Recent advances in analytic computations for one-loop amplitudes

Simon Badger[1], *Ruth Britto*[2]
[1] Deutches Elektronen-Synchrotron DESY, Platanenallee, 6, D-15738 Zeuthen, Germany
[2] Institut de Physique Théorique, Orme des Merisiers, CEA/Saclay, 91191 Gif-sur-Yvette Cedex, France

1. Introduction

Key insights of recent years have sparked progress in both analytic and numerical techniques for the computation of multi-particle one-loop amplitudes. Fully analytic computations offer the possibility of extremely fast and accurate evaluation of NLO cross-sections. Details of the analytic structure of the amplitudes, universal factorisation and cancellation of unphysical poles play an important role in the development of fully numerical methods. Further interplay between the two approaches would be useful to find the most flexible and efficient tools for NLO processes at the LHC. Achievements of new methods in numerical computation are presented elsewhere in this report. In this section we summarise recent developments in analytic computations.

Most current techniques involve unitarity cuts and generalised cuts, which are evaluated in regions of phase space where loop propagators are on shell. The expansion of the amplitude in master integrals with rational coefficients can then be evaluated algebraically in terms of tree-level quantities. Unlike in traditional reduction, individual coefficients are derived independently, based on the known analytic structure of the master integrals. A demonstration of the strength of analytic methods at one-loop was seen in the full computation of the six-gluon amplitude, whose components have been helpfully consolidated in [1]. A recent achievement, which we describe below, is the completion of all helicity amplitudes for the production of Higgs plus two jets [2–10].

2. INTEGRAL COEFFICIENTS FROM DOUBLE CUTS

The familiar "double" unitarity cuts (two propagators on shell) yield complete information about large classes of amplitudes. This cut-constructibility underlies the unitarity method of [11, 12] for finding coefficients of the master integrals without reducing Feynman integrals. "Spinor integration" methods, based on Cauchy's residue theorem applied to the unitarity cut, have recently been used to generate closed-form expressions for the coefficients of scalar master integrals [13–16]. The first such formulae [13, 14] were produced for 4-dimensional master integrals in massless theories, starting from tree-level input (the cut integrand) manifesting only physical singularities. From the integrand, the coefficients are obtained through a series of algebraic replacements. The formulae have been generalised to D-dimensional integrals in [17] and to scalar masses in [15].

The cut integrand, written analytically as a product of tree-level amplitudes, may be derived in a very compact form using "MHV diagrams" [18–26] or on-shell recursion relations, particularly in four dimensions with at least two massless particles involved in each tree amplitude [27–43]. Extensions to dimensions other than four have been explored in [44–47]. However, on-shell recursion relations typically feature unphysical singularities, "spurious poles", in their individual terms. In [16], the closed form coefficients of [15] have been generalised to allow any rational functions as cut integrands, in particular with the possible presence of spurious poles.

Current techniques of evaluating unitarity cuts permit numerous variations. As mentioned above, the cuts may be evaluated by the residue theorem each time, incorporating simplifications depending on the specific forms of the integrands; or the available closed forms for coefficients may be used blindly. Certainly, there are intermediate and related approaches as well, which are being explored for optimal efficiency, considering also numerical evaluations. A recent study [48] frames the double-cut phase space integral in terms of Stokes' theorem, bypassing spinors in favour of a conjugate pair of complex scalar integration variables. The cut is evaluated by indefinite integration in one variable followed by Cauchy's residue theorem applied to the conjugate variable. The cuts of bubble integrals are rational functions, so their coefficients may be extracted algebraically by Hermite polynomial reduction. It has also been observed that a unitarity cut, viewed as the imaginary part of the loop amplitude, may be interpreted as a Berry phase of the effective momentum space experienced by the two on-shell particles going around the loop [49]. The result of the phase-space integration is thus the flux of a 2-form given by the product of the two tree amplitudes on either side of the cut.

3. GENERALISED UNITARITY

Generalised unitarity has become an essential tool in the computation of one-loop amplitudes over the past two years. Analytic techniques have focused on generalisations to full QCD amplitudes with arbitrary internal and external masses.

Multiple cuts are well established as an efficient method for the computation of one-loop amplitudes [50]. The quadruple cut technique [51] isolates box coefficients in the one-loop basis, reducing computation to an algebraic procedure. Forde's Laurent expansion technique [52] has been widely used in numerical applications and has also been generalised to the massive case [53]. Further understanding into the analytic structure has led to the interpretation of the triple cut [54] and double cut [48] in terms of Cauchy's and Stokes's Theorem respectively.

D-dimensional cuts with generalised unitarity have also been applied to analytic computations [55] using the well known interpretation of the D-dimensional loop integral as a massive vector [56, 57]. In contrast to numerical applications [57, 58], this allows for a direct computation of the rational contributions without the need to compute quintuple cuts.

Although the D-dimensional cutting method is completely general, in some cases it is preferable to use on-shell recursion relations for the rational terms [59]. As long as a suitable analytic continuation can be found which avoids non-factorising channels, extremely compact analytic forms can be obtained [2–4, 60, 61]. Recently combinations of these techniques have been applied in the context of $H + 2j$ productions [2, 5, 6] and in preliminary studies of $t\bar{t}$ production [62]. Since the methods are all completely algebraic, they are particularly suitable for automation with compact tree-level input taken from on-shell recursion.

For massive one-loop amplitudes, the analytic structure is less understood than in the massless case. In particular, the addition of wave-function and tadpole contributions introduces complications, as these integrals lack four-dimensional branch cuts in momentum channels. A recent analysis proposes computing tadpole coefficients from coefficients of higher-point integrals by introducing an auxiliary, unphysical propagator [63]. The original tadpole integral is then related to an auxiliary integral with two propagators, which can be treated by a conventional double cut. Relations have been found giving the tadpole coefficients in terms of the bubble coefficients of both the original and auxiliary integrals, and the triangle coefficients of the auxiliary integrals. The proof of these relations is accomplished with the help of the integrand classification of [64].

Single cuts, used in conjunction with generalised cutting principles, can be an effective method for deriving full QCD amplitudes [65]. A different single-cut method, proposed as an alternative to generalised unitarity cuts, relies on a "dual" prescription for the imaginary parts of propagators [66].

4. COMPACT ANALYTIC EXPRESSIONS FOR HIGGS PLUS TWO JETS

The program of completing the computation of all helicity amplitudes for $H + 2j$ production at Hadron colliders as recently been completed. This allows for a much faster evaluation (about 10 ms for the full colour/helicity sum) of the cross-section previously available from a semi-numerical computation [7, 8]. A wide variety of the techniques listed above were employed to ensure a compact analytic form.

The calculation was performed in the large top-mass limit where the Higgs couples to the gluons through an effective dimension five operator. A complex Higgs field was decomposed into self-dual (ϕ) and anti-self-dual ($\phi^\dagger$) pieces from which the standard model amplitudes can be constructed from the sum of ϕ and parity symmetric $\phi^\dagger$ amplitudes,

$$A(H,\{p_k\}) = A(\phi,\{p_k\}) + A(\phi^\dagger,\{p_k\}). \tag{1}$$

Helicity amplitudes have been calculated using the standard 2-component Weyl spinor representations and written in terms of spinor products. Results are presented unrenormalised in the four dimensional helicity scheme.

4.1 Full analytic results

The full set of amplitudes collects together the work from a number of different groups which we summarise below:

$H \to gggg$		
Helicity	ϕ	$\phi^\dagger$
$----$	[9]	[10]
$+---$	[5]	[10]
$--++$	[3]	[3]
$-+-+$	[4]	[4]

$H \to \bar{q}qgg$		
Helicity	ϕ	$\phi^\dagger$
$-++-$	[2]	[2]
$-+-+$	[2]	[2]
$-+--$	[6]	[10]

Table 1: The set of independent ϕ and $\phi^\dagger$ helicity amplitudes contributing to $H + 2j$ production together with the references where they can be obtained.

The analytic form of the four quark squared amplitude was presented in the original semi-numerical computation [7]. The helicity amplitudes for this process were computed in reference [2]. The results where obtained using 4-dimensional cutting techniques for the cut-constructible parts. Where applicable on-shell recursion relations gave a compact representation of the rational terms. For the most complicated NMHV configuration and the "all-minus" configuration non-factorising channels in the complex plane were unavoidable and on-shell recursion was not possible. In these cases extremely compact forms were obtained from Feynman diagrams after all information from unphysical poles in the cut-constructible part had been accounted for. It was essential to make full use of the universal IR pole structure in addition to information coming from spurious poles in the logarithmic part.

This calculation relied on some non-trivial relations between terms in the amplitude:

- The rational terms in the $\phi gggg$ amplitude obey:

$$\begin{aligned}\mathcal{R}\left\{A_{4;1}(\phi;1_g,2_g,3_g,4_g)\right\} &= \left(1-\frac{N_f}{N_c}+\frac{N_s}{N_c}\right)R^{N_p}(\phi;1_g,2_g,3_g,4_g)\\ &+2\left(A_4^{(0)}(\phi,1_g,2_g,3_g,4_g)-A_4^{(0)}(\phi^\dagger,1_g,2_g,3_g,4_g)\right)\end{aligned} \tag{2}$$

- The rational terms in the $\phi\bar{q}qgg$ amplitude obey:

$$\begin{aligned}&\mathcal{R}\Big\{A_4^L(\phi;1_{\bar{q}},2_q,3_g,4_g)+A_4^R(\phi;1_{\bar{q}},2_q,3_g,4_g)+A_4^f(\phi;1_{\bar{q}},2_q,3_g,4_g)\Big\}\\ &=2\left(A_4^{(0)}(\phi,1_{\bar{q}},2_q,3_g,4_g)-A_4^{(0)}(\phi^\dagger,1_{\bar{q}},2_q,3_g,4_g)\right)\end{aligned} \tag{3}$$

- The sub-leading colour amplitudes in the $H\bar{q}qgg$ amplitude are completely determined from the leading singularities.

The identities are strongly reminiscent of cancellations seen in SUSY decompositions of pure QCD amplitudes except that they are broken by a universal factor proportional to the tree-level ϕ and $\phi^\dagger$ amplitudes.

As an example we present the colour ordered amplitude for the most complicated "NMHV" configuration in the $Hgggg$ channel [5]. The Feynman diagram representation of this amplitude consists of 739 diagrams with up to rank 4 tensor pentagon integrals. This leading colour amplitude is sufficient to give the full colour information when summed over the appropriate permutations, we refer the reader to [5] for further details.

$$\begin{aligned} A^{(1)}_{4;1}(H,1^+,2^-,3^-,4^-) &= -A^{(0)}_4(H,1^+,2^-,3^-,4^-)\sum_{i=1}^{4}\frac{1}{\epsilon^2}\left(\frac{\mu_R^2}{-s_{i,i+1}}\right)^{\epsilon} \\ &+ F_4(H,1^+,2^-,3^-,4^-)+R_4(H,1^+,2^-,3^-,4^-) \end{aligned} \tag{4}$$

where

$$\begin{aligned} &A^{(0)}_4(H,1^+,2^-,3^-,4^-) = \\ &-\frac{m_H^4\langle 24\rangle^4}{s_{124}\langle 12\rangle\langle 14\rangle\langle 2|p_H|3]\langle 4|p_H|3]}+\frac{\langle 4|p_H|1]^3}{s_{123}\langle 4|p_H|3][12][23]}-\frac{\langle 2|p_H|1]^3}{s_{134}\langle 2|p_H|3][14][34]}, \end{aligned} \tag{5}$$

and

$$\begin{aligned} &F_4(H,1^+,2^-,3^-,4^-) = \Bigg\{\frac{1}{4s_{124}}\left(\frac{\langle 3|p_H|1]^4}{\langle 3|p_H|2]\langle 3|p_H|4][21][41]}+\frac{\langle 24\rangle^4 m_H^4}{\langle 12\rangle\langle 14\rangle\langle 2|p_H|3]\langle 4|p_H|3]}\right)W^{(3)} \\ &-\frac{s_{234}^3}{4\langle 1|p_H|2]\langle 1|p_H|4][23][34]}W^{(1)}-\left(\frac{\langle 2|p_H|1]^3}{2s_{134}\langle 2|p_H|3][34][41]}+\frac{\langle 34\rangle^3 m_H^4}{2s_{134}\langle 1|p_H|2]\langle 3|p_H|2]\langle 41\rangle}\right)W^{(2)} \\ &+2C^{3\mathrm{m}}_{3;H|12|34}(1^+,2^-,3^-,4^-)\mathrm{I}_3^{3\mathrm{m}}(m_H^2,s_{12},s_{34})+\left(1-\frac{N_f}{4N_c}\right)\left(\frac{\langle 3|p_H|1]^2}{s_{124}[24]^2}\mathrm{F}^{1\mathrm{m}}(s_{12},s_{14};s_{124})\right. \\ &\left.-\frac{4\langle 24\rangle\langle 3|p_H|1]^2}{s_{124}[42]}\hat{L}_1\left(s_{124},s_{12}\right)+\frac{4\langle 23\rangle\langle 4|p_H|1]^2}{s_{123}[32]}\hat{L}_1\left(s_{123},s_{12}\right)\right)-\left(1-\frac{N_f}{N_c}+\frac{N_s}{N_c}\right)\times \\ &\left(\frac{[12][41]\langle 3|p_H|2]\langle 3|p_H|4]}{2s_{124}[24]^4}\mathrm{F}^{1\mathrm{m}}(s_{12},s_{14};s_{124})+\left(\frac{2s_{124}\langle 34\rangle^2[41]^2}{\langle 24\rangle[42]^3}-\frac{\langle 24\rangle\langle 3|p_H|1]^2}{3s_{124}[42]}\right)\hat{L}_1\left(s_{124},s_{12}\right)\right. \\ &+\frac{2s_{124}\langle 24\rangle\langle 34\rangle^2[41]^2}{3[42]}\hat{L}_3\left(s_{124},s_{12}\right)+\frac{\langle 34\rangle[41]\left(3s_{124}\langle 34\rangle[41]+\langle 24\rangle\langle 3|p_H|1][42]\right)}{3[42]^2}\hat{L}_2\left(s_{124},s_{12}\right) \\ &+\frac{\langle 3|p_H|1](4s_{124}\langle 34\rangle[41]+\langle 3|p_H|1](2s_{14}+s_{24})}{s_{124}\langle 24\rangle[42]^3}\hat{L}_0\left(s_{124},s_{12}\right)-\frac{2s_{123}\langle 23\rangle\langle 34\rangle^2[31]^2}{3[32]}\hat{L}_3\left(s_{123},s_{12}\right) \\ &\left.+\frac{\langle 23\rangle\langle 34\rangle[31]\langle 4|p_H|1]}{3[32]}\hat{L}_2\left(s_{123},s_{12}\right)+\frac{\langle 23\rangle\langle 4|p_H|1]^2}{3s_{123}[32]}\hat{L}_1\left(s_{123},s_{12}\right)\right)\Bigg\}+\Big\{(2\leftrightarrow 4)\Big\}. \end{aligned} \tag{6}$$

For convenience we have introduced the following combinations of the finite pieces of one-mass ($\mathrm{F}^{1\mathrm{m}}$) and two-mass hard ($\mathrm{F}^{2\mathrm{mh}}$) box functions,

$$\begin{aligned} W^{(1)} &= \mathrm{F}^{1\mathrm{m}}(s_{23},s_{34};s_{234})+\mathrm{F}^{2\mathrm{mh}}(s_{41},s_{234};m_H^2,s_{23})+\mathrm{F}^{2\mathrm{mh}}(s_{12},s_{234};s_{34},m_H^2) \\ W^{(2)} &= \mathrm{F}^{1\mathrm{m}}(s_{14},s_{34};s_{134})+\mathrm{F}^{2\mathrm{mh}}(s_{12},s_{134};m_H^2,s_{34})+\mathrm{F}^{2\mathrm{mh}}(s_{23},s_{134};s_{14},m_H^2) \\ W^{(3)} &= \mathrm{F}^{1\mathrm{m}}(s_{12},s_{14};s_{124})+\mathrm{F}^{2\mathrm{mh}}(s_{23},s_{124};m_H^2,s_{14})+\mathrm{F}^{2\mathrm{mh}}(s_{34},s_{124};s_{12},m_H^2). \end{aligned}$$

The bubble coefficients have been re-arranged into logarithm functions, $L_k = \frac{\log(s/t)}{(s-t)^k}$, which have smooth behaviour in the various collinear limits,

$$\begin{aligned} \hat{L}_3(s,t) &= L_3(s,t)-\frac{1}{2(s-t)^2}\left(\frac{1}{s}+\frac{1}{t}\right) \quad , \quad \hat{L}_1(s,t)=L_1(s,t), \\ \hat{L}_2(s,t) &= L_2(s,t)-\frac{1}{2(s-t)}\left(\frac{1}{s}+\frac{1}{t}\right) \quad , \quad \hat{L}_0(s,t)=L_0(s,t). \end{aligned} \tag{7}$$

Representations for the scalar integrals can be found in the literature [67–70]. The three mass triangle coefficient was obtained from Forde's Laurent expansion procedure [52],

$$C^{3\mathrm{m}}_{3;H|12|34}(1^+,2^-,3^-,4^-) = \sum_{\gamma=\gamma_\pm(p_H,p_1+p_2)} \frac{-m_\phi^4 \langle K_1^\flat 2\rangle^3 \langle 34\rangle^3}{2\gamma(\gamma+m_\phi^2)\langle K_1^\flat 1\rangle \langle K_1^\flat 3\rangle \langle K_1^\flat 4\rangle \langle 12\rangle}, \tag{8}$$

where $K_1 = p_H, K_2 = p_1 + p_2$, and

$$\begin{aligned} K_1^{\flat,\mu} &= \gamma \frac{\gamma K_1^\mu - K_1^2 K_2^\mu}{\gamma^2 - K_1^2 K_2^2}, \quad K_2^{\flat,\mu} = \gamma \frac{\gamma K_2^\mu - K_2^2 K_1^\mu}{\gamma^2 - K_1^2 K_2^2}, \\ \gamma_\pm(K_1,K_2) &= K_1 \cdot K_2 \pm \sqrt{K_1 \cdot K_2^2 - K_1^2 K_2^2}. \end{aligned} \tag{9}$$

The rational part (which incorporates the rational $A_4^{(1)}(\phi^\dagger, 1^+, 2^-, 3^-, 4^-)$ amplitude derived in [10]) is

$$\begin{aligned} R_4(H,1^+,2^-,3^-,4^-) = \quad & \Bigg\{ \Big(1 - \frac{N_f}{N_c} + \frac{N_s}{N_c}\Big) \frac{1}{2} \Big(\frac{\langle 23\rangle \langle 34\rangle \langle 4|p_H|1]\, [31]}{3 s_{123} \langle 12\rangle [21] [32]} - \frac{\langle 3|p_H|1]^2}{s_{124} [42]^2} \\ & + \frac{\langle 24\rangle \langle 34\rangle \langle 3|p_H|1]\, [41]}{3 s_{124} s_{12} [42]} - \frac{[12]^2 \langle 23\rangle^2}{s_{14} [42]^2} - \frac{\langle 24\rangle (s_{23}s_{24} + s_{23}s_{34} + s_{24}s_{34})}{3 \langle 12\rangle \langle 14\rangle [23][34][42]} \\ & + \frac{\langle 2|p_H|1]\, \langle 4|p_H|1]}{3 s_{234} [23][34]} - \frac{2[12]\langle 23\rangle [31]^2}{3[23]^2 [41][34]} \Big) \Bigg\} + \Big\{ (2 \leftrightarrow 4) \Big\}. \end{aligned} \tag{10}$$

Further study into the origin of the simplicity in the sub-leading colour amplitudes would be interesting and may shed light on possible cancellations in other processes [71]. The full results for all helicity configurations have been made available at `http://mcfm.fnal.gov`.

References

[1] D. C. Dunbar, *Nucl. Phys. Proc. Suppl.* **183** (2008) 122–136.

[2] L. J. Dixon and Y. Sofianatos, *JHEP* **08** (2009) 058, [0906.0008].

[3] S. D. Badger, E. W. N. Glover, and K. Risager, *JHEP* **07** (2007) 066, [0704.3914].

[4] E. W. N. Glover, P. Mastrolia, and C. Williams, *JHEP* **08** (2008) 017, [0804.4149].

[5] S. Badger, E. W. N. Glover, P. Mastrolia, and C. Williams, 0909.4475.

[6] S. Badger, J. M. Campbell, R. K. Ellis, and C. Williams, 0910.4481.

[7] R. K. Ellis, W. T. Giele, and G. Zanderighi, *Phys. Rev.* **D72** (2005) 054018, [hep-ph/0506196].

[8] J. M. Campbell, R. K. Ellis, and G. Zanderighi, *JHEP* **10** (2006) 028, [hep-ph/0608194].

[9] S. D. Badger and E. W. N. Glover, *Nucl. Phys. Proc. Suppl.* **160** (2006) 71–75, [hep-ph/0607139].

[10] C. F. Berger, V. Del Duca, and L. J. Dixon, *Phys. Rev.* **D74** (2006) 094021, [hep-ph/0608180].

[11] Z. Bern, L. J. Dixon, D. C. Dunbar, and D. A. Kosower, *Nucl. Phys.* **B425** (1994) 217–260, [hep-ph/9403226].

[12] Z. Bern, L. J. Dixon, D. C. Dunbar, and D. A. Kosower, *Nucl. Phys.* **B435** (1995) 59–101, [hep-ph/9409265].

[13] R. Britto and B. Feng, *Phys. Rev.* **D75** (2007) 105006, [hep-ph/0612089].

[14] R. Britto and B. Feng, *JHEP* **02** (2008) 095, [0711.4284].

[15] R. Britto, B. Feng, and P. Mastrolia, *Phys. Rev.* **D78** (2008) 025031, [0803.1989].

[16] B. Feng and G. Yang, *Nucl. Phys.* **B811** (2009) 305–352, [0806.4016].

[17] R. Britto, B. Feng, and G. Yang, *JHEP* **09** (2008) 089, [0803.3147].

[18] F. Cachazo, P. Svrcek, and E. Witten, *JHEP* **09** (2004) 006, [hep-th/0403047].

[19] G. Georgiou, E. W. N. Glover, and V. V. Khoze, *JHEP* **07** (2004) 048, [hep-th/0407027].

[20] L. J. Dixon, E. W. N. Glover, and V. V. Khoze, *JHEP* **12** (2004) 015, [hep-th/0411092].

[21] Z. Bern, D. Forde, D. A. Kosower, and P. Mastrolia, *Phys. Rev.* **D72** (2005) 025006, [hep-ph/0412167].

[22] S. D. Badger, E. W. N. Glover, and V. V. Khoze, *JHEP* **03** (2005) 023, [hep-th/0412275].

[23] C. Schwinn and S. Weinzierl, *JHEP* **05** (2005) 006, [hep-th/0503015].

[24] K. J. Ozeren and W. J. Stirling, *JHEP* **11** (2005) 016, [hep-th/0509063].

[25] R. Boels and C. Schwinn, *Phys. Lett.* **B662** (2008) 80–86, [0712.3409].

[26] C. Schwinn, *Phys. Rev.* **D78** (2008) 085030, [0809.1442].

[27] R. Britto, F. Cachazo, and B. Feng, *Nucl. Phys.* **B715** (2005) 499–522, [hep-th/0412308].

[28] R. Britto, F. Cachazo, B. Feng, and E. Witten, *Phys. Rev. Lett.* **94** (2005) 181602, [hep-th/0501052].

[29] M.-x. Luo and C.-k. Wen, *JHEP* **03** (2005) 004, [hep-th/0501121].

[30] M.-x. Luo and C.-k. Wen, *Phys. Rev.* **D71** (2005) 091501, [hep-th/0502009].

[31] J. Bedford, A. Brandhuber, B. J. Spence, and G. Travaglini, *Nucl. Phys.* **B721** (2005) 98–110, [hep-th/0502146].

[32] F. Cachazo and P. Svrcek, hep-th/0502160.

[33] R. Britto, B. Feng, R. Roiban, M. Spradlin, and A. Volovich, *Phys. Rev.* **D71** (2005) 105017, [hep-th/0503198].

[34] S. D. Badger, E. W. N. Glover, V. V. Khoze, and P. Svrcek, *JHEP* **07** (2005) 025, [hep-th/0504159].

[35] S. D. Badger, E. W. N. Glover, and V. V. Khoze, *JHEP* **01** (2006) 066, [hep-th/0507161].

[36] D. Forde and D. A. Kosower, *Phys. Rev.* **D73** (2006) 065007, [hep-th/0507292].

[37] P. Ferrario, G. Rodrigo, and P. Talavera, *Phys. Rev. Lett.* **96** (2006) 182001, [hep-th/0602043].

[38] K. J. Ozeren and W. J. Stirling, *Eur. Phys. J.* **C48** (2006) 159–168, [hep-ph/0603071].

[39] C. Schwinn and S. Weinzierl, *JHEP* **04** (2007) 072, [hep-ph/0703021].

[40] N. Arkani-Hamed and J. Kaplan, *JHEP* **04** (2008) 076, [0801.2385].

[41] A. Brandhuber, P. Heslop, and G. Travaglini, *Phys. Rev.* **D78** (2008) 125005, [0807.4097].

[42] C. Cheung, 0808.0504.

[43] J. M. Drummond and J. M. Henn, *JHEP* **04** (2009) 018, [0808.2475].

[44] C. Quigley and M. Rozali, *JHEP* **03** (2006) 004, [hep-ph/0510148].

[45] C. Cheung and D. O'Connell, *JHEP* **07** (2009) 075, [0902.0981].

[46] R. Boels, 0908.0738.

[47] T. Dennen, Y.-t. Huang, and W. Siegel, 0910.2688.

[48] P. Mastrolia, *Phys. Lett.* **B678** (2009) 246–249, [0905.2909].

[49] P. Mastrolia, 0906.3789.

[50] Z. Bern, L. J. Dixon, and D. A. Kosower, *Nucl. Phys.* **B513** (1998) 3–86, [hep-ph/9708239].

[51] R. Britto, F. Cachazo, and B. Feng, *Nucl. Phys.* **B725** (2005) 275–305, [hep-th/0412103].

[52] D. Forde, *Phys. Rev.* **D75** (2007) 125019, [0704.1835].

[53] W. B. Kilgore, 0711.5015.

[54] N. E. J. Bjerrum-Bohr, D. C. Dunbar, and W. B. Perkins, *JHEP* **04** (2008) 038, [0709.2086].

[55] S. D. Badger, *JHEP* **01** (2009) 049, [0806.4600].

[56] Z. Bern and A. G. Morgan, *Nucl. Phys.* **B467** (1996) 479–509, [hep-ph/9511336].

[57] W. T. Giele, Z. Kunszt, and K. Melnikov, *JHEP* **04** (2008) 049, [0801.2237].

[58] R. K. Ellis, W. T. Giele, Z. Kunszt, and K. Melnikov, *Nucl. Phys.* **B822** (2009) 270–282, [0806.3467].

[59] Z. Bern, L. J. Dixon, and D. A. Kosower, *Phys. Rev.* **D71** (2005) 105013, [hep-th/0501240].

[60] C. F. Berger, Z. Bern, L. J. Dixon, D. Forde, and D. A. Kosower, *Phys. Rev.* **D74** (2006) 036009, [hep-ph/0604195].

[61] C. F. Berger, Z. Bern, L. J. Dixon, D. Forde, and D. A. Kosower, *Phys. Rev.* **D75** (2007) 016006, [hep-ph/0607014].

[62] S. D. Badger, *Nucl. Phys. Proc. Suppl.* **183** (2008) 220–225, [0807.1245].

[63] R. Britto and B. Feng, *Phys. Lett.* **B681** (2009) 376–381, [0904.2766].

[64] G. Ossola, C. G. Papadopoulos, and R. Pittau, *Nucl. Phys.* **B763** (2007) 147–169, [hep-ph/0609007].

[65] E. W. Nigel Glover and C. Williams, *JHEP* **12** (2008) 067, [0810.2964].

[66] S. Catani, T. Gleisberg, F. Krauss, G. Rodrigo, and J.-C. Winter, *JHEP* **09** (2008) 065, [0804.3170].

[67] Z. Bern, L. J. Dixon, and D. A. Kosower, *Nucl. Phys.* **B412** (1994) 751–816, [hep-ph/9306240].

[68] A. Denner, U. Nierste, and R. Scharf, *Nucl. Phys.* **B367** (1991) 637–656.

[69] A. van Hameren, J. Vollinga, and S. Weinzierl, *Eur. Phys. J.* **C41** (2005) 361–375, [hep-ph/0502165].

[70] R. K. Ellis and G. Zanderighi, *JHEP* **02** (2008) 002, [0712.1851].

[71] S. Badger, N. E. J. Bjerrum-Bohr, and P. Vanhove, *JHEP* **02** (2009) 038, [0811.3405].