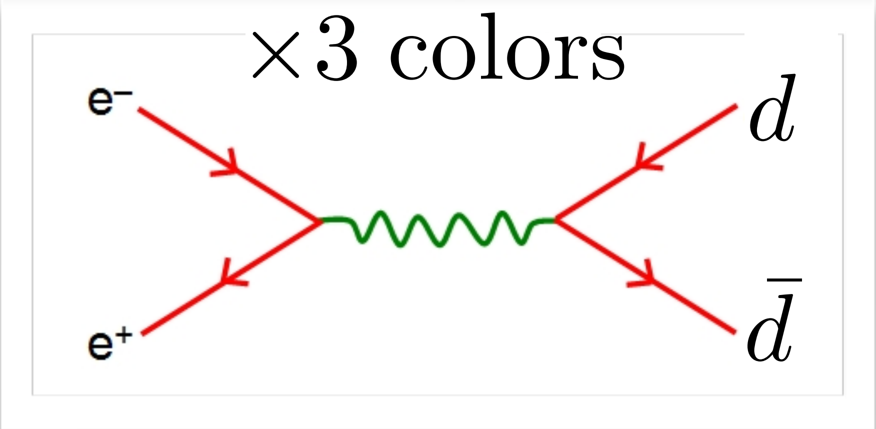




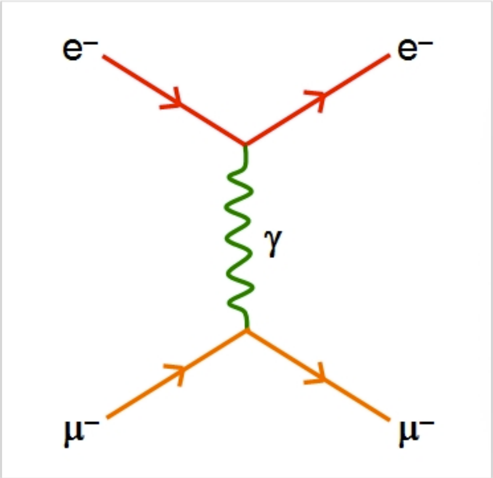
Ultimately, the total cross section is the sum over color states (3), flavor states (depends on energy) of each cross section (depends on quark charge).

$$\sigma_{\text{tot}} = \sum_{\text{c=color}} \sum_{\text{f=flavor}} \sigma_{\text{cf}}(Q_q^2)$$

Closer to QCD: Proton Scattering

If the proton didn't have structure, we could just recycle everything we did for electron-muon scattering (again!).

It turns out that the proton internal structure is interesting from the point of view of understanding QCD. So it's something we study.



$$\begin{aligned}\left\langle|\mathcal{M}|^2\right\rangle &= \frac{g_e^4}{4(p_1-p_3)^4}\left[4\left(p_1^\mu p_3^\nu + p_3^\mu p_1^\nu + (m^2 - p_1\cdot p_3)g^{\mu\nu}\right)\right] \\ &\qquad\qquad \times \left[4\left(p_{2\mu}p_{4\nu} + p_{4\mu}p_{2\nu} + (M^2 - p_2\cdot p_4)g_{\mu\nu}\right)\right. \\ &= \frac{g_e^4}{q^4}\, L^{\mu\nu}_{\text{electron}}\, L_{\mu\nu\ \text{muon}}\end{aligned}$$

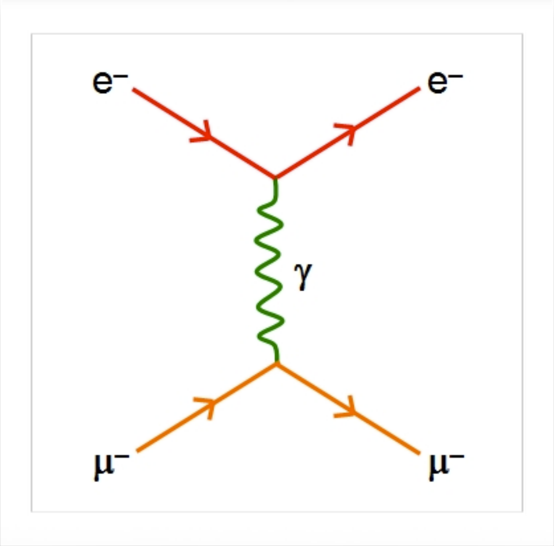
$$q = p_1 - p_3$$

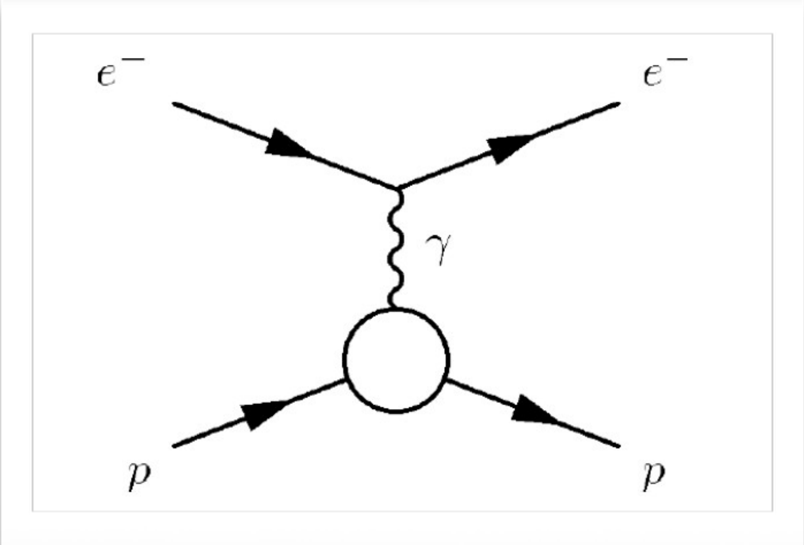
$$L^{\mu\nu}_{\text{electron}} = 2\left(p_1^\mu p_3^\nu + p_3^\mu p_1^\nu + (m^2 - p_1\cdot p_3)g^{\mu\nu}\right)$$

Closer to QCD: Proton Scattering

But the proton isn't structureless!

- Instead of just replacing $L_{\mu\nu}$ (muon) with $L_{\mu\nu}$ (proton), which assumes that the proton is a point particle, we can account for proton structure via a form factor, $K_{\mu\nu}$.
- Notice that the implied proton structure does not affect the electron-photon coupling or the photon propagator. All complications are neatly stashed within $K_{\mu\nu}$.





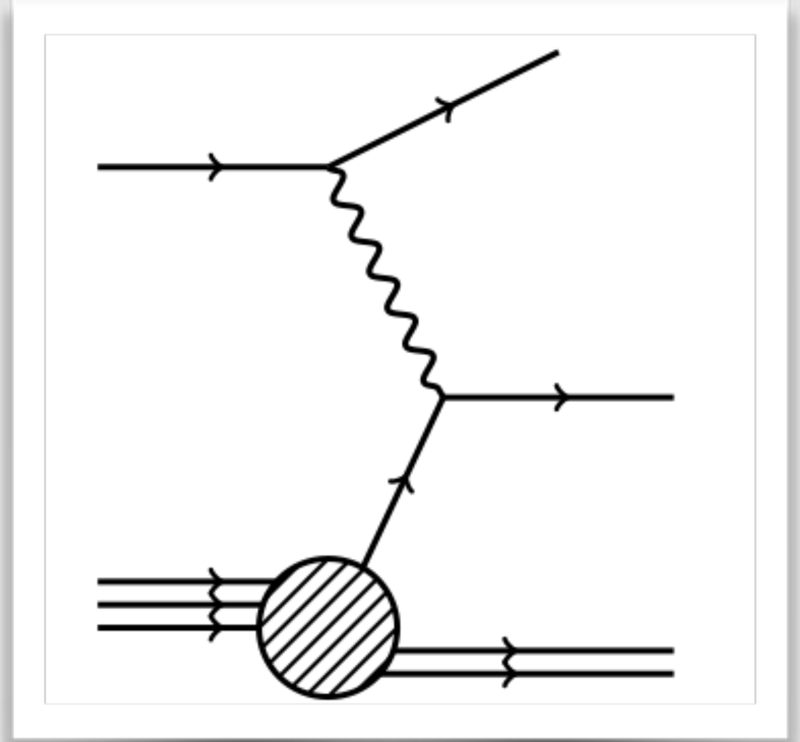
$$\left\langle |\mathcal{M}|^2 \right\rangle = \frac{g_e^4}{q^4} L^{\mu\nu}_{\text{electron}} L_{\mu\nu \text{ muon}}$$

$$\left\langle |\mathcal{M}|^2 \right\rangle = \frac{g_e^4}{q^4} L^{\mu\nu}_{\text{electron}} K_{\mu\nu \text{ proton}}$$

Deep Inelastic Scattering

If the incident electron is sufficiently energetic, it is quite unlikely that the proton will stay intact. Instead, we should be considering the more general inelastic process $e + p \rightarrow e + X$

At very high energies, this is referred to as "Deep Inelastic Scattering", with the notion that we're now probing the deep structure of the proton.



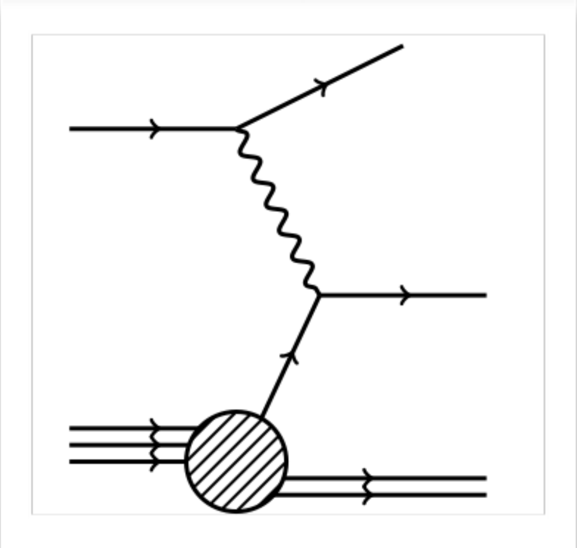
Deep Inelastic Scattering

As with the elastic case, we introduce a second-rank tensor $W_{\mu\nu}$ to describe the unknown details about the relevant subprocesses.
The electron vertex and the photon propagator are known, therefore we have:

$$\left\langle |\mathcal{M}|^2 \right\rangle = \frac{g_e^4}{q^4} L^{\mu\nu}_{\text{electron}} W_{\mu\nu}(X)$$

Together with the proton form factor, the proton structure function (W) tells us a great deal about what is happening inside the proton.

Inelastic proton collisions have shown us that there is a roiling sea of gluons and quark/anti-quark pairs inside hadrons.



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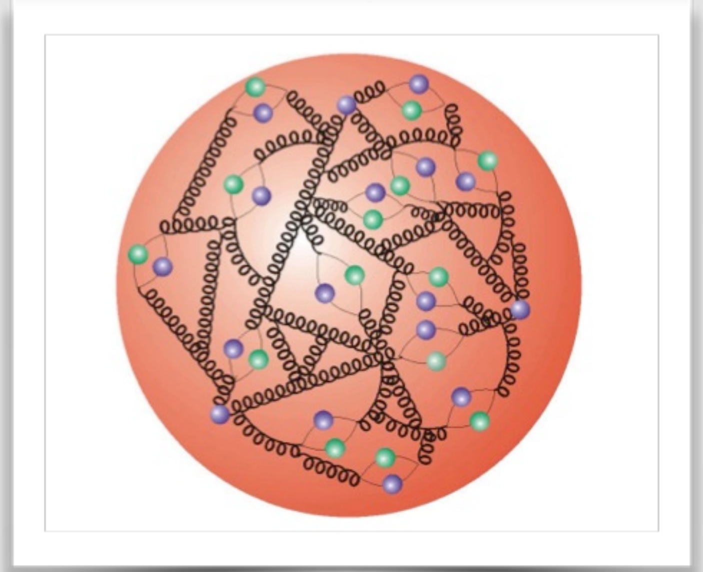
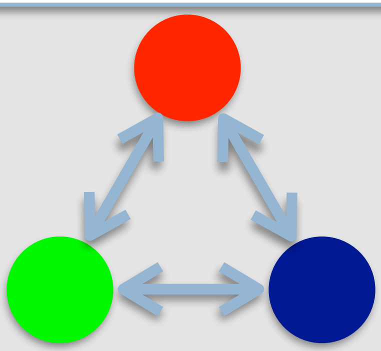


QCD and Color

- Each quark carries a color charge: red, blue or green
- The coupling strength is the same for all three colors.
- To describe a quark, use a spinor plus a color column vector:

$$r = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad b = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad g = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

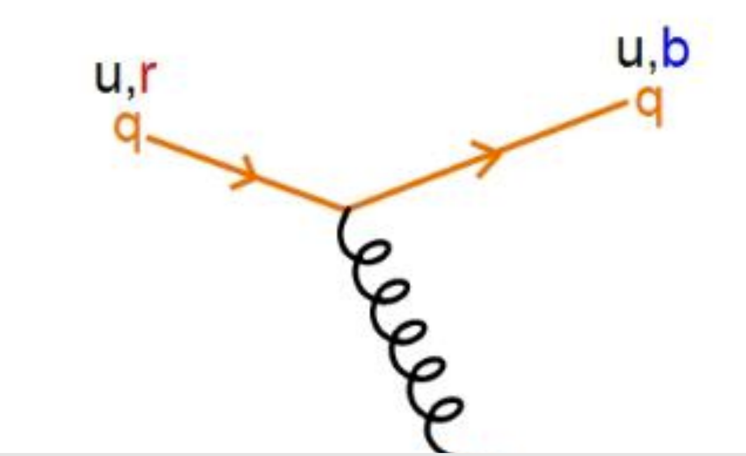
Mathematically, this is described by an SU(3) symmetry. QCD interactions are invariant under SU(3) rotations in color space:



Gluons and Color

Gluons are responsible for exchanging momentum and color between quarks.

- Gluons are massless, spin-1 particles (just like the photon!)
- Color charge is conserved at gluon vertices, as electric charge is conserved at photon vertices
- But unlike the photon, which has no electric charge, which means each gluon must contain color and anticolor.



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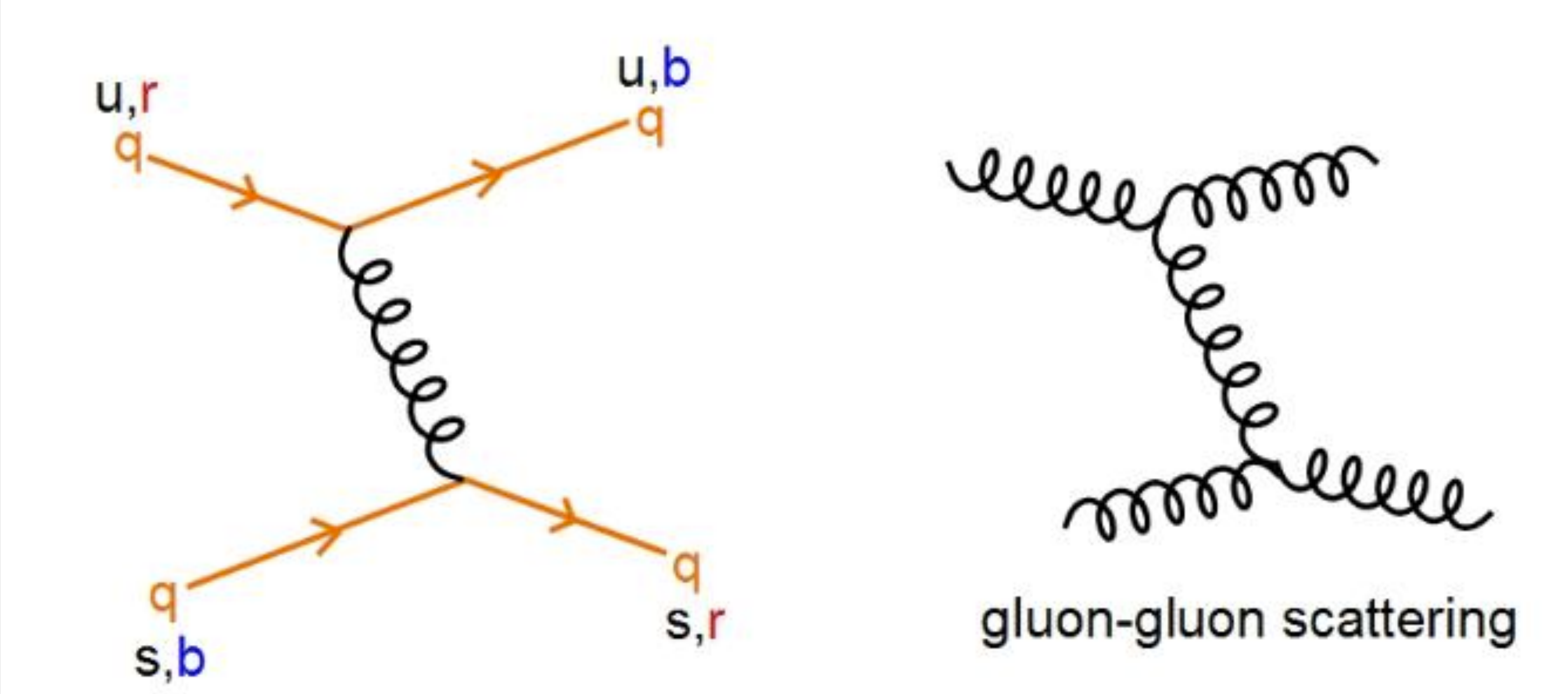
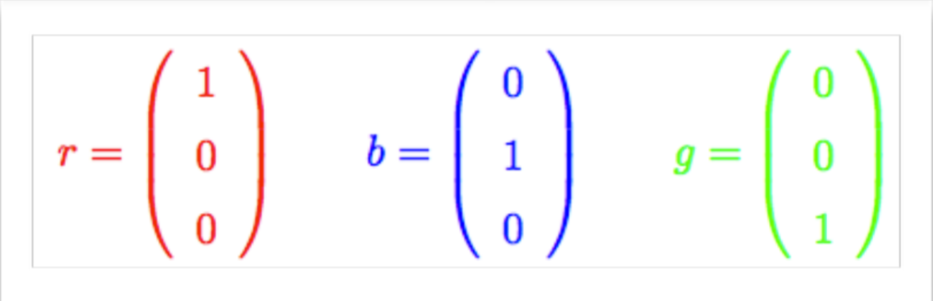


More on Color

As noted, we represent r,g,b SU(3) color states by:

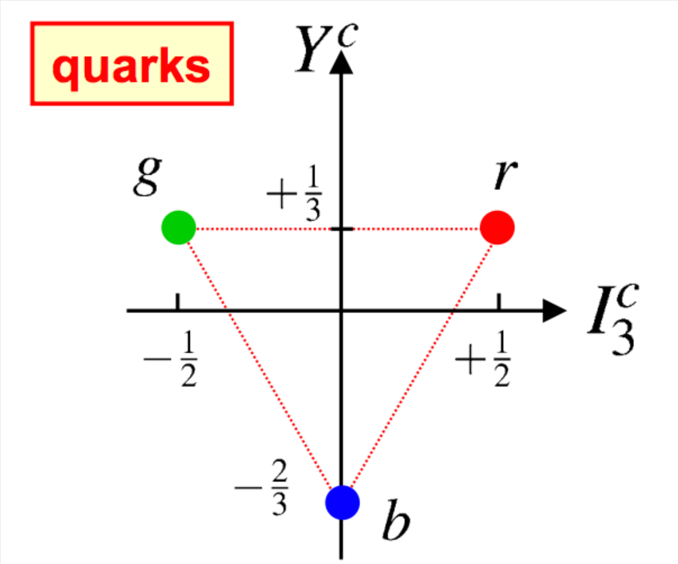


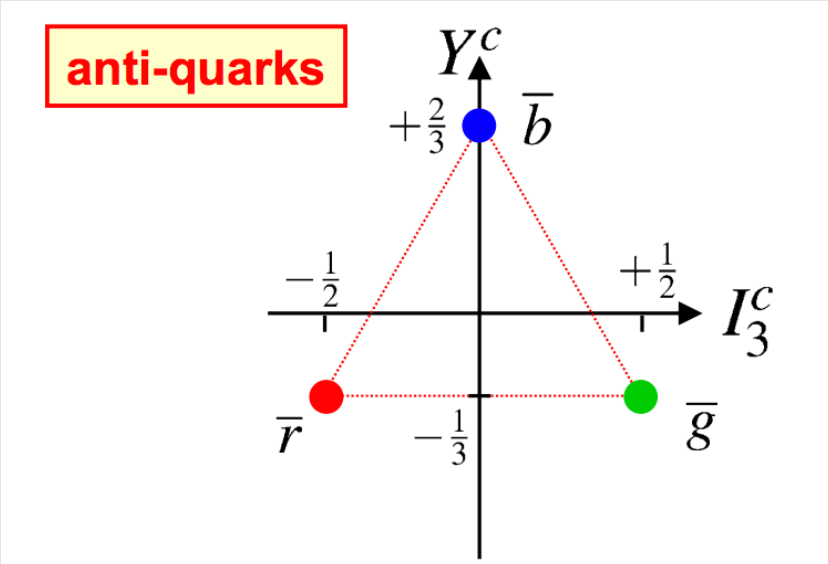
$$r = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad b = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad g = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Color states can be labelled by two quantum numbers:

color isospin I_3^c color hypercharge Y^c

Each quark (anti-quark) can have the following color quantum numbers:



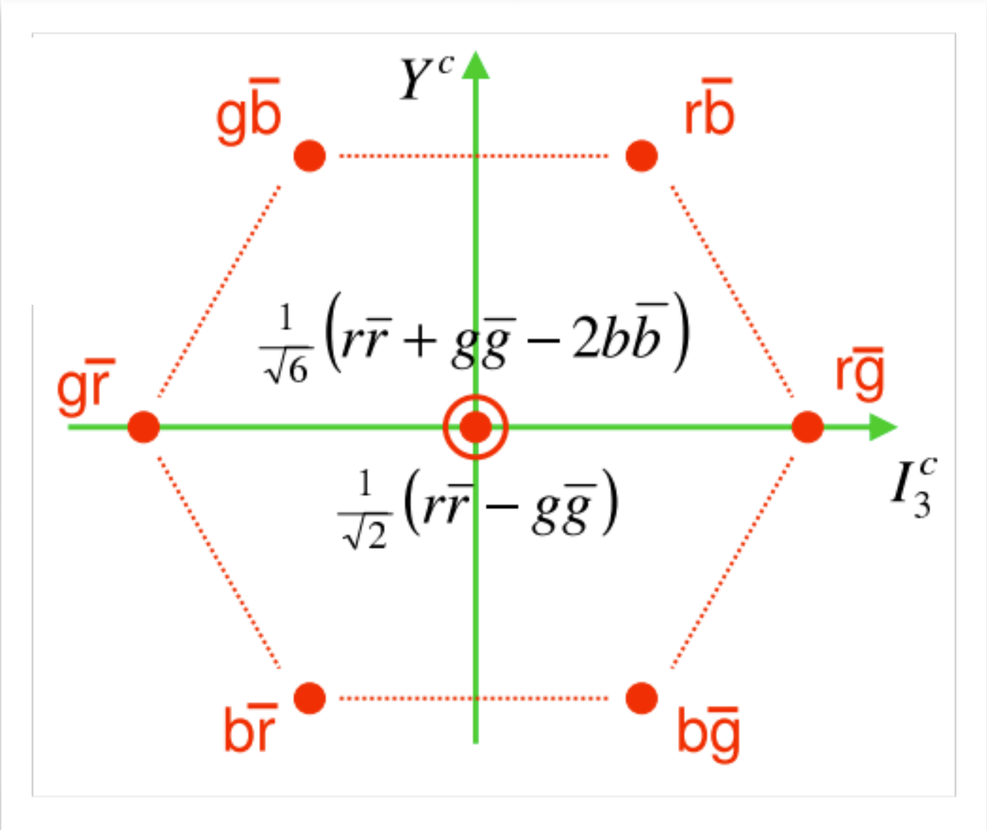


Gluons and Color

Gluons are described by the generators of the SU(3) group, giving eight linear color/anti-color combinations: a color octet

And also one combination that is symmetric upon rotations in color space.

Color Octet



Color Singlet

$$|9>=\frac{1}{\sqrt{3}}(r\bar{r}+b\bar{b}+g\bar{g})$$

This is one representation. There are others!

Color Singlets

It is important to understand what is meant by a **singlet** state.

Consider spin states obtained from two spin 1/2 particles.

Four spin combinations:

$$\uparrow\uparrow,\ \uparrow\downarrow,\ \downarrow\uparrow,\ \downarrow\downarrow$$

Gives four eigenstates of

$$\hat{S}^2,\ \hat{S}_z$$

$$|1,+1\rangle = \uparrow\uparrow$$
$$|1,0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$$
$$|1,-1\rangle = \downarrow\downarrow$$

spin-1 triplet

$$|0,0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$$

spin-0 singlet

The singlet state is "spinless": it has zero angular momentum, is invariant under SU(2) spin transformations and spin ladder operators yield zero.

$$S_{\pm}|0,0\rangle = 0$$

In the same way color singlets are "colorless" combinations:

- they have zero color quantum numbers
- invariant under SU(3) color transformations
- NOT sufficient to have $I_3^c = 0,\ Y^c = 0$: does not mean that state is a singlet

Hadron Color Singlets

The structure of SU(3) requires that only color singlets can exist as free particles.

This gives rise to our observed hadron spectrum: mesons and baryons

Notice carefully the difference between “colorless” and “color singlet”

$$\frac{1}{\sqrt{3}}(r\bar{r} + b\bar{b} + g\bar{g})$$

Meson color singlet

What about a singlet gluon??

$$\frac{1}{\sqrt{6}}(rgb - grb + gbr - bgr + brg - rbg)$$

Baryon color singlet

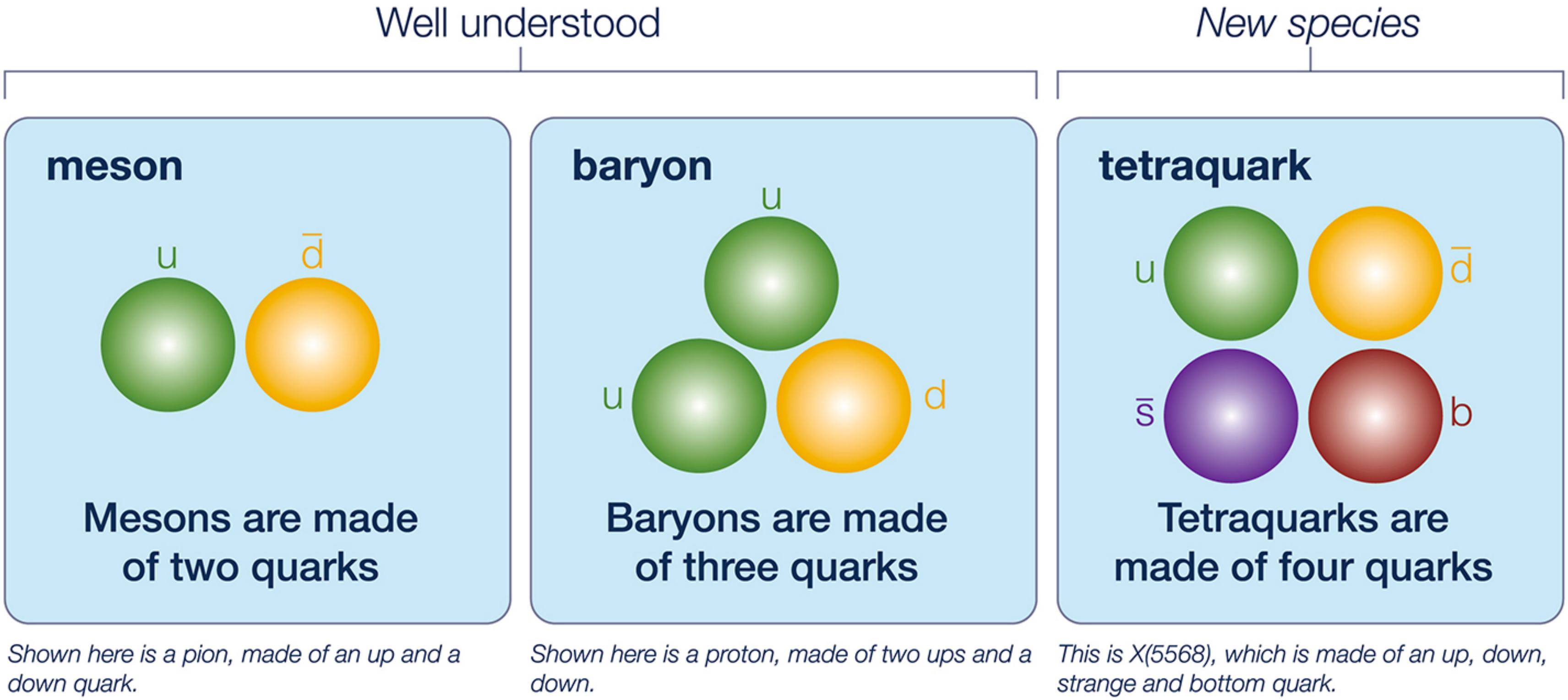
$$q\bar{q} \qquad qqq \qquad \underline{q\bar{q}q\bar{q}} \qquad \underline{qqqq\bar{q}}$$

There are many singlet possibilities.

Observed?

Not (yet?) observed

Tetraquarks?



Mesons are made of two quarks

Baryons are made of three quarks

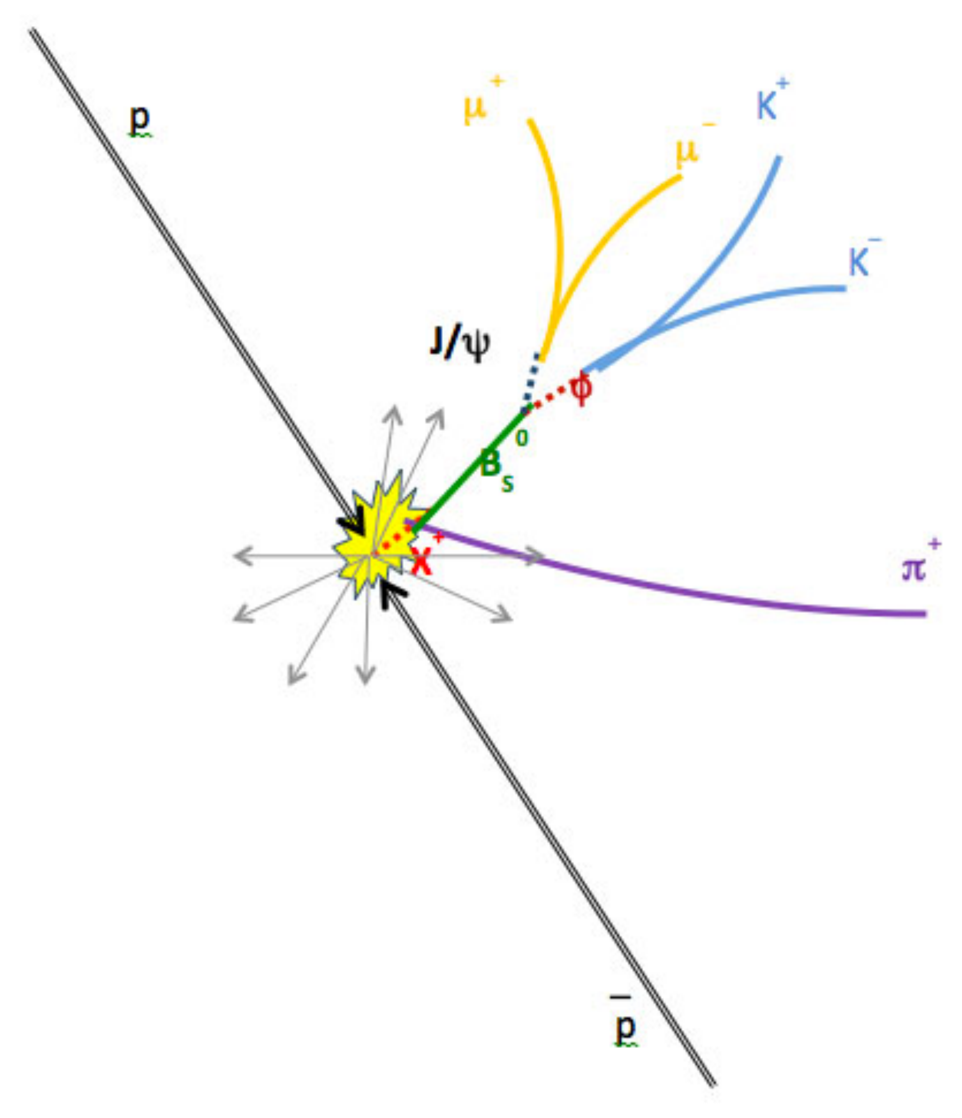
Tetraquarks are made of four quarks

Shown here is a pion, made of an up and a down quark.

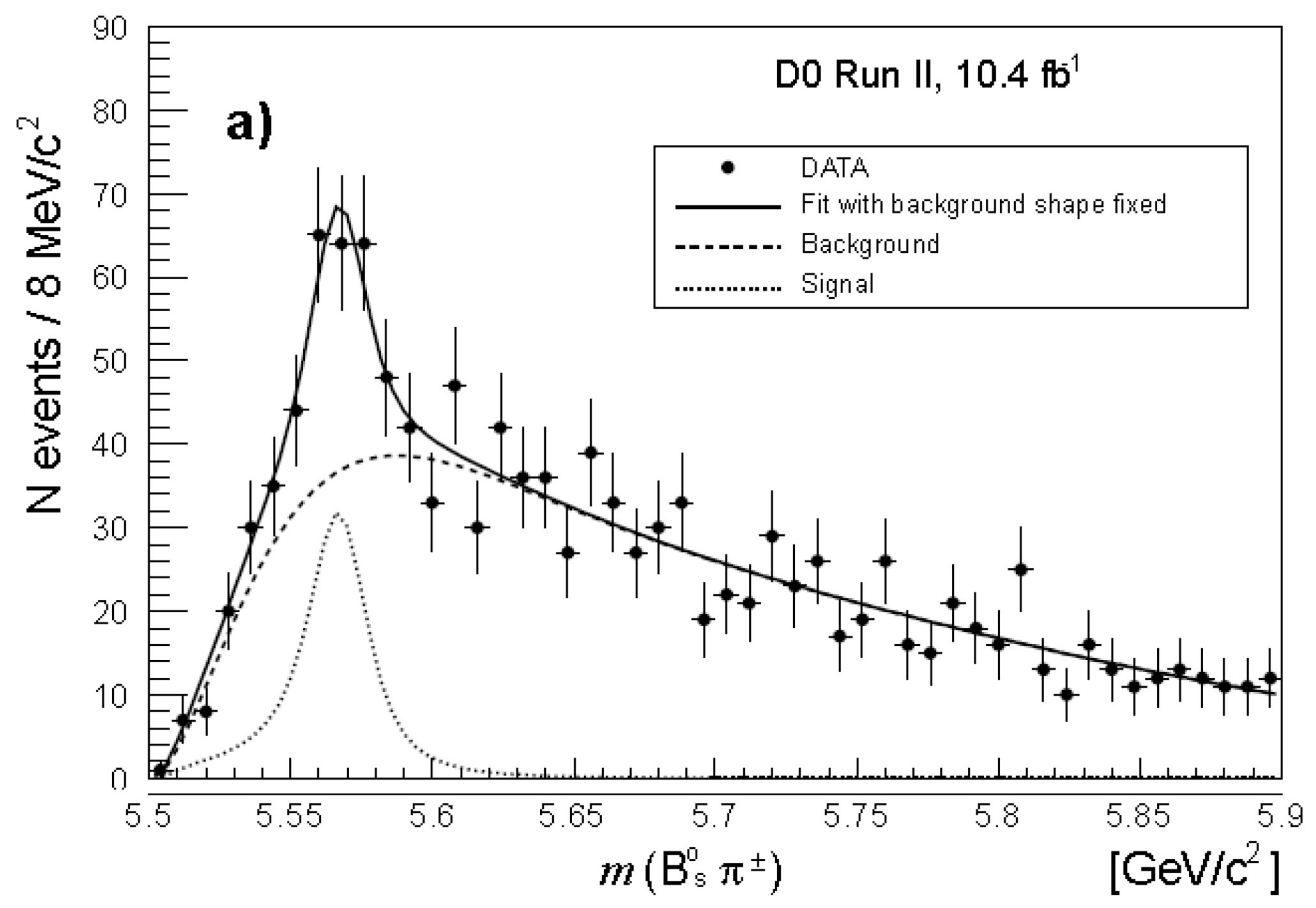
Shown here is a proton, made of two ups and a down.

This is X(5568), which is made of an up, down, strange and bottom quark.

Tetraquarks?

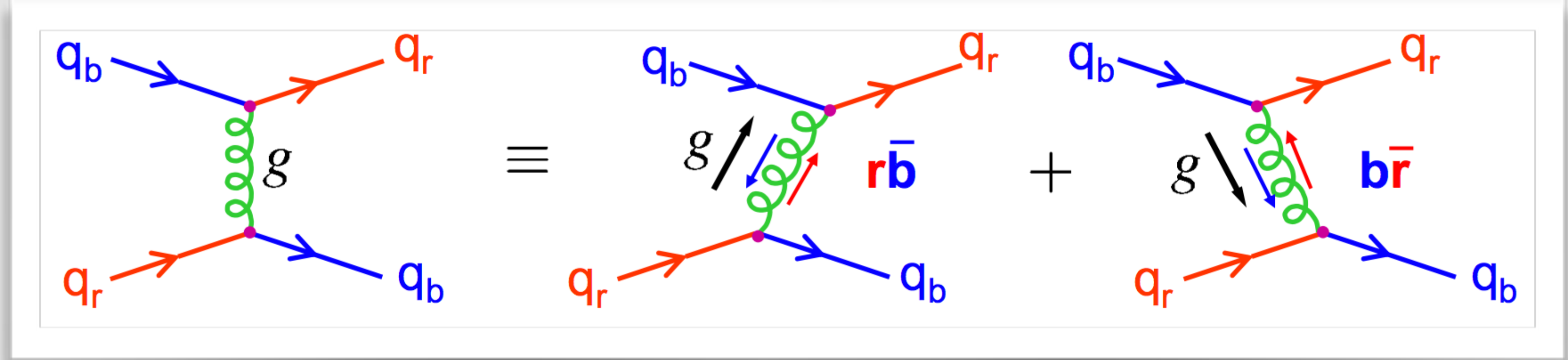


Tetraquarks?



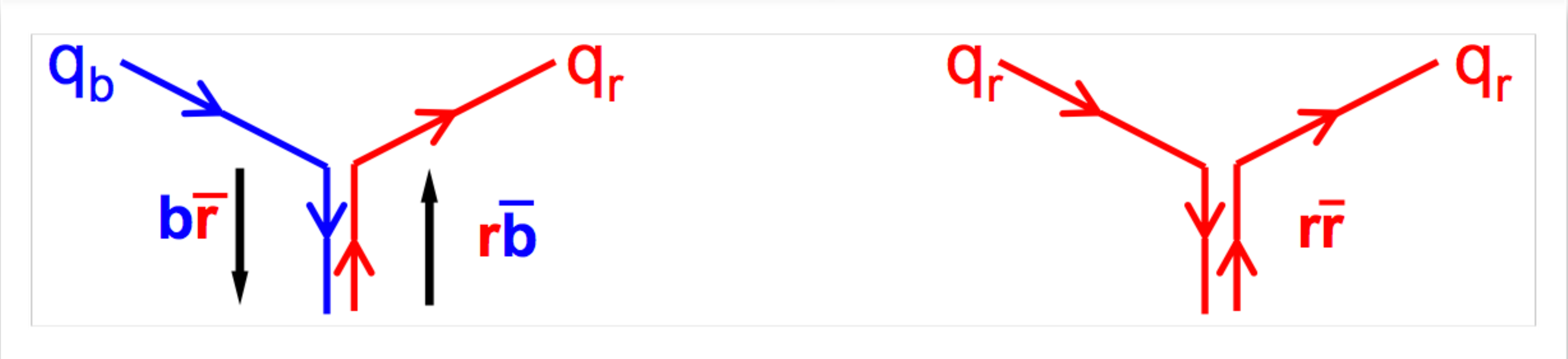
Gluons and Color

In QCD quarks interact by exchanging virtual massless gluons



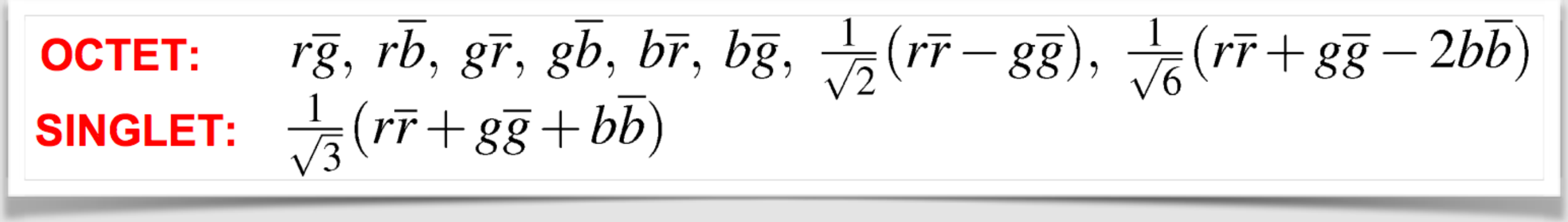
Gluons carry both color and anti-color.

Useful to think of this in terms of color flow or a "color current".



Gluons and Color

We could justifiably have expected 9 physical gluons of color/anti-color format



OCTET: $r\bar{g},\ r\bar{b},\ g\bar{r},\ g\bar{b},\ b\bar{r},\ b\bar{g},\ \frac{1}{\sqrt{2}}(r\bar{r}-g\bar{g}),\ \frac{1}{\sqrt{6}}(r\bar{r}+g\bar{g}-2b\bar{b})$

SINGLET: $\frac{1}{\sqrt{3}}(r\bar{r}+g\bar{g}+b\bar{b})$

But consider the color confinement hypothesis:

only colour singlet states can exist as free particles ⇨ **Colour singlet gluon would be unconfined. It would behave like a strongly interacting photon ➡ infinite range Strong force.**

Empirically, the strong force is short range and therefore know that the physical gluons are confined. The color singlet state does not exist in nature !

This is not entirely ad hoc!

In the context of field theory the strong interaction arises from a fundamental SU(3) symmetry. The gluons arise from the generators of the symmetry group (the Gell-Mann matrices). These 8 matrices give rise to 8 gluons.

> Had nature "chosen" a U(3) symmetry, we'd get 9 gluons. The extra gluon would be the color singlet state and QCD would be an unconfined long-range force.

Gluons and Color

The Gell-Mann matrices describe the allowed color configurations of gluons.

The Gell-Mann matrices are the generators of the SU(3) symmetry.

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}}(r\bar{r} - g\bar{g})$$

$\mathrm{r}\bar{\mathrm{g}}$ $\quad$ $\bar{\mathrm{r}}\mathrm{g}$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$\mathrm{r}\bar{\mathrm{b}}$ $\quad$ $\bar{\mathrm{r}}\mathrm{b}$ $\quad$ $\mathrm{g}\bar{\mathrm{b}}$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \frac{1}{\sqrt{6}}(r\bar{r} + b\bar{b} - 2g\bar{g})$$

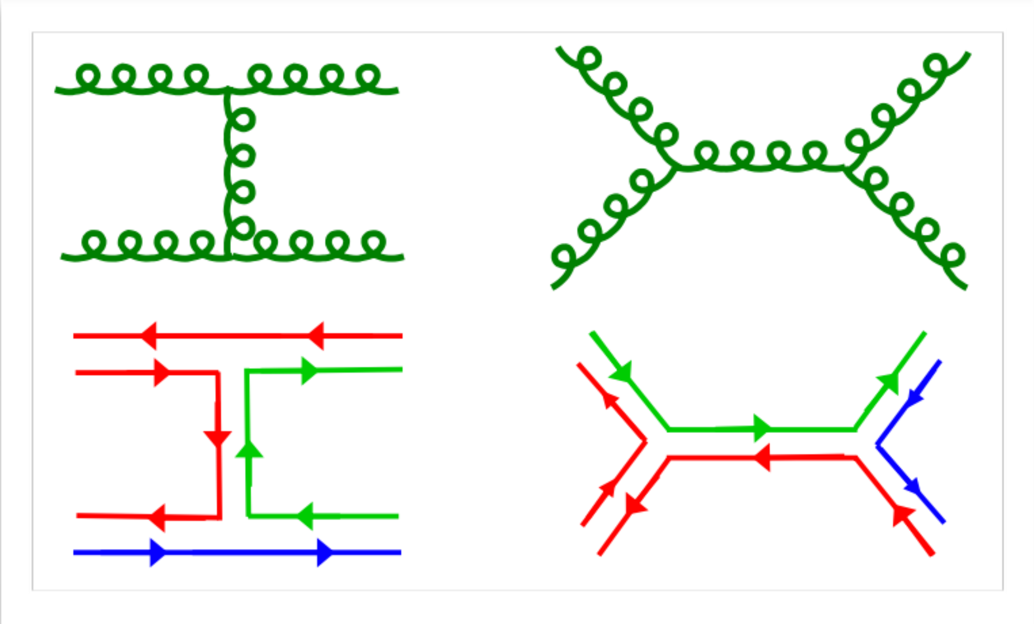
$\mathrm{b}\bar{\mathrm{g}}$

Thus the gluon octet can simply be described as a contraction of the Gell-Mann matrices over color/anti-color indices

$$g^i = \begin{pmatrix} \mathrm{r} & \mathrm{g} & \mathrm{b} \end{pmatrix} \lambda^i \begin{pmatrix} \bar{\mathrm{r}} \\ \bar{\mathrm{g}} \\ \bar{\mathrm{b}} \end{pmatrix}$$

Gluon-Gluon Interactions

Gluons have color, so they participate in the QCD interaction with other gluons!
We typically refer to this as "self-interaction". There are two diagrams:

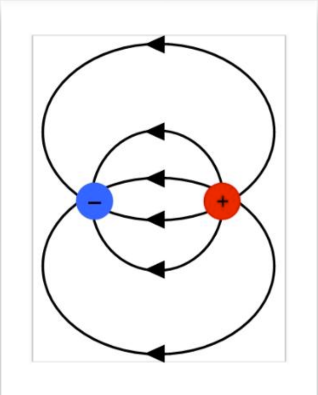


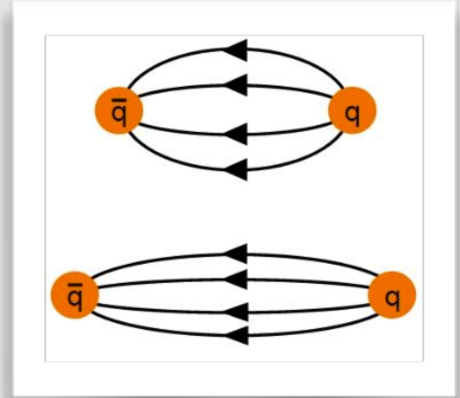
In addition to quark-quark scattering, therefore can have gluon-gluon scattering

Gluons and Confinement

Gluon self-interactions are believed to give rise to color confinement.
The qualitative picture can be envisioned by comparing QED to QCD:

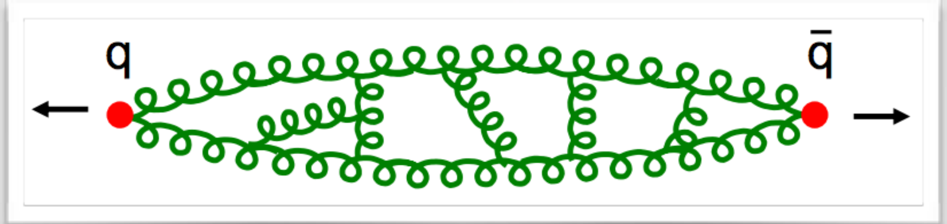
In QCD gluon self-interactions squeeze lines of force into a "flux tube"





What happens when we try to separate two colored objects

- A gluon flux tube of interacting gluons is formed. Energy ~1 GeV/fm.
- We need infinite energy to separate colored objects to infinity
- Colored quarks and gluons are always confined within colorless states

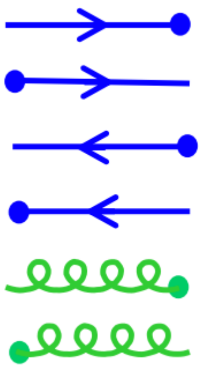


$$V(r) \sim \lambda r$$

Feynman Rules for QCD

External lines:

spin 1/2	incoming quark	$u(p)$
	outgoing quark	$\overline{u}(p)$
	incoming anti-quark	$\overline{v}(p)$
	outgoing anti-quark	$v(p)$
spin 1	incoming gluon	$\varepsilon^{\mu}(p)$
	outgoing gluon	$\varepsilon^{\mu}(p)^{*}$



Internal lines (propagators):

spin 1/2 quark

$$\boxed{\frac{i(\not{q}+m)}{q^2-m^2}}$$

spin 1 gluon

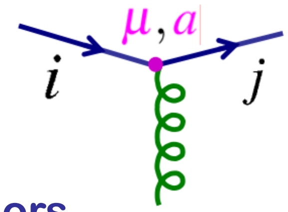
$$\boxed{\frac{-ig_{\mu\nu}}{q^2}\delta^{ab}}$$

a, b = 1,2,…,8 are gluon color indices

Vertex factors:

spin 1/2 quark

$$\boxed{-ig_s\tfrac{1}{2}\lambda^{a}_{ji}\gamma^{\mu}}$$



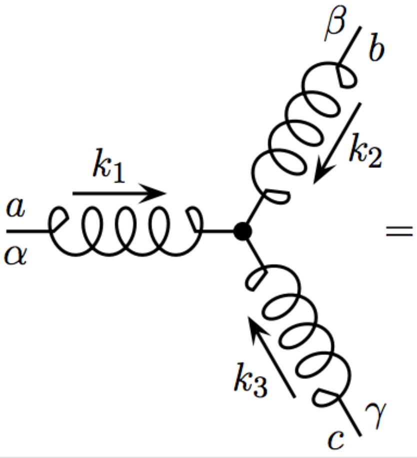
i, j = 1,2,3 are quark colors,

λ^a a = 1,2,..8 are the Gell-Mann SU(3) matrices

Feynman Rules for QCD

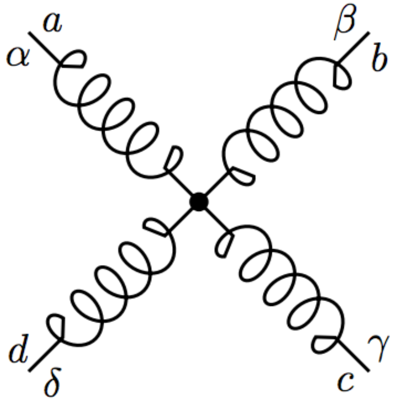
We also have the 3- and 4-gluon vertex factors.

We will look at these here, but not focus on them in this course.



$$= -g f^{abc}\left[g^{\alpha\beta}(k_1-k_2)^{\gamma} + g^{\beta\gamma}(k_2-k_3)^{\alpha} + g^{\gamma\alpha}(k_3-k_1)^{\beta}\right]$$

$$[\lambda^{\alpha}, \lambda^{\beta}] = 2i\, f^{\alpha\beta\gamma}\, \lambda^{\gamma}$$



$$= -ig^2 \left[\begin{array}{l} f^{abe} f^{cde}(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma}) \\ + f^{ace} f^{bde}(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\delta}g^{\gamma\beta}) \\ + f^{ade} f^{bce}(g^{\alpha\beta}g^{\delta\gamma} - g^{\alpha\gamma}g^{\delta\beta}) \end{array}\right]$$

The Quark-Gluon Interaction

We need to modify the spin-1/2 solutions to the Dirac equation with color vectors:

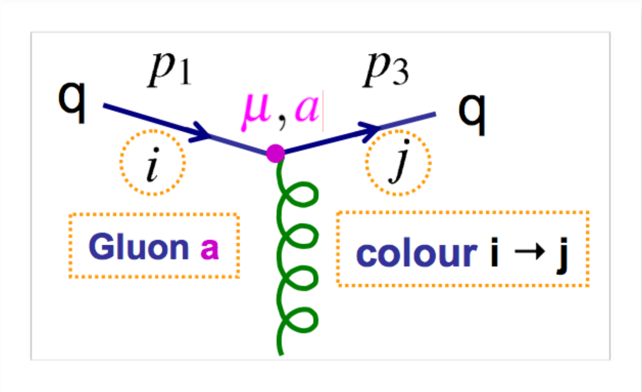
$$r = c_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad g = c_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad b = c_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$u(p) \longrightarrow c_i u(p)$$

The QCD qgq vertex is written:

$$\bar{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1)$$

Only difference w.r.t. QED is the insertion of the 3x3 SU(3) Gell-Mann matrices



Isolating the color part

$$c_j^\dagger \lambda^a c_i = c_j^\dagger \begin{pmatrix}\lambda^a_{1i}\\ \lambda^a_{2i}\\ \lambda^a_{3i}\end{pmatrix} = \lambda^a_{ji}$$

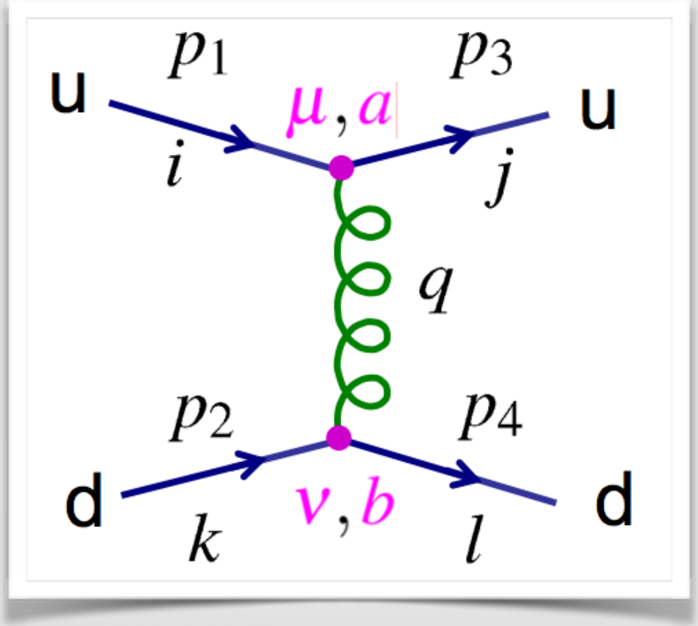
The fundamental quark - gluon QCD interaction can be written as:

$$\bar{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1) \equiv \bar{u}(p_3)\{-\tfrac{1}{2}ig_s\lambda^a_{ji}\gamma^\mu\}u(p_1)$$

Quark-Quark Scattering

Consider scattering of an up and a down quark.

- The incoming and out-going quark colors are labelled by **i,j,k,l = {1,2,3} or {r,g,b}**
- Thus in terms of color scattering, this is: **ik→jl**
- The 8 possible gluons are accounted for by the color indices: **a,b = 1,2,….,8**
- **NOTE:** The delta function in the propagator ensures **a=b**. The gluon emitted at a is the same as is absorbed at b.



Apply Feynman rules:

$$\mathcal{M} = i[\bar{u}_u c_3^\dagger]\left[-i\frac{g_s}{2}\lambda^a_{ji}\gamma^\mu\right][u_u c_1]\left(\frac{-ig_{\mu\nu}\delta^{ab}}{q^2}\right)[\bar{u}_d c_4^\dagger]\left[-i\frac{g_s}{2}\lambda^a_{lk}\gamma^\nu\right][u_d c_2]$$

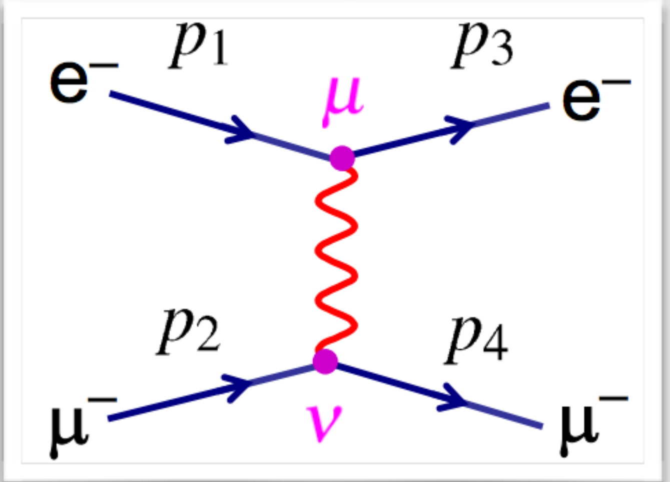
$$\mathcal{M} = -\frac{g_s^2}{4q^2}[\bar{u}_u\gamma^\mu u_u][\bar{u}_d\gamma_\mu u_d](c_j^\dagger\lambda^a c_i)(c_l^\dagger\lambda^a c_k)$$

$$\mathcal{M} = -\frac{g_s^2}{4q^2}[\bar{u}_u\gamma^\mu u_u][\bar{u}_d\gamma_\mu u_d](\lambda^a_{ji}\lambda^a_{lk})$$

Comparing QED and QCD

Lepton scattering:

$$\mathcal{M} = -\frac{g_e^2}{q^2}[\bar{u}_e \gamma^\alpha u_e][\bar{u}_\mu \gamma_\beta u_\mu]$$



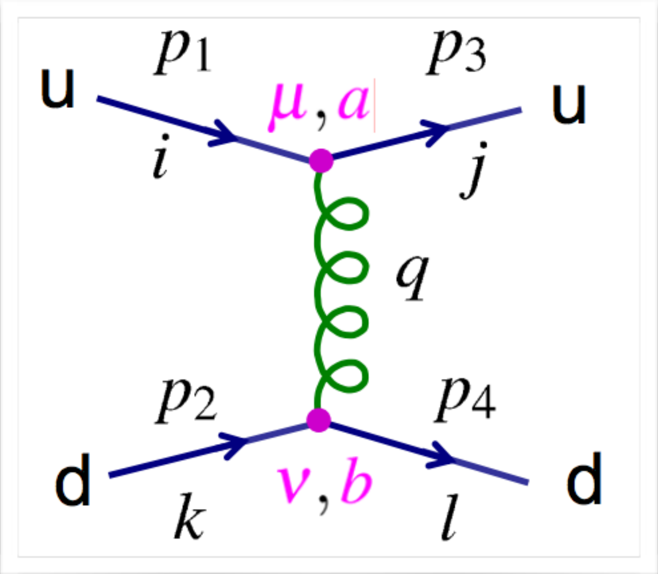
Quark scattering:

$$\mathcal{M} = -\frac{g_s^2}{4q^2}[\bar{u}_u \gamma^\mu u_u][\bar{u}_d \gamma_\mu u_d](\lambda_{ji}^a \lambda_{lk}^a)$$

QCD Matrix Element = QED Matrix Element with:

Strong coupling constant $\alpha = \frac{e^2}{4\pi} \rightarrow \alpha_s = \frac{g_s^2}{4\pi}$

Color Factor $C(ik \rightarrow jl) \equiv \frac{1}{4} \sum_{a=1}^{8} \lambda_{ji}^a \lambda_{lk}^a$

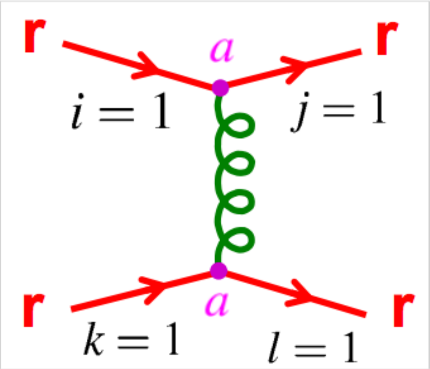


Quark-Gluon Color Factors

QCD color factors reflect the gluon states that are involved

$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ $\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$	$\lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$	$\lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$
Gluons: $r\bar{g}\,,\, g\bar{r}$	$r\bar{b}\,,\, b\bar{r}$	$g\bar{b}\,,\, b\bar{g}$	$\frac{1}{\sqrt{2}}(r\bar{r} - g\bar{g})$ $\frac{1}{\sqrt{6}}(r\bar{r} + g\bar{g} - 2b\bar{b})$

Consider a single color



$$C(rr \to rr) = \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{11}\lambda^a_{11} = \frac{1}{4}(\lambda^3_{11}\lambda^3_{11} + \lambda^8_{11}\lambda^8_{11})$$

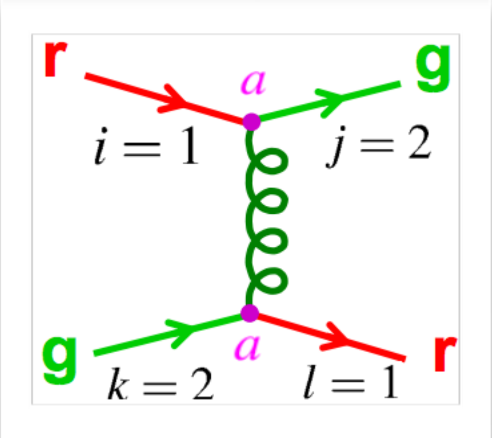
$$= \frac{1}{4}\left(1 + \frac{1}{3}\right) = \frac{1}{3}$$

$$C(rr \to rr) = C(gg \to gg) = C(bb \to bb) = \frac{1}{3}$$

Quark-Gluon Color Factors

$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ $\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$	$\lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$	$\lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$
Gluons: $r\bar{g}\,,\, g\bar{r}$	$r\bar{b}\,,\, b\bar{r}$	$g\bar{b}\,,\, b\bar{g}$	$\frac{1}{\sqrt{2}}(r\bar{r} - g\bar{g})$ $\frac{1}{\sqrt{6}}(r\bar{r} + g\bar{g} - 2b\bar{b})$

Quark Color Swap



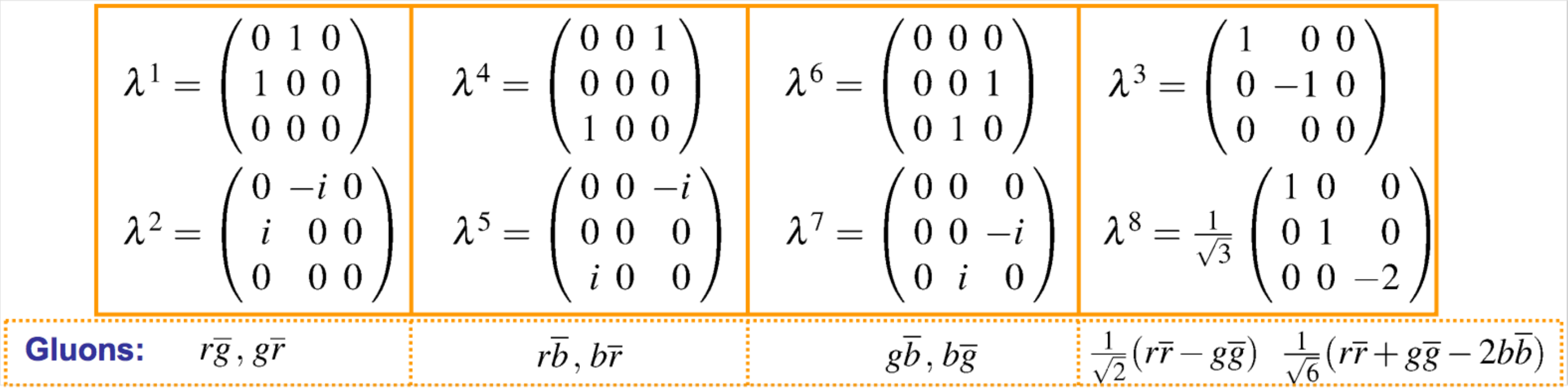
Only matrices with non-zero entries in 12 and 21 position are involved

$$C(rg \to gr) = \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{21}\lambda^a_{12} = \frac{1}{4}(\lambda^1_{21}\lambda^1_{12} + \lambda^2_{21}\lambda^2_{12})$$

$$= \frac{1}{4}(i(-i) + 1) = \frac{1}{2}$$

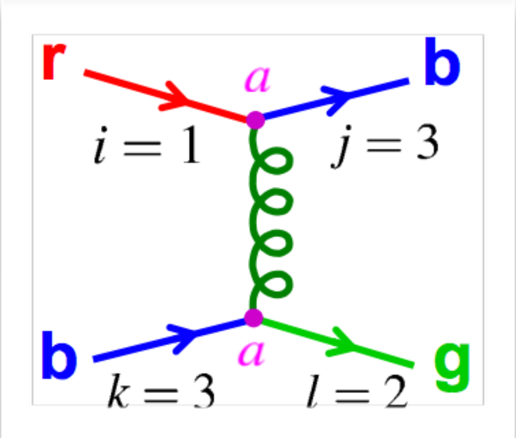
$$C(rb \to br) = C(rg \to gr) = C(gr \to rg) = C(gb \to bg) = C(br \to rb) = C(bg \to gb) = \frac{1}{2}$$

Quark-Gluon Factors



$\lambda^1 = \begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}$	$\lambda^4 = \begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}$	$\lambda^6 = \begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}$	$\lambda^3 = \begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}$
$\lambda^2 = \begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix}$	$\lambda^5 = \begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix}$	$\lambda^7 = \begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix}$	$\lambda^8 = \frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}$
Gluons: $r\bar{g}\,,\, g\bar{r}$	$r\bar{b}\,,\, b\bar{r}$	$g\bar{b}\,,\, b\bar{g}$	$\frac{1}{\sqrt{2}}(r\bar{r}-g\bar{g})$ $\frac{1}{\sqrt{6}}(r\bar{r}+g\bar{g}-2b\bar{b})$

3-Color Interchange



$$C(rb \to bg) = \frac{1}{4}\sum_{a=1}^{8}\lambda^a_{31}\lambda^a_{23} = \frac{1}{4}(\lambda^?_{31}\lambda^?_{23})$$

???

=0!!

Anti-Quark Color Factors

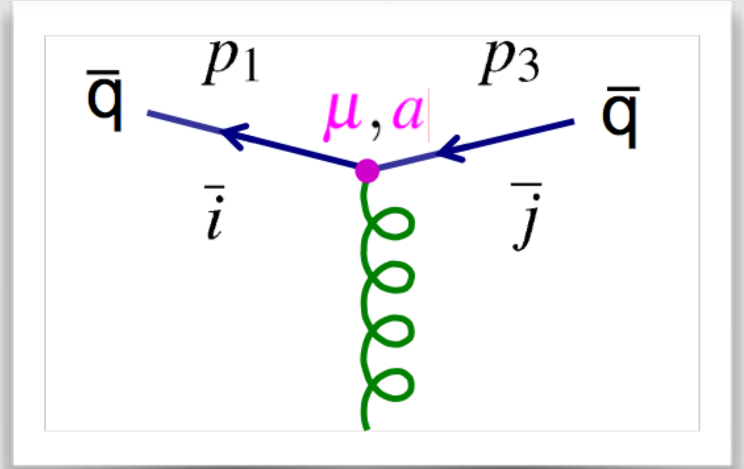
Anti-quarks get somewhat different color factors, giving a subtly different result

Recall the quark-gluon-quark vertex:

$$\bar{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_iu(p_1)$$

Consider the equivalent anti-quark vertex:

$$\bar{v}(p_1)c_i^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_jv(p_3)$$



Note that the incoming anti-particle now enters on the LHS of the expression

Quark vertex:

$$\bar{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_iu(p_1) \equiv \bar{u}(p_3)\{-\tfrac{1}{2}ig_s\lambda^a_{ji}\gamma^\mu\}u(p_1)$$

Anti-quark vertex:

$$\bar{v}(p_1)c_i^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_jv(p_3) \equiv \bar{v}(p_1)\{-\tfrac{1}{2}ig_s\lambda^a_{ij}\gamma^\mu\}v(p_3)$$

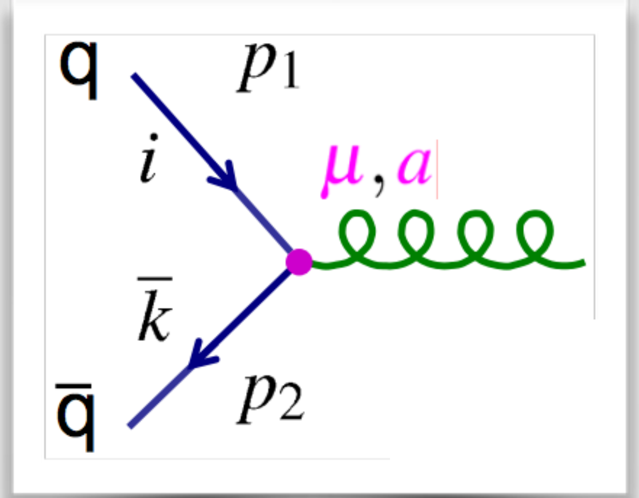
Quark / Anti-Quark Annihilation

Quark-antiquark vertex:

$$\bar{v}(p_2)c_k^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1)$$

Color contraction:

$$c_k^\dagger\lambda^a c_i = \lambda^a_{ki}$$



Quark-antiquark vertex:

$$\bar{v}(p_2)c_k^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1) \equiv \bar{v}(p_2)\{-\tfrac{1}{2}ig_s\lambda^a_{ki}\gamma^\mu\}u(p_1)$$

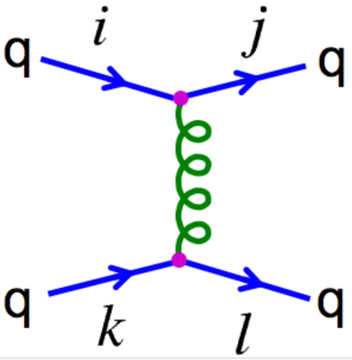
Quark vertex:

$$\bar{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1) \equiv \bar{u}(p_3)\{-\tfrac{1}{2}ig_s\lambda^a_{ji}\gamma^\mu\}u(p_1)$$

Anti-quark vertex:

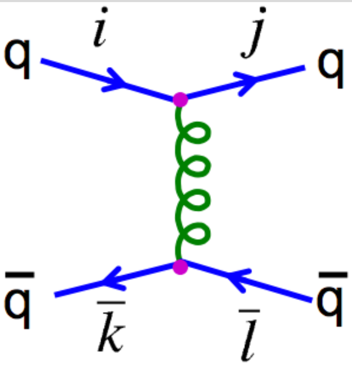
$$\bar{v}(p_1)c_i^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_j v(p_3) \equiv \bar{v}(p_1)\{-\tfrac{1}{2}ig_s\lambda^a_{ij}\gamma^\mu\}v(p_3)$$

Color Factor Summary



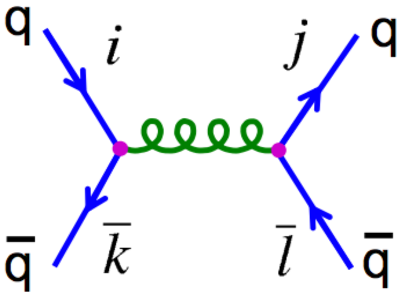
$$\boxed{C(ik \to jl) \equiv \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{ji}\lambda^a_{lk}}$$

$$C(rr \to rr) = \tfrac{1}{3}$$
$$C(rg \to rg) = -\tfrac{1}{6}$$
$$C(rg \to gr) = \tfrac{1}{2}$$



$$\boxed{C(i\bar{k} \to j\bar{l}) \equiv \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{ji}\lambda^a_{kl}}$$

$$C(r\bar{r} \to r\bar{r}) = \tfrac{1}{3}$$
$$C(r\bar{g} \to r\bar{g}) = -\tfrac{1}{6}$$
$$C(r\bar{r} \to g\bar{g}) = \tfrac{1}{2}$$



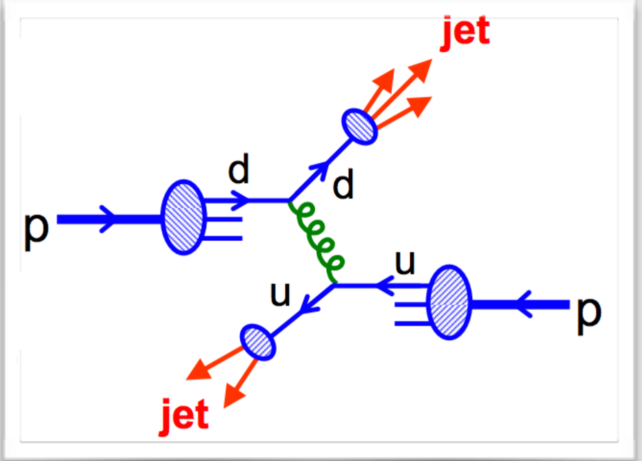
$$\boxed{C(i\bar{k} \to j\bar{l}) \equiv \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{ki}\lambda^a_{jl}}$$

$$C(r\bar{r} \to r\bar{r}) = \tfrac{1}{3}$$
$$C(r\bar{g} \to r\bar{g}) = \tfrac{1}{2}$$
$$C(r\bar{r} \to g\bar{g}) = -\tfrac{1}{6}$$

Quark-Antiquark Scattering

Consider the $u + d \to u + d$ scattering process

- There are nine possible color configurations of the colliding quarks which are all equally likely.
- We need to determine the average matrix element which is the sum over all possible colors divided by the number of possible initial color states



$$\langle |M_{fi}|^2\rangle = \frac{1}{3}\cdot\frac{1}{3}\sum_{i,j,k,l=1}^{3}|M_{fi}(ij\to kl)|^2$$

$$\langle|C|^2\rangle = \frac{1}{9}\sum_{i,j,k,l=1}^{3}|C(ij\to kl)|^2$$

$qq \to qq$

rr→rr,.. **rb→rb,..** **rb→br,..**

$$\langle|C|^2\rangle=\frac{1}{9}\left[3\times\left(\frac{1}{3}\right)^2+6\times\left(-\frac{1}{6}\right)^2+6\times\left(\frac{1}{2}\right)^2\right]=\frac{2}{9}$$

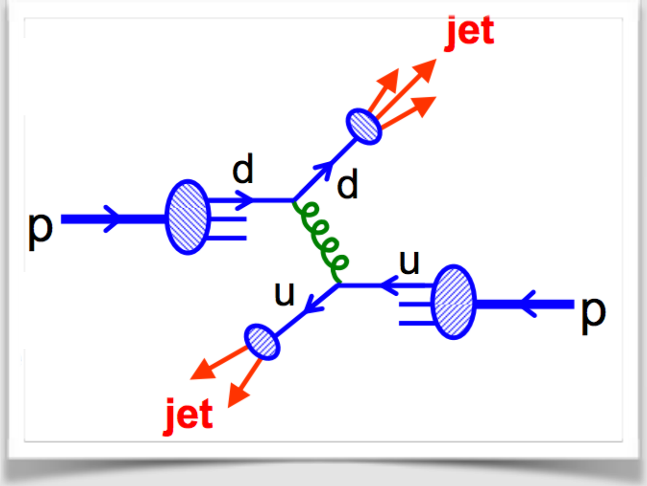
Quark-Antiquark Scattering

We've seen this before in electron-muon scattering!

- The cross section can thus be recycled

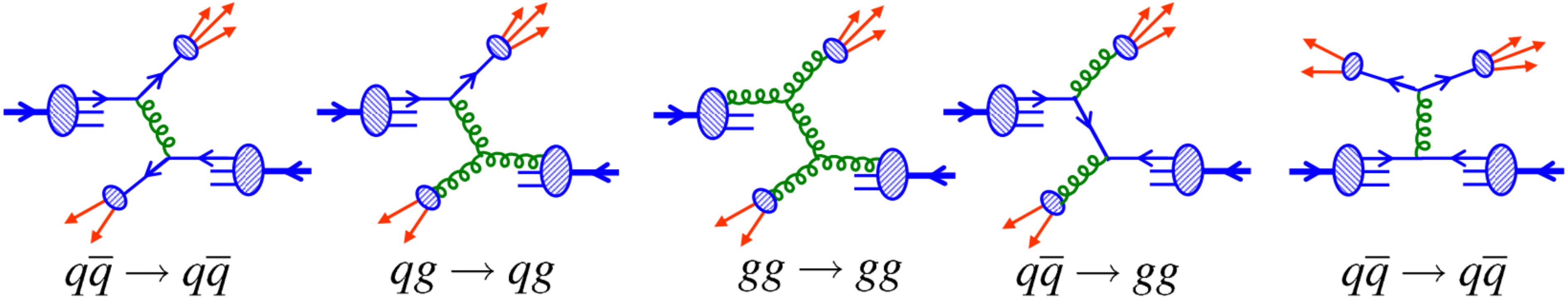
$$\sigma(e^+e^- \rightarrow \mu^+\mu^-) = \frac{4\pi\alpha^2}{E^4}$$

$$\sigma(ud \rightarrow ud) = \frac{2}{9}\frac{4\pi\alpha_s^2}{E^4}$$



The calculation of hadron-hadron scattering is very involved, need to include parton structure functions and include all possible interactions.

For example, generic two jet production



Recap / Up Next

This time:

Quantum Chromodynamics
- Quarks in QED
- QCD Feynman Rules
- Asymptotic Freedom

Next time:

Quantum Flavor Dynamics
- Weak Interactions
- Weak Decays
- Electroweak Unification

Proton Form Factors

Even without assuming anything about the proton substructure, we know that the structure function is a second-rank tensor.

There are only so many ways to do this in a symmetric manner (ie, in μ and ν)

$$g^{\mu\nu} \qquad p^\mu p^\nu \qquad q^\mu q^\nu \qquad (p^\mu q^\nu + p^\nu q^\mu) \qquad \cancel{(p^\mu q^\nu - p^\nu q^\mu)}$$

After a few identities and some algebra:

$$K^{\mu\nu}_{\text{proton}} = K_1 \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) + \frac{K_2}{M^2} \left(p^\mu + \frac{q^\mu}{2} \right) \left(p^\nu + \frac{q^\nu}{2} \right)$$

The goal is then to measure these form factors experimentally and to try to calculate them theoretically.

$$\frac{d\sigma}{d\Omega} = \left(\frac{\alpha}{4ME\sin^2(\theta/2)} \right)^2 \frac{E'}{E} \left[2K_1 \sin^2(\theta/2) + K_2 \cos^2(\theta/2) \right]$$