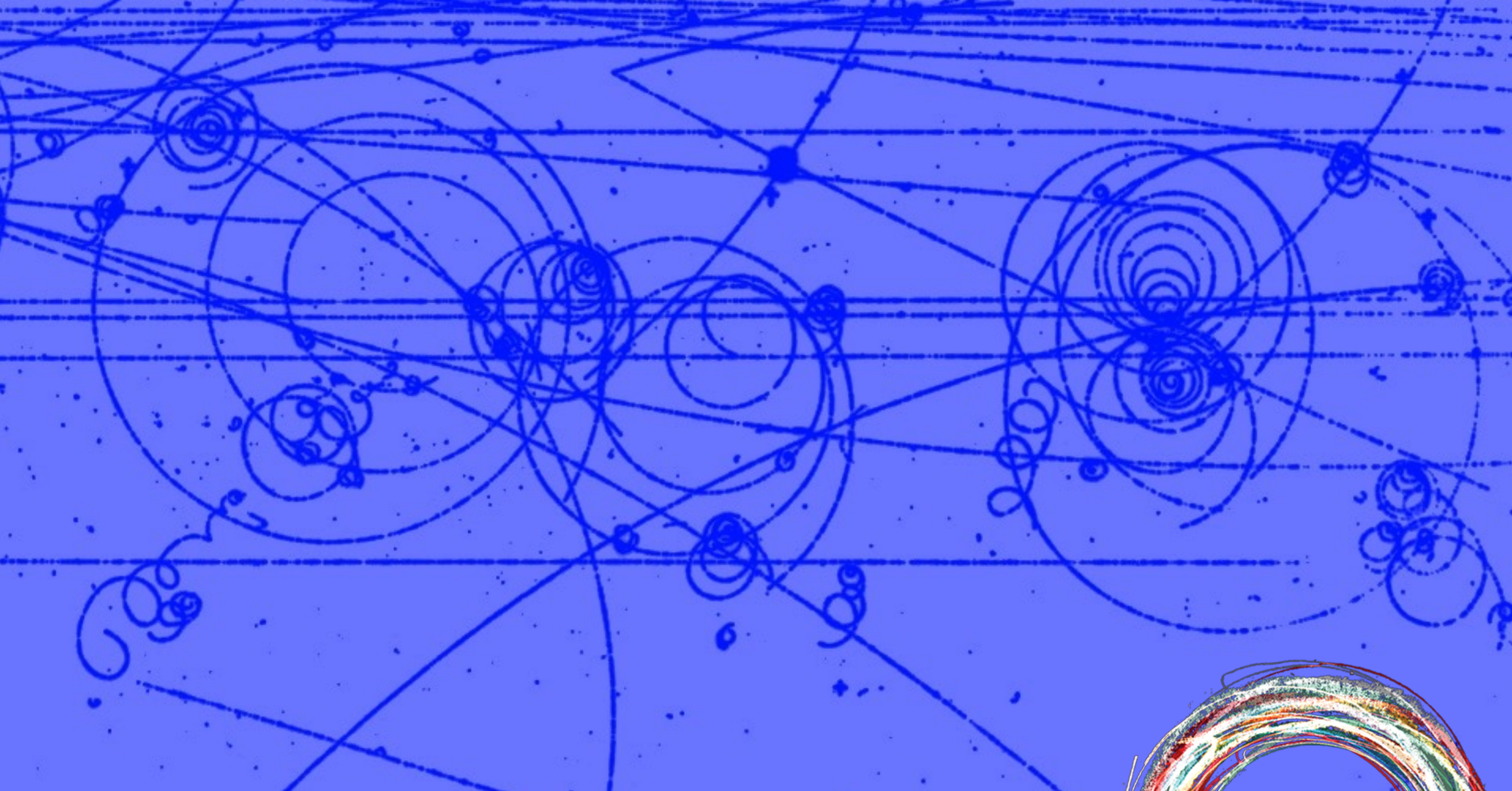


LECTURE 7
FEYNMAN RULES

PHY 493/803, 2017

Recap / Up Next

<u>Last time:</u>

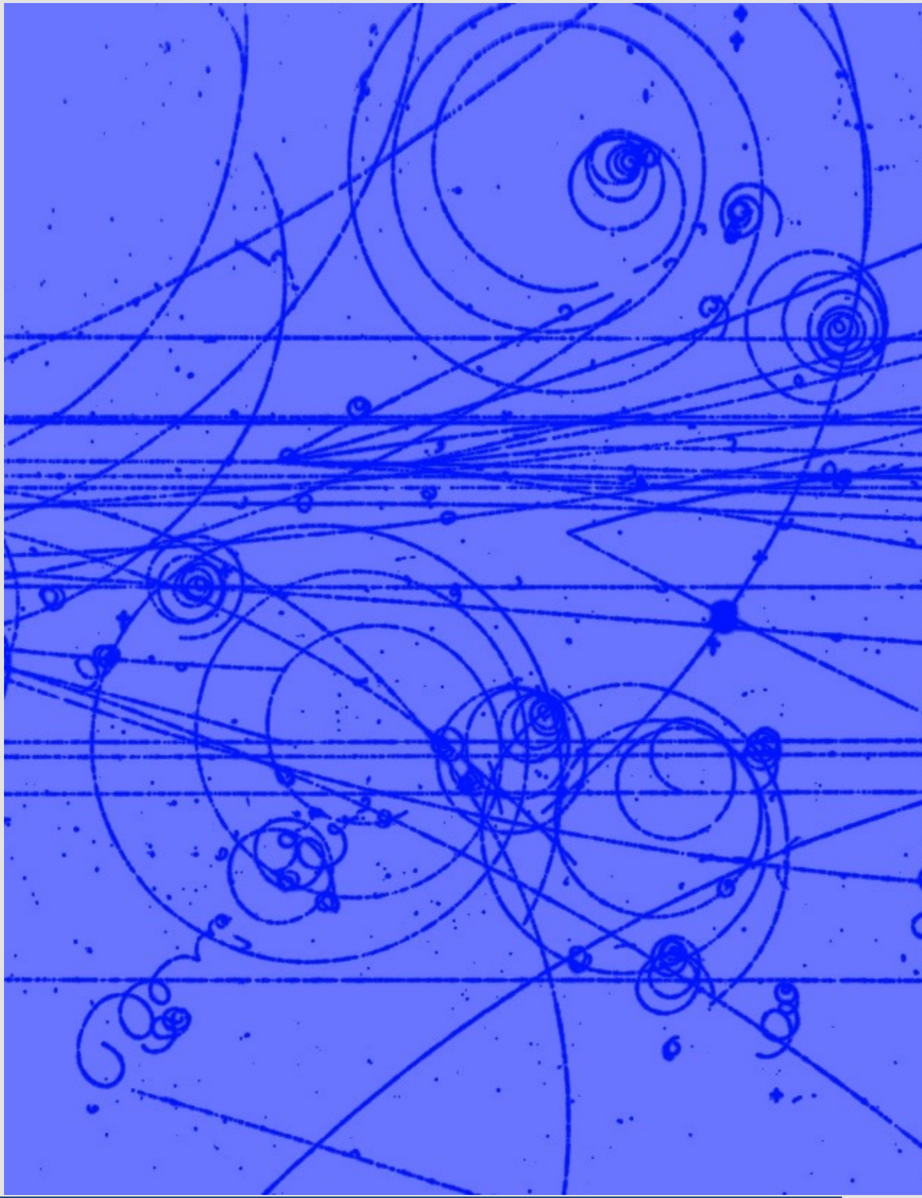
Bound States

- Hydrogen
- ‘Onium
- Mesons/Baryons

<u>This time:</u>

Feynman Calculus

- Decays/Scattering
- The Golden Rule
- Feynman Rules



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Decay rate (Γ): Describes how quickly particles disappear.

Lifetime (τ): Describes how long particles stick around

$$\tau = 1/\Gamma$$

Decay rates & rules

Generally, all particles are unstable and can decay. Rough rules:

1) The final state cannot have more total mass, to conserve energy.
2) If there is not a lower energy/mass state, the particle cannot decay.
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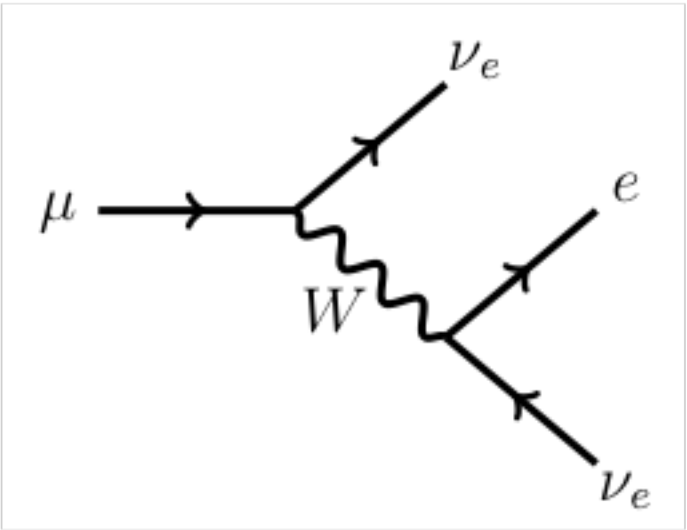
Consider muon decay:

- The final state mass is much smaller than the initial state mass: 0.5 MeV vs 106 MeV
- The W boson must be virtual (M_W~80 GeV)
- The final state momentum is:
  - Shared amongst 3 particles
  - Determined by the difference in mass between initial and final states.



Decay rate and Lifetime

Particles have no concept of history

The probability a particle will decay in a fixed period is constant.

A population of decaying particles can be described as a function of time.

$$\begin{aligned} \Delta N(t) &= N(t_a) - N(t_b) \\ &= p\, N(t_b)\, (t_a - t_b) \end{aligned}$$

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Written differentially, we have:

$$\mathrm{d}N = -\Gamma N \mathrm{d}t$$

$$N(t) = N_0 e^{-\Gamma t} = N_0 e^{-t/\tau}$$

$$\Gamma = 1/\tau$$

Summing Decay Rates

What if a particle has more than one path to decay?

The total decay rate just becomes the sum of individual decay rates.

$$W^+ \to e^+ + \nu_e$$

$$W^+ \to \mu^+ + \nu_\mu$$

$$W^+ \to \tau^+ + \nu_\tau$$

$$W^+ \to u + \bar{d}$$

$$W^+ \to c + \bar{s}$$

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$$W^+ \to u + \bar{d}$$
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$$\mathrm{d}N_W = -\left(\Gamma_{e\nu} + \Gamma_{c\bar{s}} + \cdots\right) N_W \mathrm{d}t$$

$$\Gamma_{tot} = \sum \Gamma_i$$

Branching Fraction/Ratio

The fraction of a certain particle's decays to a given final state is referred to as the Branching Fraction or Branching Ratio

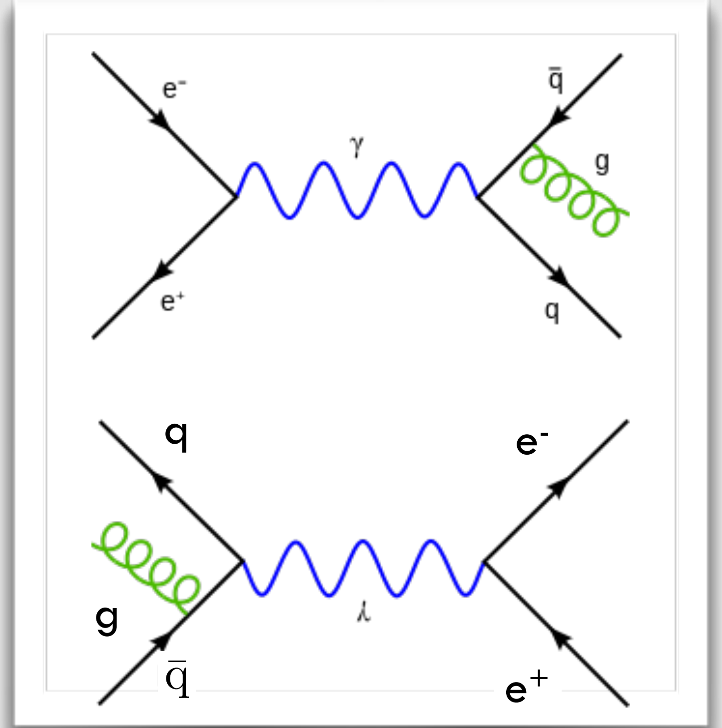
It's trivially calculated!

$$BR(W \to e\nu) = \frac{\Gamma_{W \to ev}}{\Gamma_{\text{tot}}}$$

Decay & Production

We will see that particle decay and particle production are intrinsically related

- You would have guessed this from the idea of CPT and also from how we were able to rotate Feynman diagrams
- The decay rate for a process is also referred to as the "width" of the production rate "peak".



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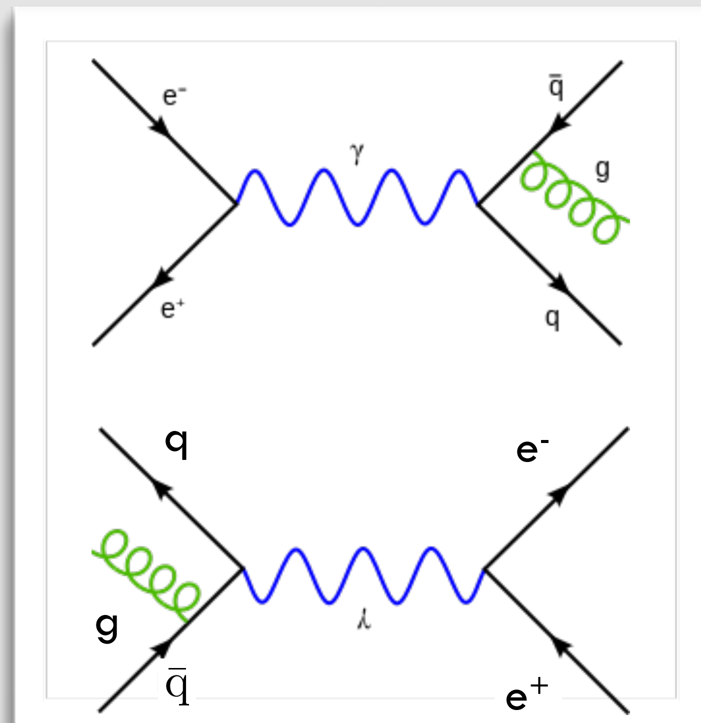

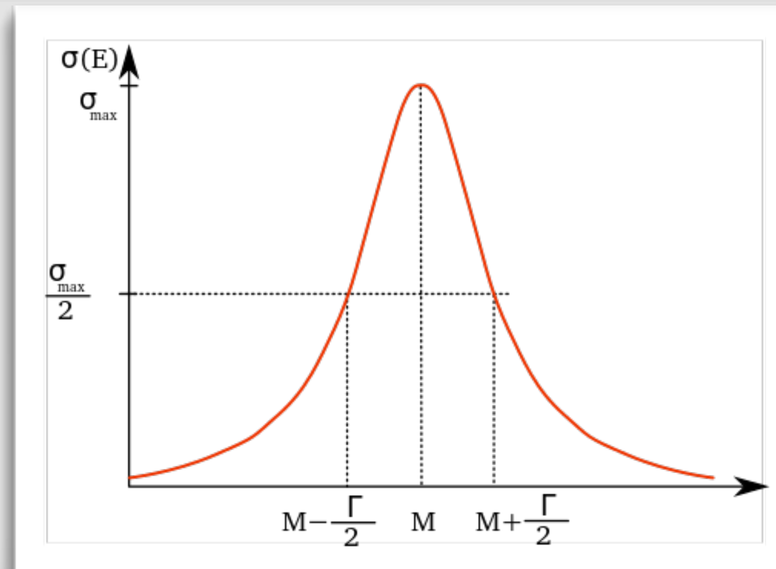

$$P(E) = \frac{1}{2\pi} \frac{\Gamma}{(E - N)^2 + \Gamma^2/4}$$

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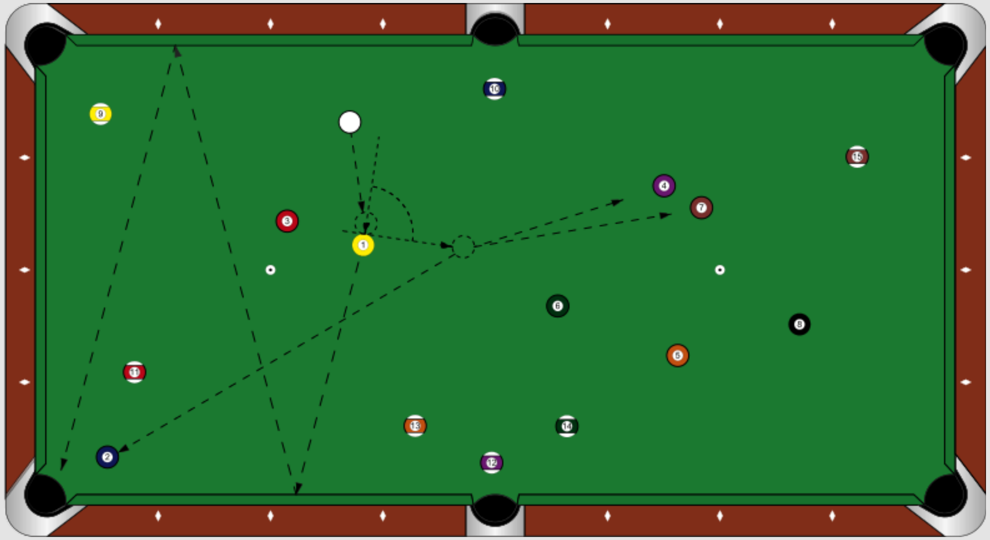
Cross section (σ): Describes the probability for an interaction to occur.

Differential cross section ($\mathrm{d}\sigma/\mathrm{d}X$): Describes the probability for an interaction with a particular final state.

Cross Section

We need to connect the idea of a unit area (classical) and the probability for an interaction to occur.

Classically, the cross section is inherently related to the size of an object.



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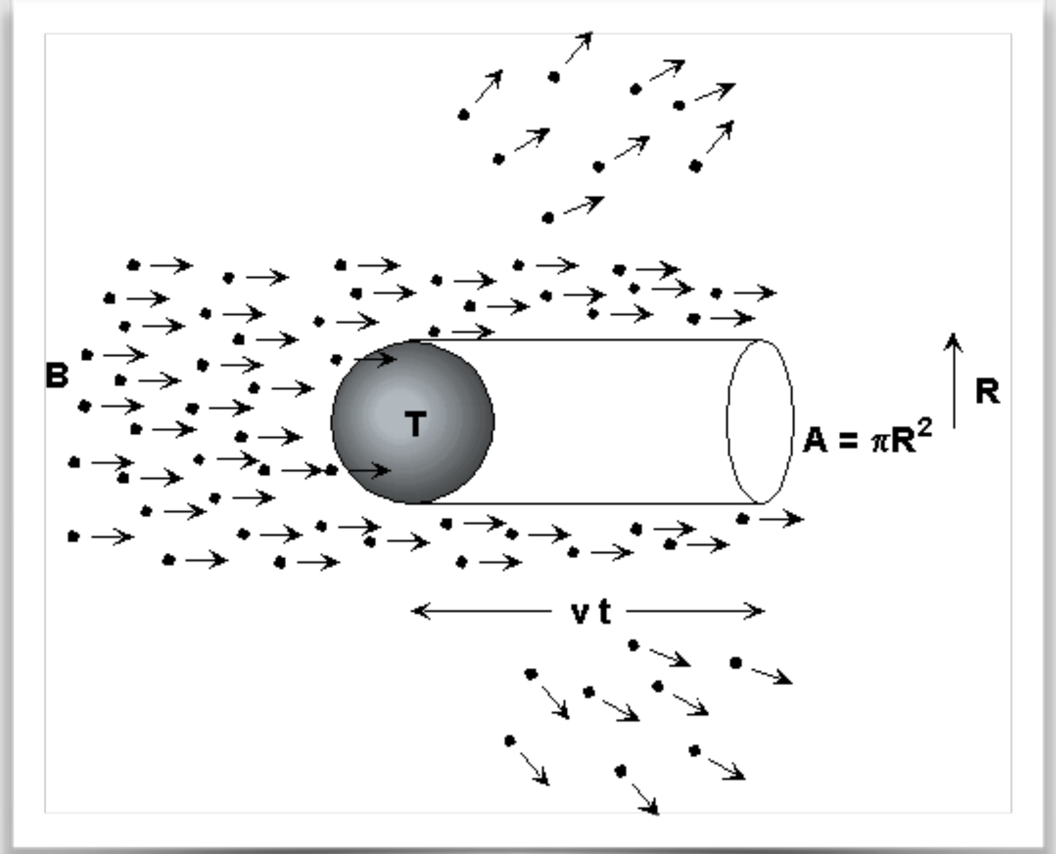


In the classical sense, scattering occurs when the particles in a beam overlap their path with the target's cross section

Hard Sphere Scattering

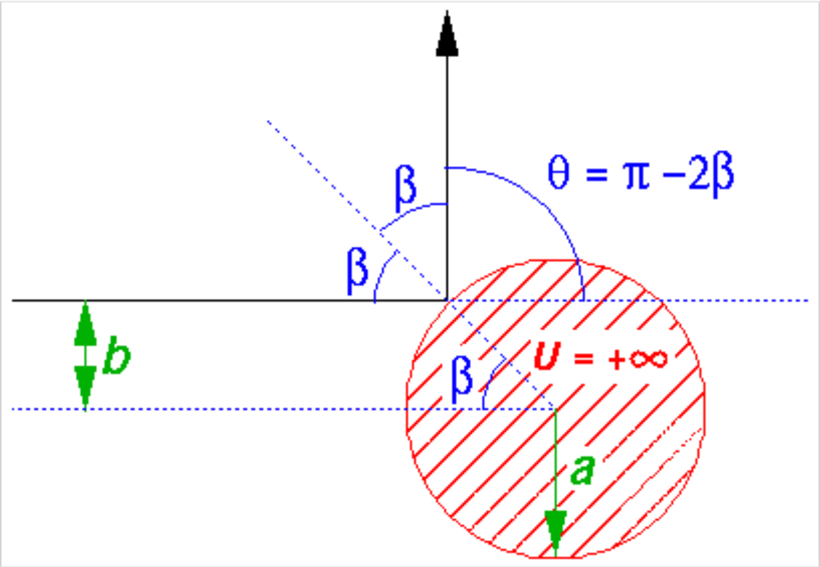
We can start with the classical hard sphere calculation

$$b = R\sin(\beta)$$

$$\theta = \pi - 2\beta$$

$$\sin(\beta) = \cos(\theta/2)$$

$$b = R\cos(\theta/2)$$



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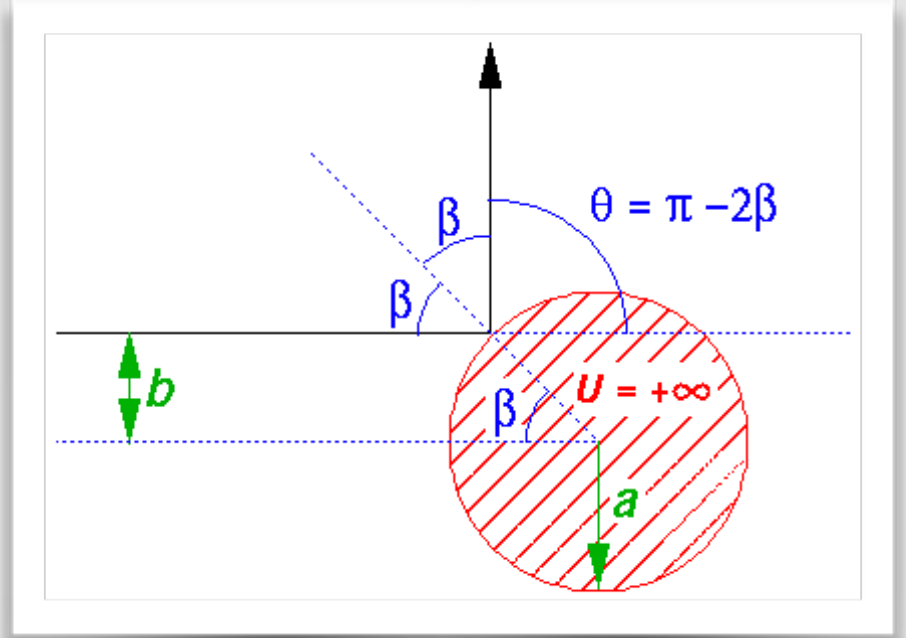

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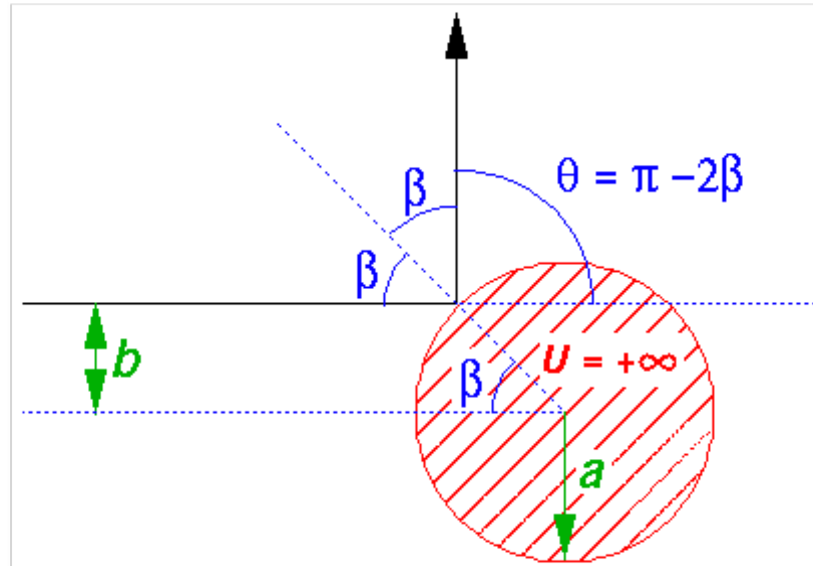

$$d\sigma = 2\pi\, b\, db$$
$$d\Omega = \sin(\theta)\, d\theta\, d\phi$$

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin(\theta)}\frac{db}{d\theta}$$

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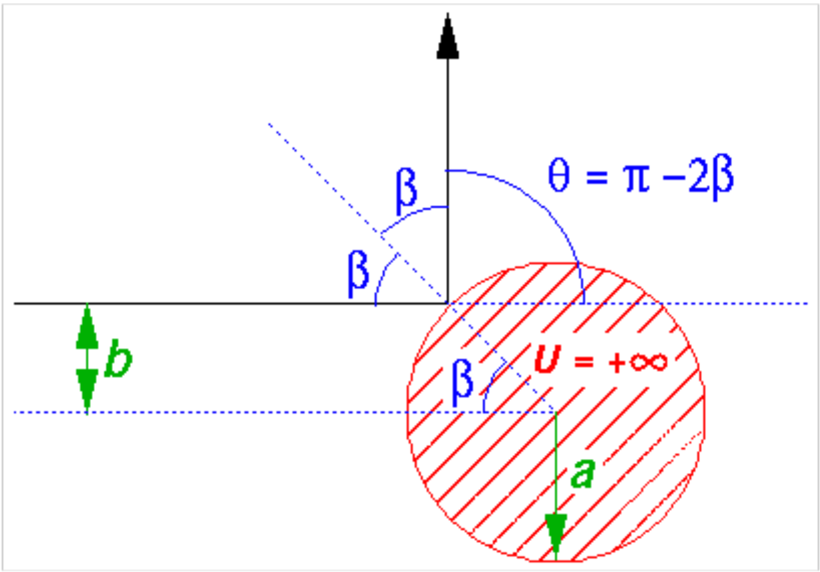

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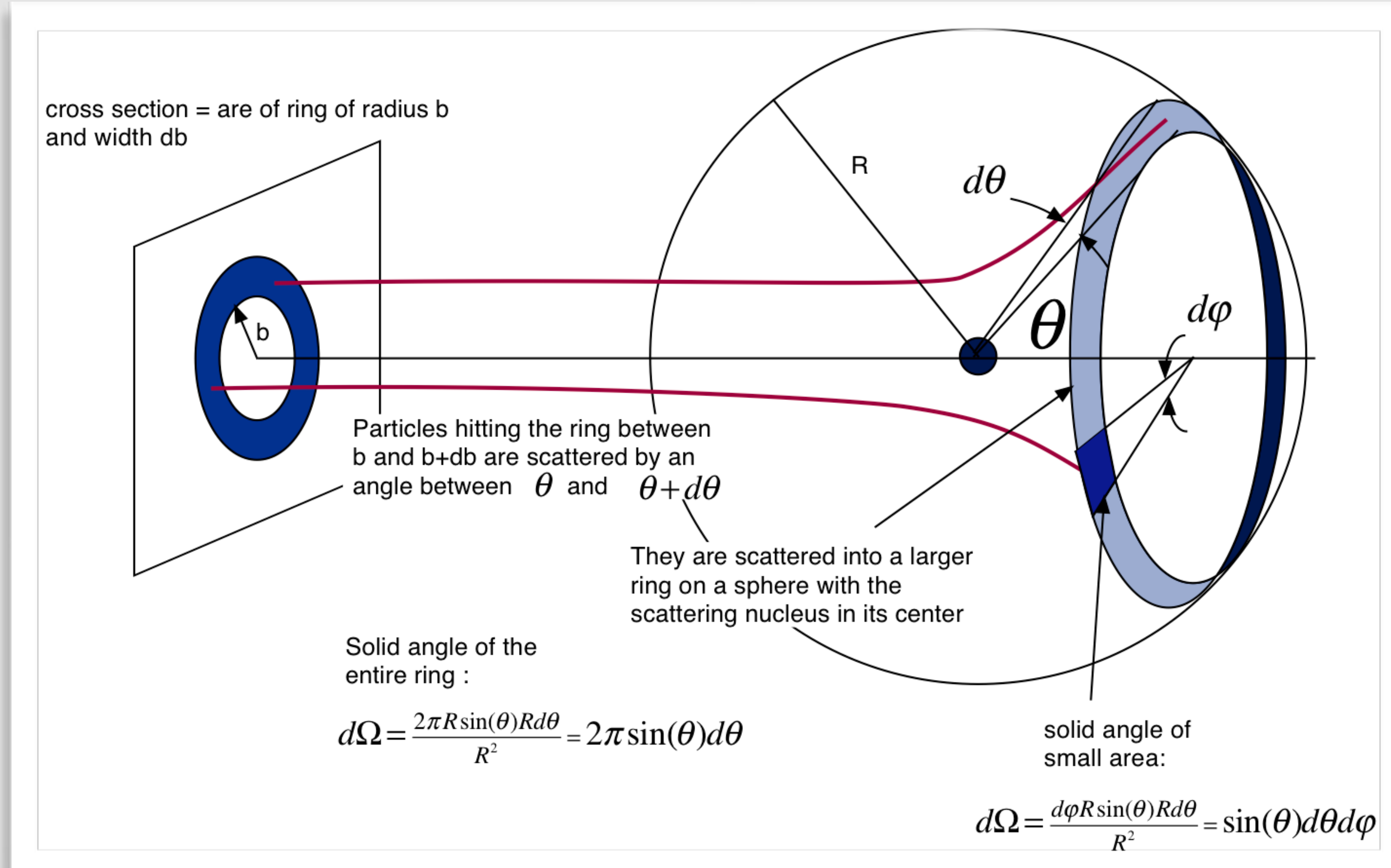
$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{b}{\sin(\theta)}\frac{\mathrm{d}b}{\mathrm{d}\theta}$$

$$\frac{\mathrm{d}b}{\mathrm{d}\theta} = -\frac{R}{2}\sin(\frac{\theta}{2}) \qquad \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{R^2}{4}$$

Reminder: Rutherford Scattering

Rutherford scattering describes classical scattering off a central potential



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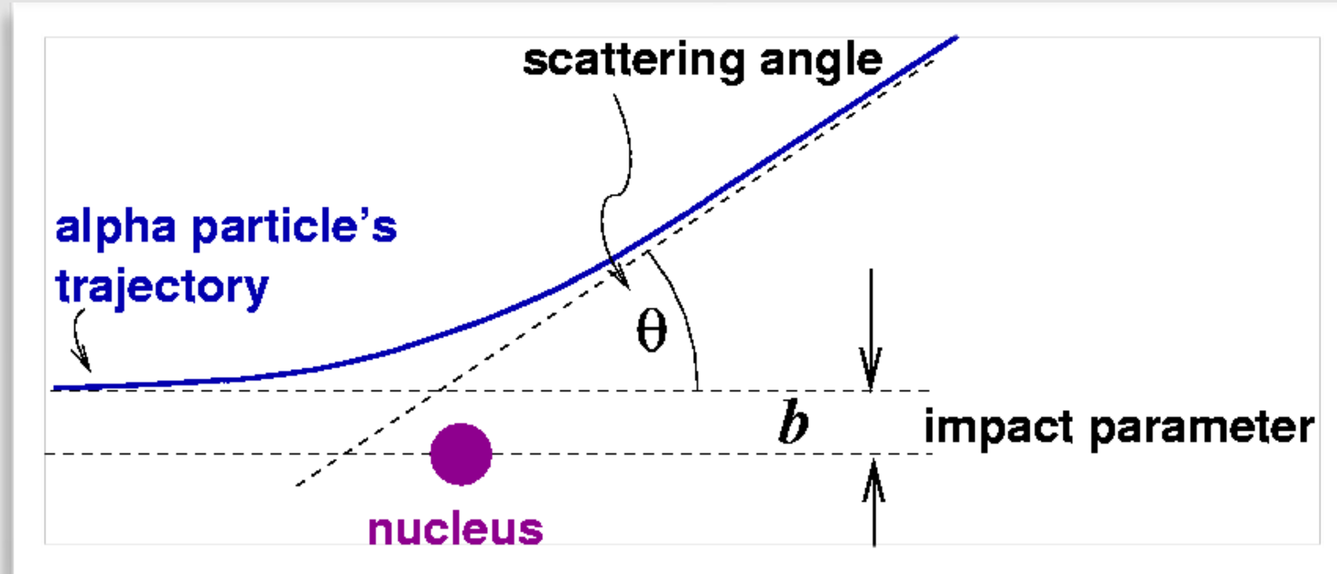

$$b = \frac{q_1 q_2}{2E} \cot(\theta/2) \qquad \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{q_1\, q_2}{4E\sin^2(\theta/2)}\right)^2$$

The QFT Cross Section

In quantum field theory, the interaction cross section doesn't correspond to the physical size of an object.

Many particles are interpreted as point objects (zero size!)

We now refer to the probability of particles interacting with each other.

We've seen a form of this in the context of spin & angular momentum:

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$$|1,\ 1 > |1/2,\ -1/2 >= \sqrt{\frac{1}{3}}\,|3/2,\ 1/2 > + \sqrt{\frac{2}{3}}\,|1/2,\ 1/2 >$$

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Be careful with this particular analogy, though. But we can consider:

The cross section for the $|3/2, 1/2>$ state is 1/3 of the total cross section.

The cross section for anything but $|3/2, 1/2>$ or $|1/2, 1/2>$ is 0.

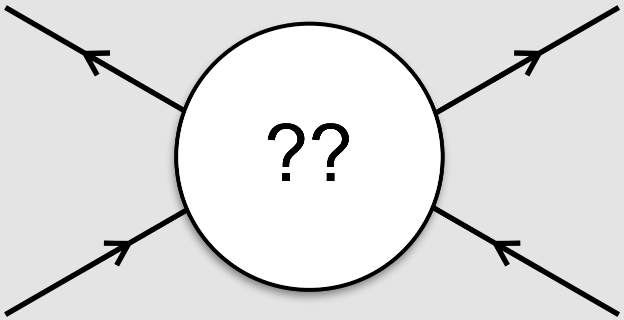
However, we don't know the total cross section!

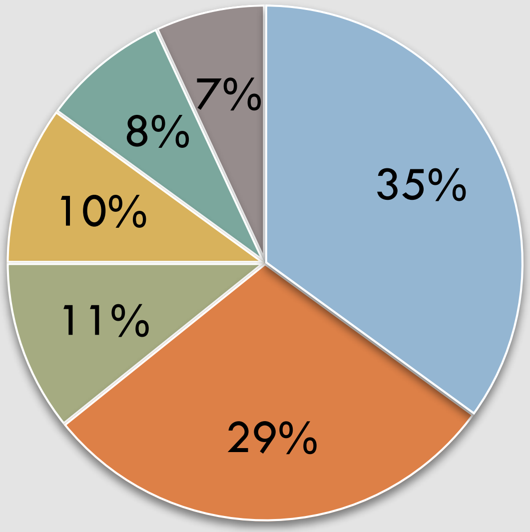
We'll talk about that in the context of the Golden Rule.

Total Cross Section

As we saw with the decay rate, the cross section can be subdivided into constituent pieces.

To find the constituents, we have to consider all ways to produce the final state.





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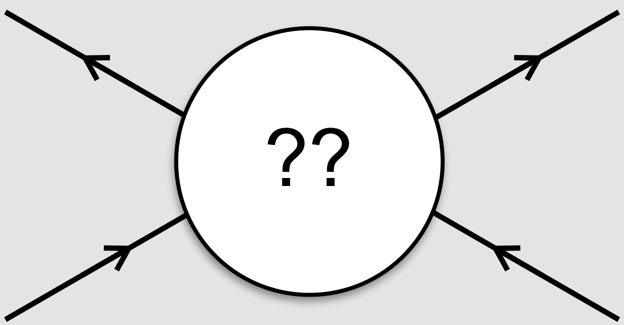

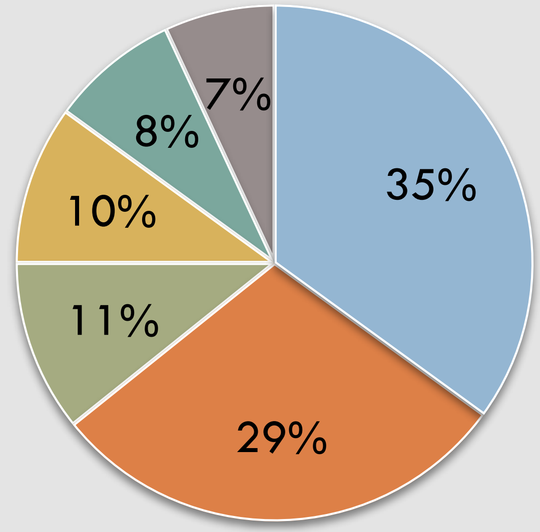

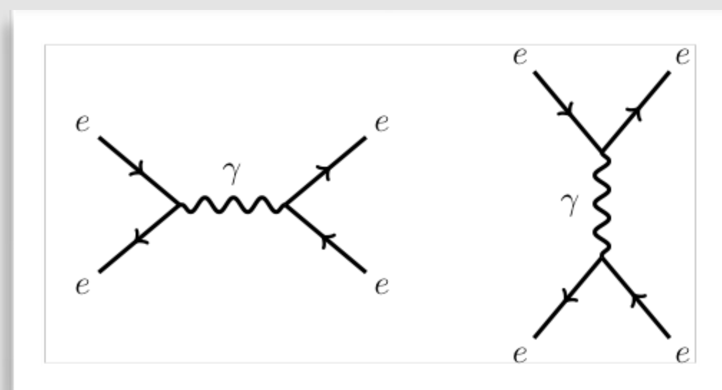

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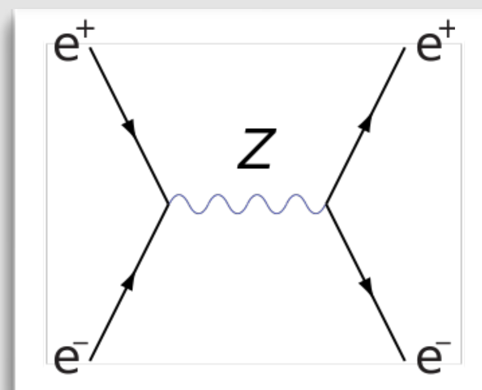


$$\sigma_{\text{tot}} = \sum \sigma_i$$

Reminder: From Rutherford Scattering

Cross section (σ) : the probability of a collision of occurring between two particles (beam and target)

Reaction rate (W)

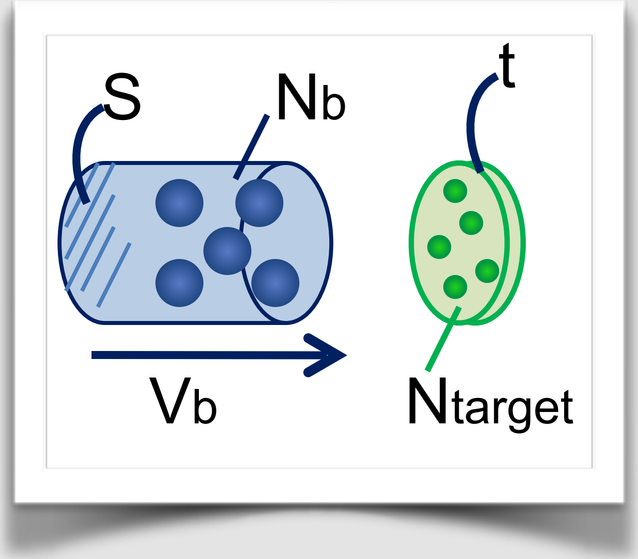
- $\propto$ σ
- $\propto$ N_{target} : number of target illuminated by the beam
- $\propto$ Flux J : beam rate per unit area

$$J = n_b \cdot V_b$$

(n_b : number density of beam particles)

(V_b: beam velocity)

Beam intensity I = J S (S: beam area)



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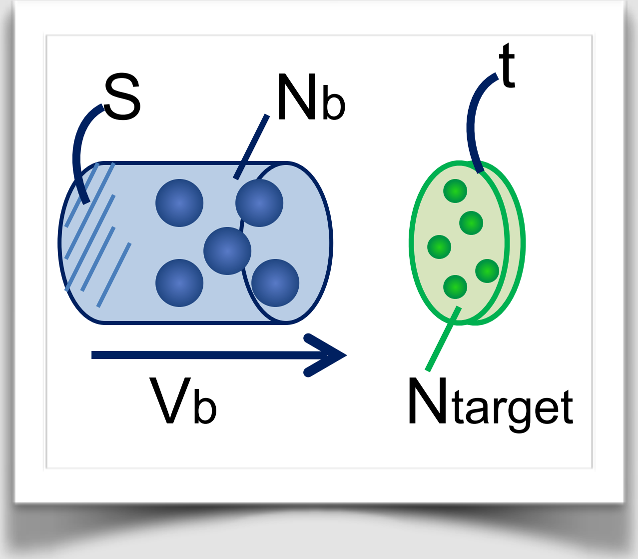

$$W = \sigma \cdot N_{\text{target}} \cdot J$$
$$= \sigma \cdot N_{\text{target}} \cdot I/S$$
$$= \sigma \cdot (n_t \cdot V) \cdot I/S$$
$$= \sigma \cdot (n_t \cdot t) \cdot I$$
$$= \sigma \cdot \rho \cdot (N_A/M_A) \cdot t \cdot I$$

(n_t: number of target particles/unit volume)

(t: thickness of target)

(ρ: target density)

(N_A: Avogadro's constant, M_A: mass)

The Golden Rule

Fermi's Golden Rule states:

The transition rate for a system depends on two fundamental quantities

1) The "Matrix Element" or "Transition Amplitude"
2) The final state phase space

$$W = \frac{2\pi}{\hbar} \left| \mathcal{M}_{if} \right|^2 \rho(E)$$

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Phase Space or Density of States

Density of States

Density of states*

Also known as the available phase space or just phase space

Describes how many equivalent ways a final state can be configured

1) Assume we have a particle with quantized momentum confined to volume V

$$p = \hbar k = h/\lambda$$

→ Smallest element of phase space in any coordinate is **h**!

*Some people like to use the particle-in-a-box example. It's fine, but I find it confusing for our purposes here.

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2) The number of equivalent states, N_i, can be calculated by dividing the total phase space volume by the elemental phase space volume:

$$N_i = \frac{1}{(2\pi\hbar)^3} \int dx\, dy\, dz\, dp_x\, dp_y\, dp_z = \frac{V}{(2\pi\hbar)^3} \int d^3p$$

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3) Simplify life by calculating over a unit volume (V=1 of your favorite unit). Also consider the option to have **n** particles in the final state:

$$N_n = \frac{1}{(2\pi\hbar)^{3n}} \int \prod_{i=1}^{n} d^3 p_i$$

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4) Impose conservation of momentum in the final state, which means there is one less degree of freedom in the distribution of momenta:

$$N_n = \frac{1}{(2\pi\hbar)^{3n}} \int \prod_{i=1}^{n} d^3 p_i \, \delta \left[p_n - (p_0 - \sum_{i=1}^{n-1} p_i) \right]$$

$$= \frac{1}{(2\pi\hbar)^{3n}} \int \prod_{i=1}^{n-1} d^3 p_i$$

Dirac Delta Function

I hope this is review. If not, go study up on this!

The Dirac delta function is simply the derivative of the Heaviside step function

$$\theta(x) = \begin{cases} 0, & (x < 0) \\ 1, & (x > 0) \end{cases} \longrightarrow \delta(x) = \begin{cases} 0, & (x \neq 0) \\ \infty, & (x = 0) \end{cases}$$

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The integral of the delta function has a valuable property

The delta function selects out the zeros of its argument

$$\int_{-\infty}^{+\infty} \delta(x) = 1 \longrightarrow \int_{-\infty}^{+\infty} f(x)\delta(x-a) = f(a)$$

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What we did is just fine, but it's not Lorentz Invariant

- But we can fix that by taking into account the what we've learned
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2) Impose Lorentz invariance by fixing the Lorentz invariant inner product for each final state particle:

$$N_n = \frac{1}{(2\pi\hbar)^{4n}} \int \prod_{i=1}^{n} d^4 p_i \, \underline{\delta \left[(p^\mu p_\mu)_i - m_i^2 c^2 \right]}$$

Lorentz Invariant Phase Space

While we're at it, we may as well introduce "sanity checks"

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4) Introduce a Heaviside function to remove negative final state energies:

$$N_n = \frac{1}{(2\pi\hbar)^{4n}} \int \delta^4(p_i - \sum p_f) \prod_{j=1}^{n} d^4p_j \, \delta\left[(p^\mu p_\mu)_j - m_j^2 c^2\right] \, \underline{\theta(E_j)}$$

Lorentz Invariant Phase Space

We're almost there!! One more thing to do…

- Recall, our integrals were not truly over continuous momentum space.
- Momentum is quantized, so we're actually integrating integer k space!

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→ Every delta function gets a 2π in the integral.

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6) We done! We call the energy derivative the "differential Lorentz Invariant phase Space" or just dLIPS.

$$\rho(E) = \frac{\partial N_n}{\partial E} = \frac{(2\pi)^{n+4}}{(2\pi\hbar)^{4n}} \delta^4(p_i - \sum p_f) \prod_{j=1}^{n} d^4 p_j \, \delta\left[(p^\mu p_\mu)_j - m_j^2 c^2\right] \, \theta(E_j)$$

The Golden Rule

Fermi's Golden Rule states:

The transition rate for a system depends on two fundamental quantities

1) The "Matrix Element" or "Transition Amplitude"
2) The final state phase space

$$W = \frac{2\pi}{\hbar} \left|\mathcal{M}_{if}\right|^2 \rho(E)$$

The transition rate (decays or interactions)

The Matrix Element. We may not actually know this.

Phase Space or Density of States

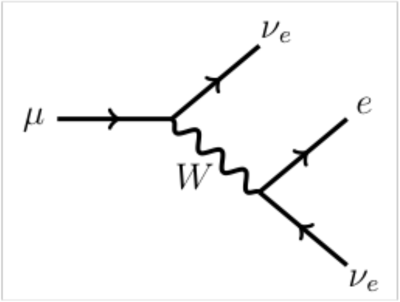
Golden Rule for Decays

We can now build the form of the decay rate

Assume we have a particle at rest decaying to n particles: 1→2,3,4…n

Here we assume the form for the decay rate and insert elements as needed

$$\Gamma = \frac{S}{2\hbar m_1} \int |\mathcal{M}|^2 \rho(E)$$



The mysterious S factor takes into account final state particle interchange

If the final state has N identical particles, we include a factor of N!

Thus S = 1/N!

Golden Rule for Decays

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Assume we have a particle at rest decaying to n particles: 1→2,3,4…n

Here we assume the form for the decay rate and insert elements as needed

$$\Gamma = \frac{S}{2\hbar m_1}\int |\mathcal{M}|^2 \, \rho(E)$$

$$\Gamma = \frac{S}{2\hbar m_1}\int |\mathcal{M}|^2 \, (2\pi)^4 \, \delta^4(p_1 - p_2 - p_3 \cdots - p_n) \times \prod_{j=2}^{n} 2\pi \, \delta(p_j^2 - m_j^2 c^2) \, \theta(E_j) \, \frac{d^4 p_i}{(2\pi)^4}$$

Following the book or in-class derivation for 2-particle decays (1→2,3):

$$\Gamma = \frac{S\,|p|}{8\pi\hbar\, m_1^2 c}|\mathcal{M}|^2$$

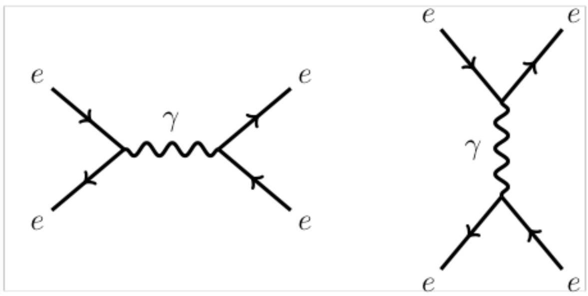
Golden Rule for Scattering

We can also build the form for particle scattering

Suppose we have two particle scattering: 1,2 → 3,4,…,n

Don't worry about the inertial frame just yet.

$$\sigma = \frac{S\,\hbar^2}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2}} \int |\mathcal{M}|^2 \rho(E)$$



Following the book's derivation for 2-body scattering (1,2→3,4):

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2}{(E_1 + E_2)^2} \frac{|p_f|}{|p_i|}$$

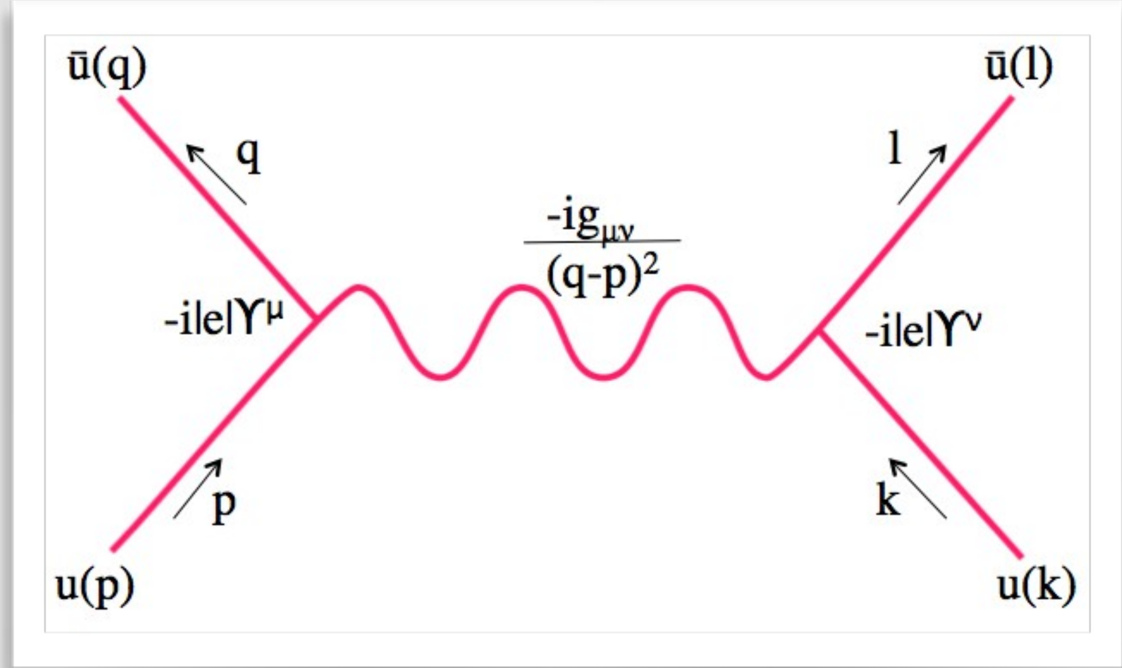
Feynman Rules

We've been completely ignoring the dynamical part of the equation

- All of the specific interaction dynamics are bound up in the **Matrix Element**
- We need to now build up the rules for calculating it!

The Feynman diagram will be our guide in most cases

- We will assign kinematical features to each part of the diagram, depending on what type of particle it represents.

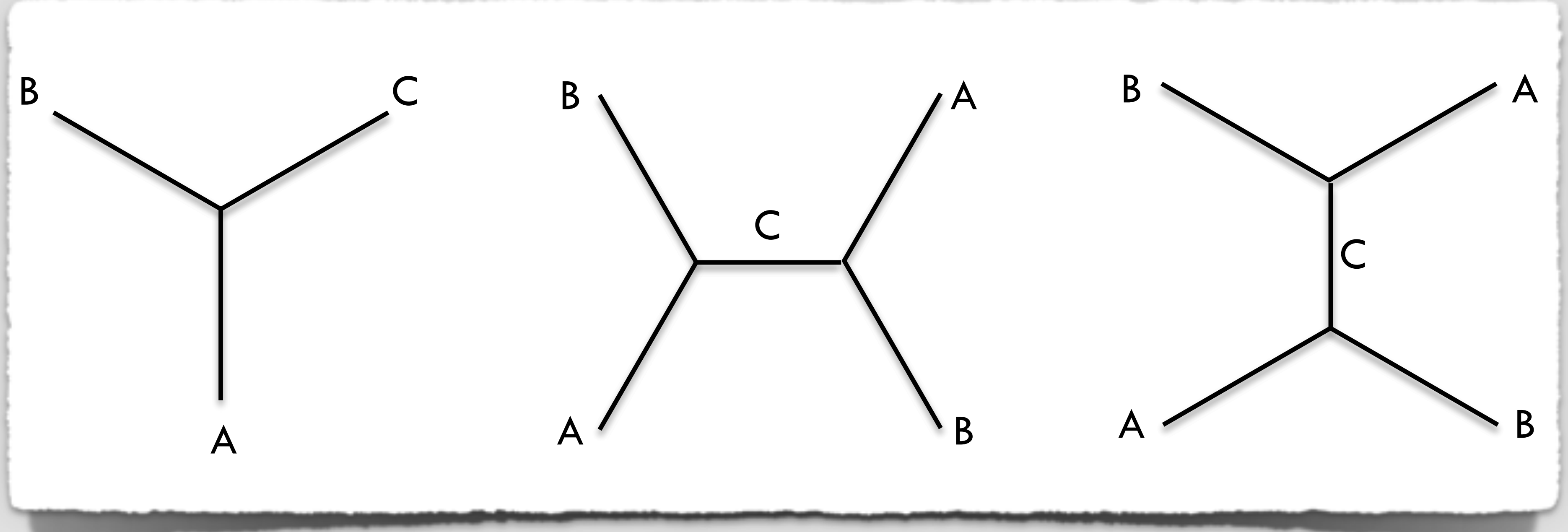


The Feynman diagram will give rise to the matrix element.

ABC Theory

To exercise the Feynman rules, we'll use a toy theory: ABC theory

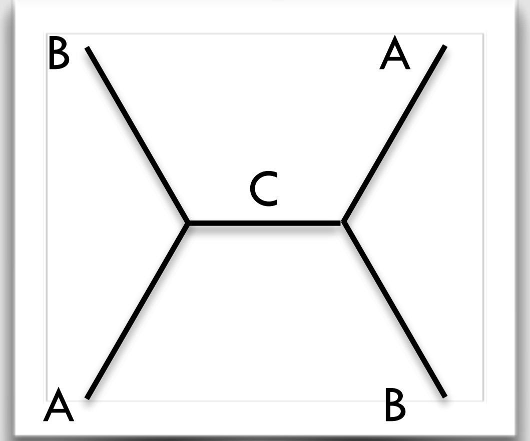
- There are 3 spinless particles: A, B, C
- Each particle is its own antiparticle (No arrows!)
- Only 1 vertex exists and includes all three particles (ABC). Eg, (AAA) is forbidden.
- If $m_A > m_B + m_C$, then A can decay to B and C.



Feynman Rules for ABC Theory

The Feynman rules provide the recipe for constructing an amplitude (Matrix Element $\mathcal{M}$) from a Feynman diagram.

Rule 1:
Draw the Feynman diagram with the minimum number of vertices. There may be more than 1.



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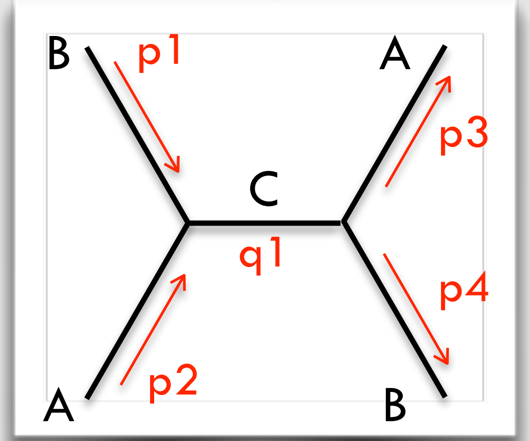
Rule 1:
Draw the Feynman diagram with the minimum number of vertices. There may be more than 1.

Rule 2:
Label the four-momentum of each line (with arrows), enforcing four-momentum conservation at each vertex.

p1, p2, … external momenta: need arrows

q1, q2, … internal momenta: arrow is arbitrary

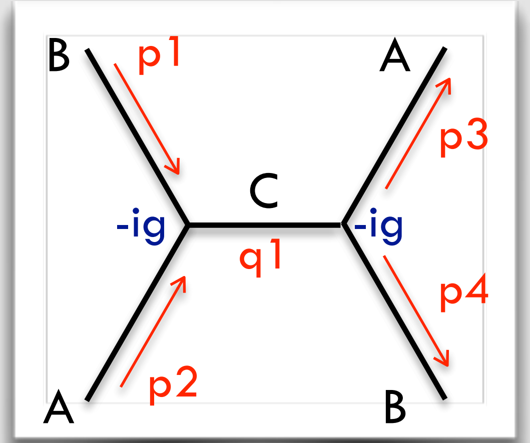
We'll keep track of arrows into/out of vertices.



Feynman Rules for ABC Theory

Rule 3:

Each vertex contributes a factor of $(-ig)$, where g is referred to as the coupling constant. It specifies the strength of the ABC interaction.



Feynman Rules for ABC Theory

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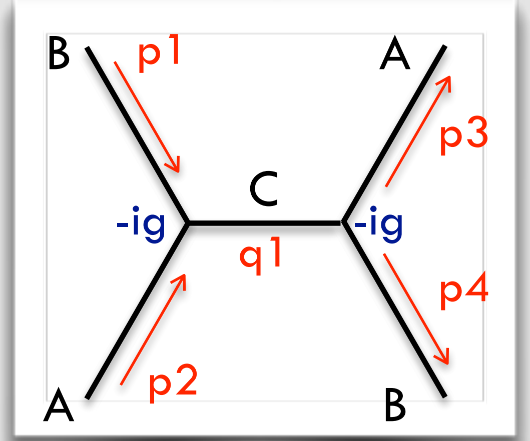
Rule 4:

Each internal line, or propagator, with mass **m** and four-momentum **q** gets a factor of:

$$\frac{i}{q^2 - m^2c^2}$$

Note: q^2 doesn't have to equal m^2. These are virtual particles!



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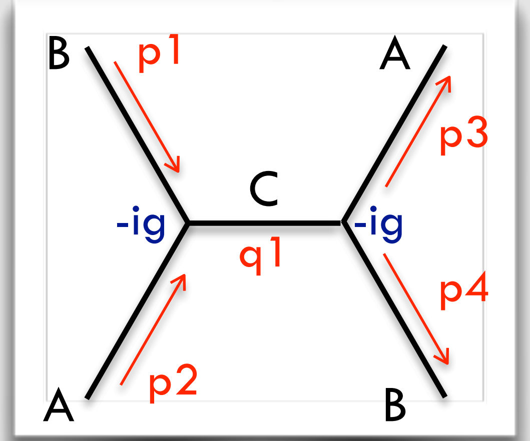
Rule 5:

Each vertex contributes a delta function to conserve energy and momentum. The $\mathbf{k_i}$ are the momenta coming into the vertex:

$$(2\pi)^4\ \delta^4(k_1 + k_2 + k_3)$$



Feynman Rules for ABC Theory

<u>Rule 6:</u>

Build up the proto-Matrix Element from the previous factors & add an **i**:

$$\mathcal{M} = i\ (\text{vertex factors})\ (\text{propagator factors})\ (\text{momentum conservation})$$

Feynman Rules for ABC Theory

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Integrate over the internal momenta:

$$\frac{1}{(2\pi)^4}\, d^4 q_i$$

This connects initial/final state momenta via the delta functions.

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This connects initial/final state momenta via the delta functions.

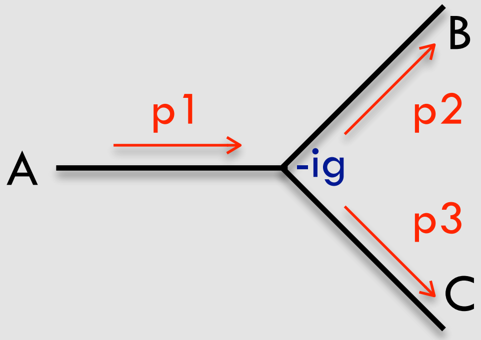
Rule 8:

Drop the extra delta function. We do this because the Matrix Element gets squared, and that would double-count the delta function. Don't worry, it gets put back in the Golden Rule Equation!

The result of step 8 is the matrix element!

Decay of the A

We are now in a position to calculate something! The easiest is the lifetime and/or decay rate of A→BC.



We've already done the kinematics!

$$\Gamma = \frac{S\,|p|}{8\pi\, m_A^2}|\mathcal{M}|^2$$

$\hbar = c = 1$

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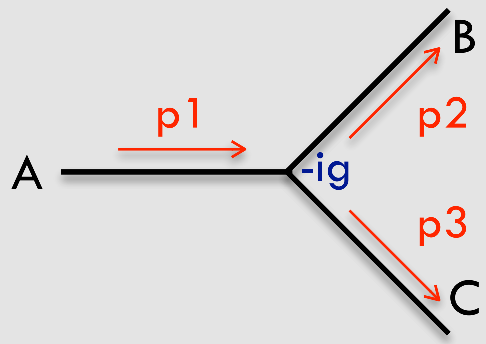

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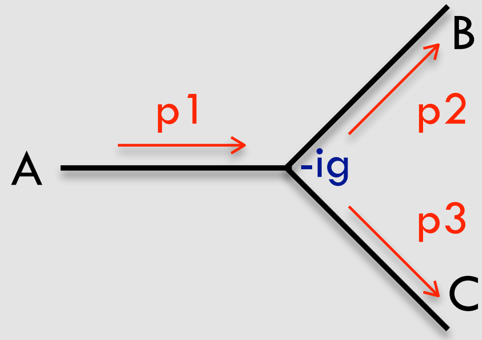

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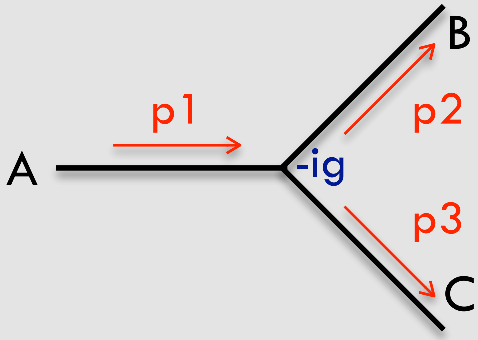

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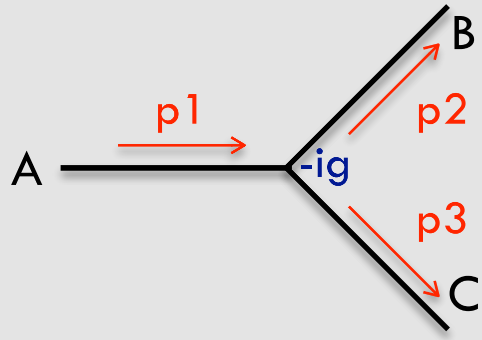

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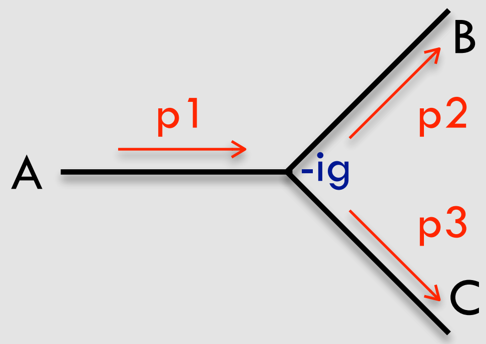

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Momentum: $|\mathbf{p}_B| = |\mathbf{p}_C| = p$ is uniquely determined by M_A, M_B, & M_C.

See problem 3.19: $|p| = \frac{1}{2m_A}\sqrt{m_A^4 + m_B^4 + m_C^4 - 2m_A^2 m_B^2 - 2m_A^2 m_C^2 - 2m_B^2 m_C^2}$

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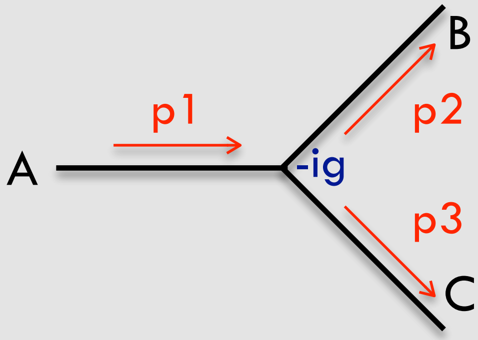


We've already done the kinematics!

$$\Gamma = \frac{S\,|p|}{8\pi\, m_A^2}|\mathcal{M}|^2$$

$\hbar = c = 1$

$$\Gamma(A \to BC) = \frac{g^2\,|p|}{8\pi\, m_A^2} \qquad \tau(A \to BC) = \frac{8\pi\, m_A^2}{g^2\,|p|}$$

See problem 3.19: $|p| = \frac{1}{2m_A}\sqrt{m_A^4 + m_B^4 + m_C^4 - 2m_A^2 m_B^2 - 2m_A^2 m_C^2 - 2m_B^2 m_C^2}$

Example: $\pi^0 \to \gamma\gamma$ Decays

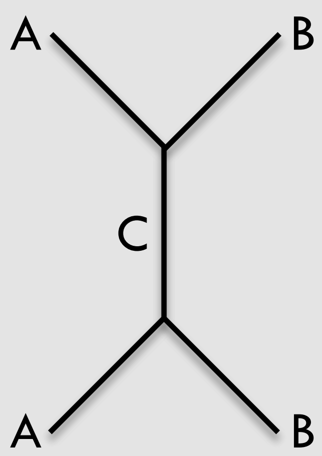
Example done in class.

A+A→B+B Scattering

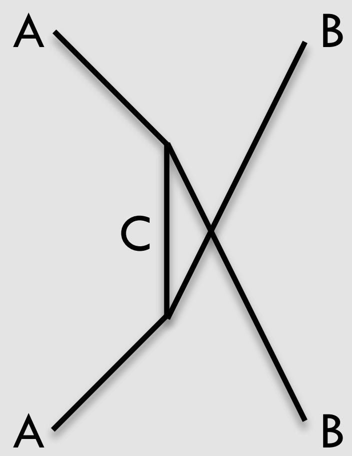
Now we can try something more complicated.

The scattering of A+A into B+B (or, equivalently A+A) is highly relevant
In ABC theory, this occurs via the exchange of particle C.

We have the t-channel diagram



But also the u-channel diagram!



Why two diagrams?

- The u-channel diagram is necessary because we cannot tell which outgoing particles connect to which vertex. So we have to include both possibilities in our matrix element and, thus, in the integrations.
- But don't worry about double-counting, as our S-factor will now be S=1/2!=1/2. By doing so, we effectively average over the two diagrams

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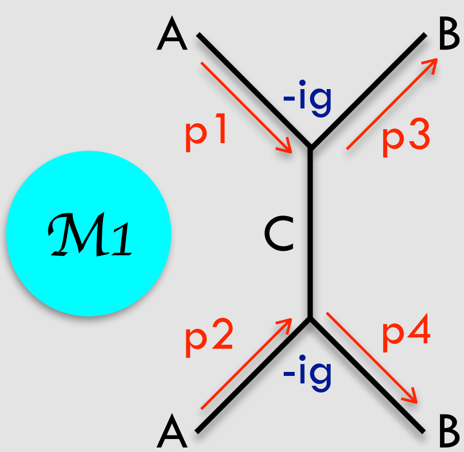

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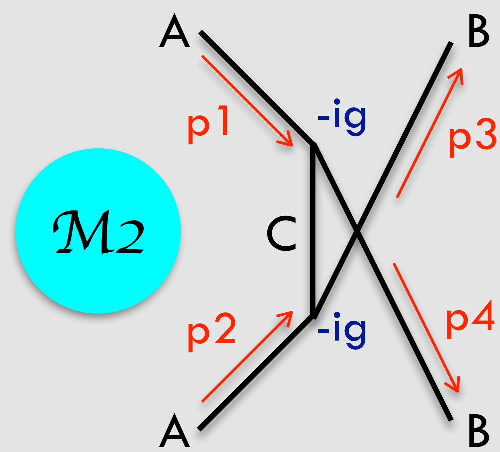
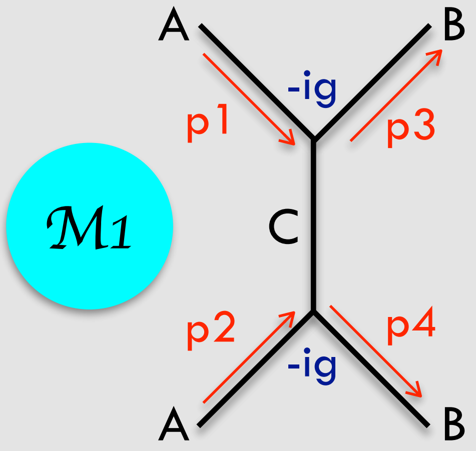


We've already done the kinematics, just fix S=1/2:

$$\frac{d\sigma}{d\Omega} = \left(\frac{1}{8\pi}\right)^2 \frac{|\mathcal{M}|^2}{2(E_1+E_2)^2}\frac{|p_f|}{|p_i|} \qquad \hbar = c = 1$$

A+A→B+B Scattering

First let's calculate the t-channel matrix element.



Rule 3: Two vertices, so we get two factors of $(-ig)$

Rule 4: One propagator gets us a factor of:

$$\frac{i}{q^2 - m_C^2}$$

Rule 5: We pick up two delta functions for the momenta in/out of the vertices

$$(2\pi)^4\ \delta^4(p_1 - q - p_3) \qquad (2\pi)^4\ \delta^4(p_2 + q - p_4)$$

Rule 7: We have to integrate over the phase space of the C propagator (we'll do Rule 6 next)

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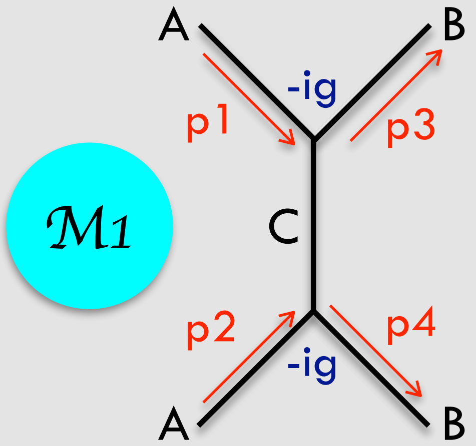

Rule 3: Vertex Factors

Rule 5: Vertex E/p conservation

Rule 6:

$$\mathcal{M}_1 = i \int (-ig)^2 \left(\frac{i}{q^2 - m_C^2} \right) (2\pi)^4 \delta^4(p_1 - q - p_3)\, (2\pi)^4 \delta^4(p_2 + q - p_4)\, \frac{d^4 q}{(2\pi)^4}$$

Rule 4: Propagator Factor

Rule 7: Vertex E/p conservation

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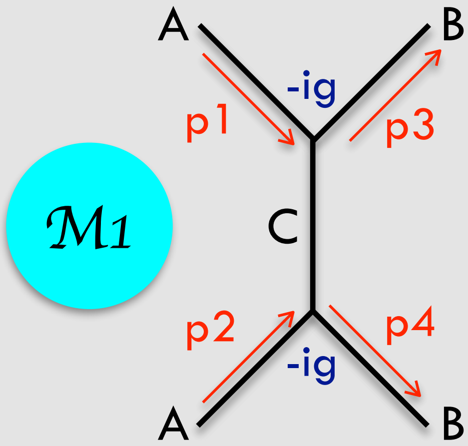


The propagator integral sends either:

$$q \rightarrow p_4 - p_2$$

or:

$$q \rightarrow p_3 - p_1$$

Rule 6:

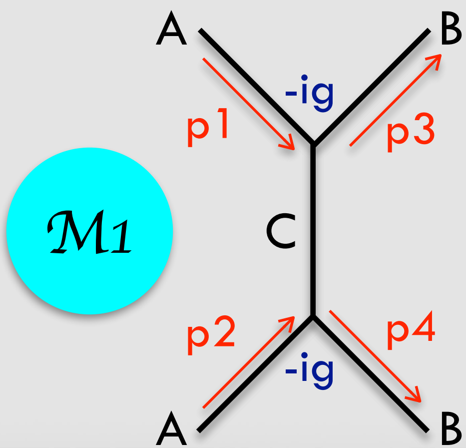
$$\mathcal{M}_1 = i \int (-ig)^2 \left(\frac{i}{q^2 - m_C^2} \right) (2\pi)^4 \delta^4(p_1 - q - p_3)\, (2\pi)^4 \delta^4(p_2 + q - p_4)\, \frac{d^4 q}{(2\pi)^4}$$

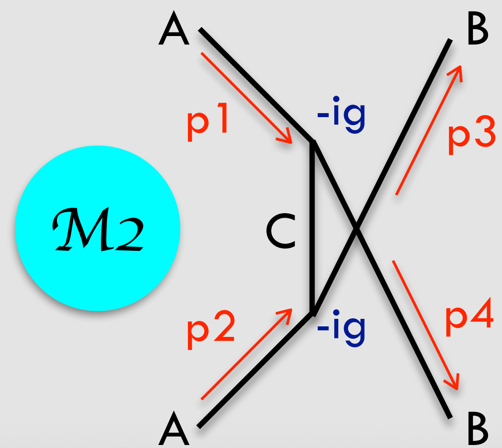
$$\mathcal{M}_1 = \frac{g^2}{(p_4 - p_2)^2 - m_C^2} (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)$$

A+A→B+B Scattering

Now we can consider the u-channel matrix element.

No work required, just swap out the relevant momenta





$$\mathcal{M}_1 = \frac{g^2}{(p_4 - p_2)^2 - m_C^2} \qquad \mathcal{M}_2 = \frac{g^2}{(p_3 - p_2)^2 - m_C^2}$$

$$\mathcal{M}_{\text{tot}} = \mathcal{M}_1 + \mathcal{M}_2$$

A+A→B+B Scattering

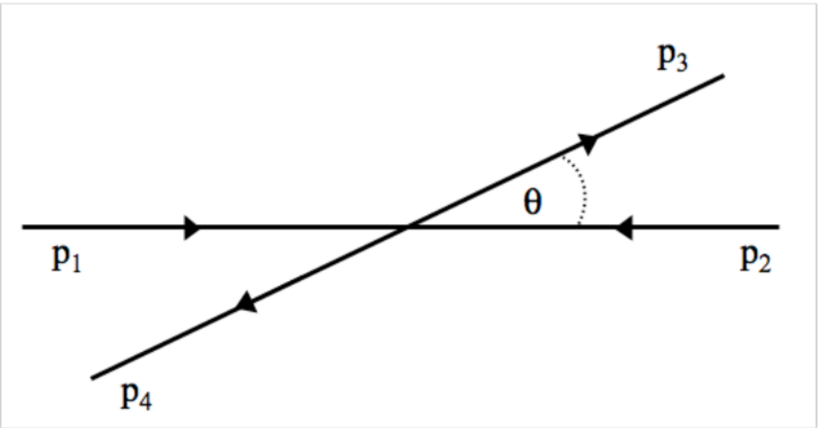
The A+A→B+B scattering matrix element is Lorentz Invariant

But the evaluated quantities may depend on the inertial frame

Let's consider the CoM frame and let $M_A = M_B = m$ and $M_C = 0$

$$\mathcal{M}_1 = \frac{g^2}{(p_4 - p_2)^2 - m_C^2}$$

$$\mathcal{M}_2 = \frac{g^2}{(p_3 - p_2)^2 - m_C^2}$$



$$t = (p_4 - p_2)^2 = p_4^2 + p_2^2 - 2p_2 \cdot p_4 = -2p^2(1 - \cos\theta)$$

$$u = (p_3 - p_2)^2 = p_3^2 + p_2^2 - 2p_3 \cdot p_2 = -2p^2(1 + \cos\theta)$$

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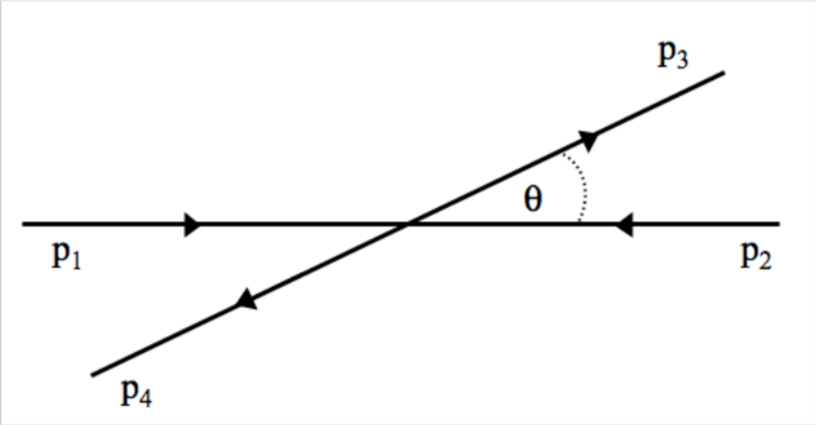
$$\mathcal{M}_2 = \frac{g^2}{(p_3 - p_2)^2 - m_C^2}$$

$$\mathcal{M} = -\frac{g^2}{p^2 \sin^2\theta}$$



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A+A→B+B Scattering

Finally, we can evaluate the differential cross section!

We worked out the 2-body kinematics earlier

We just calculated the matrix element

$$\frac{d\sigma}{d\Omega} = \left(\frac{1}{8\pi}\right)^2 \frac{|\mathcal{M}|^2}{2(E_1+E_2)^2} \frac{|p_f|}{|p_i|}$$

$$\mathcal{M} = -\frac{g^2}{p^2 \sin^2\theta}$$

From the CoM condition we have:

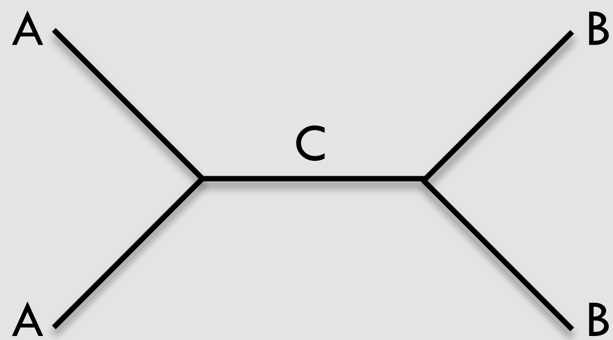
$$E_1 = E_2$$
$$|p_f| = |p_i|$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{2}\left(\frac{g^2}{16\pi\ E\ |p|^2 \sin^2\theta}\right)^2$$

A+A→B+B Scattering

A great question! Why didn't we consider this diagram?

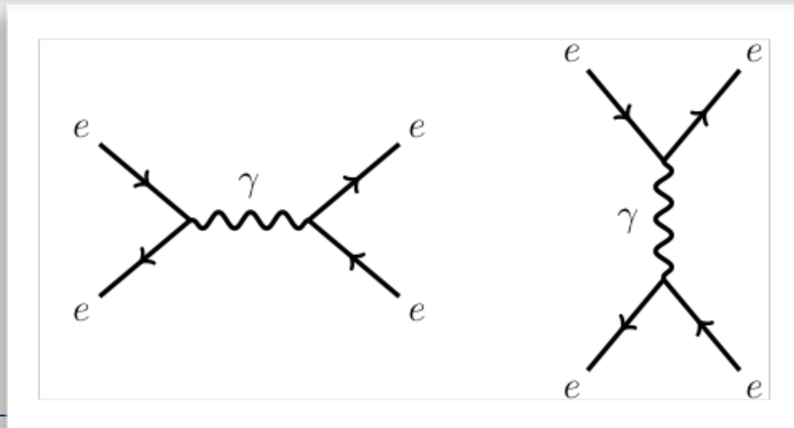
Be careful to not confuse or read too deep into ABC theory!



Answer:

There are no AAC or CBB vertices in ABC theory.

But this diagram does exist in QED, for example.



Another question

Earlier we said cross sections sum:

$$\sigma_{\text{tot}} = \sum \sigma_i$$

And we just said matrix elements sum:

$$\mathcal{M}_{\text{tot}} = \mathcal{M}_1 + \mathcal{M}_2$$

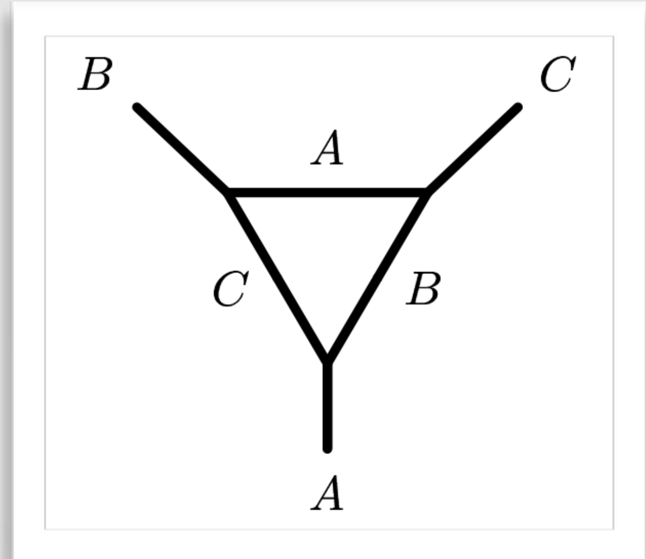
The first statement is clearly only true if there is no interference amongst the matrix elements!

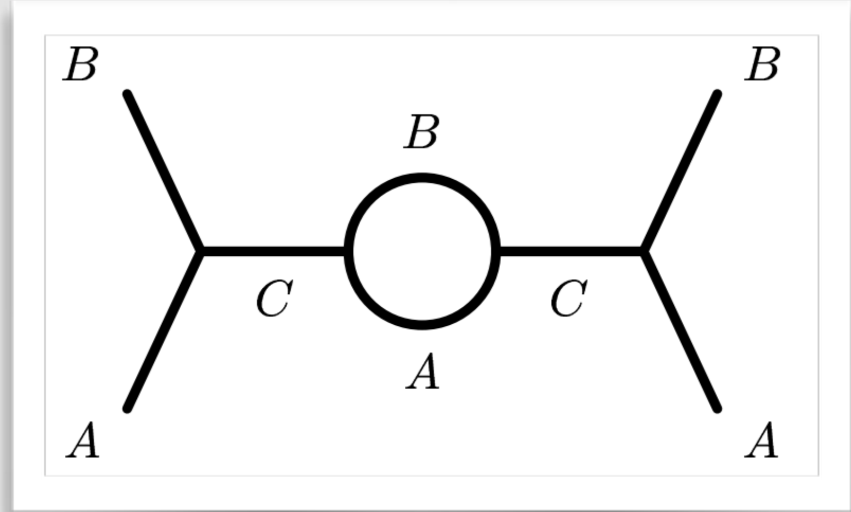
$$|\mathcal{M}|^2 = |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 + 2|\mathcal{M}_1 \cdot \mathcal{M}_2|^2$$

Higher Order Diagrams

These interference effects frequently (but not always) arise due to higher order diagrams

- ABC theory limits the number of such diagrams, so continue to be wary
- We're not going to dig deep into this right now, but be aware they exist





Recap / Up Next

<u>This time:</u>

Feynman Calculus

- Decays/Scattering
- The Golden Rule
- Feynman Rules

<u>Next time:</u>

Particle Accelerators

- Cyclotrons
- Synchrotrons
- The Large Hadron Collider