Lecture 2
In 4-dimensions, the contribution of the real diagrams can be written (ignoring diagrams with incoming gluons for simplicity)

\[
|M(\bar{u}d \rightarrow W^+ g)|^2 \sim g^2 C_F \left[ \frac{\hat{u}}{\hat{t}} + \frac{\hat{t}}{\hat{u}} + \frac{2Q^2 \hat{s}}{\hat{u}\hat{t}} \right]
\]

\[
\sim g^2 C_F \left[ \left( \frac{1 + z^2}{1 - z} \right) \left( \frac{-\hat{s}}{\hat{t}} + \frac{-\hat{s}}{\hat{u}} \right) - 2 \right]
\]

where

\[
z = \frac{Q^2}{s} \quad \text{and} \quad \hat{s} + \hat{t} + \hat{u} = Q^2
\]

Note that the real diagrams contain collinear singularities, \( u \rightarrow 0 \), \( t \rightarrow 0 \), and soft singularities, \( z \rightarrow 1 \)

...thanks to Keith Ellis for the next few slides
Now do the dimension trick for the real part

- Working in $4-2\varepsilon$ dimensions, to control the divergences (dimensional reduction)

\[
\sigma_{\text{real}} = \frac{\alpha_s}{2\pi} C_F \left( \frac{\mu^2}{Q^2} \right)^\varepsilon c_\Gamma \left[ \left( \frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} - \frac{\pi^2}{3} \right) \delta(1-z) - \frac{2}{\varepsilon} P_{qq}(z) - 2(1-z) + 4\left(1 + z^2\right) \left[ \ln(1-z) \right]_+ - 2 \frac{1 + z^2}{1 - z} \ln z \right]
\]

- with

\[
c_\Gamma = \frac{(4\pi)^\varepsilon}{\Gamma(1-\varepsilon)}
\]

We get $1/\varepsilon$ terms from individual soft and collinear singularities

We get $1/\varepsilon^2$ terms for overlapping IR singularities.
\[ \sigma_{virt} = \delta(1 - z) \left[ 1 + \frac{\alpha_s}{2\pi} C_F \left( \frac{\mu^2}{Q^2} \right) \varepsilon c'_\Gamma \left( -\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - 6 + \pi^2 \right) \right] \]

where

\[ c'_\Gamma = c_\Gamma + O(\varepsilon^3) \]

We also get UV divergences when the loop momenta go off to infinity. The summation of these singularities leads to the running of the strong couplings, i.e. we define the sum of all such contributions (scales \( > \mu_{UV} \)) as the physical renormalized coupling, \( \alpha_s \). Some more details in extra slides.
Now add real and virtual

\[ \sigma_{\text{real+virt}} = \frac{\alpha_s}{2\pi} C_F \left( \frac{\mu^2}{Q^2} \right)^\epsilon c_F \left[ \left( \frac{2\pi^2}{3} - 6 \right) \delta(1 - z) - \frac{2}{\epsilon} P_{qq}(z) - 2(1 - z) + 4(1 + z^2) \right] \left[ \frac{\ln(1 - z)}{1 - z} \right] + 2 \frac{1 + z^2}{1 - z} \ln z \]

- Notice that the \( \epsilon^2 \) terms cancel
- The divergences, proportional to the branching probabilities, are universal
- We can factorize them into the parton distributions, performing mass factorization by subtracting the counterterm (MSbar scheme)

\[ 2 \frac{\alpha_s}{2\pi} C_F \left[ -\frac{c_F}{\epsilon} P_{qq}(z) - (1 - z) + \delta(1 - z) \right] \]

- To get

\[ \hat{\sigma}_{\text{real+virt}} = \frac{\alpha_s}{2\pi} C_F \left[ \left( \frac{2\pi^2}{3} - 8 \right) \delta(1 - z) + 4(1 + z^2) \right] \left[ \frac{\ln(1 - z)}{1 - z} \right] + 2 \frac{1 + z^2}{1 - z} \ln z + 2 P_{qq}(z) \ln \frac{Q^2}{\mu^2} \]

- Plus a similar correction for incoming gluons
- That works for the total cross section, but we need differential distributions for comparisons to data, so we need a general subtraction procedure at NLO, using Monte Carlo techniques
In general

- That works for the total cross section, but we need differential distributions for comparisons to data, so we need a general subtraction procedure at NLO, using Monte Carlo techniques.
- For incoming partons \(a\) and \(b\), producing \(m\) outgoing partons

\[
\sigma_{ab} = \sigma_{ab}^{LO} + \sigma_{ab}^{NLO}
\]

\[
\sigma_{ab}^{LO} = \int d\sigma_{ab}^{Born}
\]

\[
\sigma_{ab}^{NLO} = \int_{m+1} d\sigma_{ab}^{real} + \int_{m} d\sigma_{ab}^{virt}
\]

- Construct a series of counter-terms

\[
d\sigma_{ct} = \sum_{ct} \int_{m} d\sigma_B \otimes \int_{1} dV_{ct}
\]

- Where \(\sigma_B\) denotes the appropriate color and spin projection of the Born level cross section, and the counter-terms are independent of the details of the process under consideration.
Catani-Seymour subtraction

- These counter-terms cancel all non-integrable singularities in $d\sigma^{\text{real}}$, so that one can write
  \[
  \sigma^{\text{NLO}}_{ab} = \int_{m+1} d\sigma^{\text{real}}_{ab} - d\sigma^{\text{ct}}_{ab} + \int_{m+1} d\sigma^{\text{ct}}_{ab} + \int_{m} d\sigma^{\text{virt}}_{ab}
  \]

- The phase space integration in the first term can now be performed numerically in 4 dimensions

- The integral in the 2\textsuperscript{nd} term can be done easily and analytically
Consider matrix element counter-event for W production

In soft limit ($p_5 \to 0$), we have

$$|M_1(p_1, p_2, p_3, p_4, p_5)|^2 = g^2 C_F \frac{p_1 \cdot p_2}{p_1 \cdot p_5 p_2 \cdot p_5} |M_0(p_1, p_2, p_3, p_4)|^2$$

The eikonal factor can be associated with radiation from a given leg by partial fractioning

$$\frac{p_1 \cdot p_2}{p_1 \cdot p_5 p_2 \cdot p_5} = \left[ \frac{p_1 \cdot p_2}{p_1 \cdot p_5 + p_2 \cdot p_5} \right] \left[ \frac{1}{p_1 \cdot p_5} + \frac{1}{p_2 \cdot p_5} \right]$$

Including the collinear contributions, singular as $p_1 \cdot p_5 \to 0$, the matrix element for the counter-event has the structure

$$|M_1(p_1, p_2, p_3, p_4, p_5)|^2 = \frac{g^2}{x_a p_1 \cdot p_5} \hat{P}_{qq}(x_a) |M_0(p_1, p_2, p_3, p_4)|^2$$

where

$$1 - x_a = \frac{p_1 \cdot p_5 + p_2 \cdot p_5}{p_1 \cdot p_2} \quad \hat{P}_{qq}(x_a) = C_F \frac{1 + x^2}{1 - x}$$

These are the Catani-Seymour dipoles (actually single collinear poles produced by partial fractioning); Keith Ellis thinks Catani and Seymour should be shot for calling these dipoles; maybe we can take a vote.
Programs that do NLO calculations, such as MCFM, are parton-level Monte Carlo generators in which (weighted) events and counter-events are generated

- for complicated processes, such as $W + 2$ jets, there can be many counter-events (24), corresponding to the Catani-Seymour subtraction terms, for each event

- only the sum of all events (events + counter-events) is meaningful, since many positive and negative weights need to cancel against each other; if too few events are generated, or if the binning is too small, can have negative results

- in general, cannot connect these complex NLO matrix elements to parton showering…although that’s the dream/plan

  - processes such as $W, Z, Higgs, t\bar{t},$ single top,… have been included in NLO parton shower Monte Carlo programs like MC@NLO, Powheg
Many processes available at LO and NLO

- Note these are partonic level only

Option for ROOT output (see later)

mcfm.fnal.gov

<table>
<thead>
<tr>
<th>Process 1</th>
<th>Process 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p\bar{p} \rightarrow W^{\pm}/Z$</td>
<td>$p\bar{p} \rightarrow W^+ + W^-$</td>
</tr>
<tr>
<td>$p\bar{p} \rightarrow W^{\pm} + Z$</td>
<td>$p\bar{p} \rightarrow Z + Z$</td>
</tr>
<tr>
<td>$p\bar{p} \rightarrow W^{\pm} + \gamma$</td>
<td>$p\bar{p} \rightarrow W^\pm/Z + H$</td>
</tr>
<tr>
<td>$p\bar{p} \rightarrow W^{\pm} + g^* (\rightarrow b\bar{b})$</td>
<td>$p\bar{p} \rightarrow Zb\bar{b}$</td>
</tr>
<tr>
<td>$p\bar{p} \rightarrow W^{\pm}/Z + 1 \text{ jet}$</td>
<td>$p\bar{p} \rightarrow W^\pm/Z + 2 \text{ jets}$</td>
</tr>
<tr>
<td>$p\bar{p}(gg) \rightarrow H$</td>
<td>$p\bar{p}(gg) \rightarrow H + 1 \text{ jet}$</td>
</tr>
<tr>
<td>$p\bar{p}(VV) \rightarrow H + 2 \text{ jets}$</td>
<td>$p\bar{p} \rightarrow t + X$</td>
</tr>
<tr>
<td>$pp \rightarrow t + W$</td>
<td></td>
</tr>
</tbody>
</table>
State of the art

- LO: well under control, even for multiparticle final states
- NLO: well understood for 2->1, 2->2 and 2->3; first calculations of 2->4 (W +3 jets, ttbb)
- NNLO: known for inclusive and exclusive 2->1 (i.e. Higgs, Drell-Yan); work on 2->2 (Higgs + 1 jet)

<table>
<thead>
<tr>
<th>Relative order</th>
<th>2-&gt;1</th>
<th>2-&gt;2</th>
<th>2-&gt;3</th>
<th>2-&gt;4</th>
<th>2-5</th>
<th>2-&gt;6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>LO</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\alpha_s$</td>
<td>NLO</td>
<td>LO</td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>$\alpha_s^2$</td>
<td>NNLO</td>
<td>NLO</td>
<td>LO</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\alpha_s^3$</td>
<td></td>
<td>NNLO</td>
<td>NLO</td>
<td>LO</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\alpha_s^4$</td>
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<td></td>
<td>NLO</td>
<td>LO</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\alpha_s^5$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>LO</td>
</tr>
</tbody>
</table>
Some issues/questions

- Once we have the calculations, how do we (experimentalists) use them?
- Best is to have NLO partonic level calculation interfaced to parton shower/hadronization
  - but that has been done only for relatively simple processes and is very (theorist) labor intensive
    ▲ still waiting for inclusive jets in MC@NLO, for example

- Even with partonic level calculations, need public code and/or ability to write out ROOT ntuples of parton level events
  - so that can generate once with loose cuts and distributions can be re-made without the need for the lengthy re-running of the predictions
  - what is done for example with MCFM for CTEQ4LHC
    ▲ but 10’s of Gbytes

- See discussion later
K-factors may differ from one because of new subprocesses/contributions at higher order and/or differences between LO and NLO pdf’s
Some rules-of-thumb

NLO corrections are larger for processes in which there is a great deal of color annihilation

- \( gg \rightarrow \text{Higgs} \)
- \( gg \rightarrow \gamma\gamma \)
- \( K(\text{gg-}\rightarrow t\bar{t}) > K(\text{qQ-}\rightarrow t\bar{t}) \)
- these \( gg \) initial states want to radiate like crazy (see Sudakovs)

NLO corrections decrease as more final-state legs are added

- \( K(\text{gg-}\rightarrow \text{Higgs + 2 jets}) < K(\text{gg-}\rightarrow \text{Higgs + 1 jet}) < K(\text{gg-}\rightarrow \text{Higgs}) \)
- unless can access new initial state gluon channel

Can we generalize for uncalculated HO processes?

What about effect of jet vetoes on \( K \)-factors? Signal processes compared to background

### Table 2: \( K \)-factors for various processes at the Tevatron and the LHC calculated using a selection of input parameters. In all cases, the CTEx6 PDF set is used at NLO. \( K \) uses the CTEQ6L1 set at leading order, whilst \( K' \) uses the same set, CTEQ6M, as at NLO. For most of the processes listed, jets satisfy the requirements \( p_T > 15 \text{ GeV/c} \) and \( |\eta| < 2.5 \) (5.0) at the Tevatron (LHC). For \( \text{Higgs+1,2jets} \), a jet cut of 40 GeV/c and \( |\eta| < 4.5 \) has been applied. A cut of \( p_T^{\text{jet}} > 20 \text{ GeV/c} \) has been applied for the \( t+\text{jet} \) process, and a cut of \( p_T^{\text{jet}} > 50 \text{ GeV/c} \) for \( WW+\text{jet} \). In the \( W(\text{Higgs})+\text{2jets} \) process the jets are separated by \( \Delta R > 0.52 \), whilst the VBF calculations are performed for a Higgs boson of mass 120 GeV. In each case the value of the \( K \)-factor is compared at two often-used scale choices, where the scale indicated is used for both renormalization and factorization scales.

<table>
<thead>
<tr>
<th>Process</th>
<th>Typical scales</th>
<th>Tevatron ( K )-factor</th>
<th>LHC ( K )-factor</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \mu_0 )</td>
<td>( \mu_3 )</td>
<td>( K(\mu_0) )</td>
</tr>
<tr>
<td>( W )</td>
<td>( m_W )</td>
<td>( 2m_W )</td>
<td>1.33</td>
</tr>
<tr>
<td>( W+1\text{jet} )</td>
<td>( m_W )</td>
<td>( p_T^{\text{jet}} )</td>
<td>1.42</td>
</tr>
<tr>
<td>( W+2\text{jets} )</td>
<td>( m_W )</td>
<td>( p_T^{\text{jet}} )</td>
<td>1.16</td>
</tr>
<tr>
<td>( WW+\text{jet} )</td>
<td>( m_W )</td>
<td>( 2m_W )</td>
<td>1.19</td>
</tr>
<tr>
<td>( t\bar{t} )</td>
<td>( m_t )</td>
<td>( 2m_t )</td>
<td>1.08</td>
</tr>
<tr>
<td>( t\bar{t}+1\text{jet} )</td>
<td>( m_t )</td>
<td>( 2m_t )</td>
<td>1.13</td>
</tr>
<tr>
<td>( b\bar{b} )</td>
<td>( m_b )</td>
<td>( 2m_b )</td>
<td>1.20</td>
</tr>
<tr>
<td>( \text{Higgs} )</td>
<td>( m_H )</td>
<td>( p_T^{\text{jet}} )</td>
<td>2.33</td>
</tr>
<tr>
<td>( \text{Higgs via VBF} )</td>
<td>( m_H )</td>
<td>( p_T^{\text{jet}} )</td>
<td>1.07</td>
</tr>
<tr>
<td>( \text{Higgs+1jet} )</td>
<td>( m_H )</td>
<td>( p_T^{\text{jet}} )</td>
<td>2.02</td>
</tr>
<tr>
<td>( \text{Higgs+2jets} )</td>
<td>( m_H )</td>
<td>( p_T^{\text{jet}} )</td>
<td>--</td>
</tr>
</tbody>
</table>

Simplistic rule

\[ C_{i1} + C_{i2} - C_{f,\text{max}} \]

L. Dixon

Casimir color factors for initial state

Casimir for biggest color representation final state can be in
## An experimenter’s wishlist

<table>
<thead>
<tr>
<th>Single Boson</th>
<th>Diboson</th>
<th>Triboson</th>
<th>Heavy Flavour</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W^+ \leq 5j$</td>
<td>$WW^+ \leq 5j$</td>
<td>$WWW^+ \leq 3j$</td>
<td>$t\bar{t}^+ \leq 3j$</td>
</tr>
<tr>
<td>$W + b\bar{b} \leq 3j$</td>
<td>$W + b\bar{b}^+ \leq 3j$</td>
<td>$WWW + b\bar{b}^+ \leq 3j$</td>
<td>$t\bar{t} + \gamma^+ \leq 2j$</td>
</tr>
<tr>
<td>$W + c\bar{c} \leq 3j$</td>
<td>$W + c\bar{c}^+ \leq 3j$</td>
<td>$WWW + \gamma\gamma^+ \leq 3j$</td>
<td>$t\bar{t} + W^+ \leq 2j$</td>
</tr>
<tr>
<td>$Z^+ \leq 5j$</td>
<td>$ZZ^+ \leq 5j$</td>
<td>$Z\gamma\gamma^+ \leq 3j$</td>
<td>$t\bar{t} + Z^+ \leq 2j$</td>
</tr>
<tr>
<td>$Z + b\bar{b}^+ \leq 3j$</td>
<td>$Z + b\bar{b}^+ \leq 3j$</td>
<td>$ZZZ^+ \leq 3j$</td>
<td>$t\bar{t} + H^+ \leq 2j$</td>
</tr>
<tr>
<td>$Z + c\bar{c}^+ \leq 3j$</td>
<td>$ZZ + c\bar{c}^+ \leq 3j$</td>
<td>$ZZZ^+ \leq 3j$</td>
<td>$t\bar{b} \leq 2j$</td>
</tr>
<tr>
<td>$\gamma^+ \leq 5j$</td>
<td>$\gamma\gamma^+ \leq 5j$</td>
<td>$ZZZ^+ \leq 3j$</td>
<td>$b\bar{b}^+ \leq 3j$</td>
</tr>
<tr>
<td>$\gamma + b\bar{b} \leq 3j$</td>
<td>$\gamma\gamma + b\bar{b} \leq 3j$</td>
<td>single top</td>
<td></td>
</tr>
<tr>
<td>$\gamma + c\bar{c} \leq 3j$</td>
<td>$\gamma\gamma + c\bar{c} \leq 3j$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$WZ^+ \leq 5j$</td>
<td></td>
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</tr>
<tr>
<td></td>
<td>$WZ + b\bar{b} \leq 3j$</td>
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<td>$WZ + c\bar{c} \leq 3j$</td>
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<tr>
<td></td>
<td>$W\gamma^+ \leq 3j$</td>
<td></td>
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<tr>
<td></td>
<td>$Z\gamma^+ \leq 3j$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Identifying important missing processes at the LHC

- In 2005 (and then continued in 2007), a priority list of processes to be calculated at NLO was created.
- Each of these calculations requires a tremendous amount of work, and thus the need for an experimental priority list:
  - the calculation can be done
  - the calculation needs to be done
- The Les Houches wishlist from 2005/2007 is filling up slowly but progressively.
- In 2009, we added tttt, Wbbj, Z+3j, W+4j plus an extra column for each process indicating the level of precision required by the experiments:
  - to see for example if EW corrections may need to be calculated
- In order to be most useful, decays for final state particles (t,W,H) need to be provided in the codes as well.

<table>
<thead>
<tr>
<th>Process</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V \in { Z,W,\gamma } )</td>
<td>Calculations completed since Les Houches 2005</td>
</tr>
<tr>
<td>1. ( pp \rightarrow VV ) jet</td>
<td>( WW ) jet completed by Dittmaier/Kallweit/Uwer, Campbell/ Ellis/Zanderighi and Binoth/Karg/Kauer/Sanguinetti (in progress)</td>
</tr>
<tr>
<td>2. ( pp \rightarrow Higgs+2jets )</td>
<td>NLO QCD to the ( gg ) channel completed by Campbell/Ellis/Zanderighi; NLO QCD+EW to the VBF channel completed by Ciccolini/Denner/Dittmaier</td>
</tr>
<tr>
<td>3. ( pp \rightarrow VV )</td>
<td>( ZZ ) completed by Lazopoulos/Melnikov/Petriello and ( WW ) by Hankele/Zeppenfeld</td>
</tr>
<tr>
<td>4. ( pp \rightarrow tt ) b\bar{b}</td>
<td>relevant for ( ttH )</td>
</tr>
<tr>
<td>5. ( pp \rightarrow t\bar{t}+2jets )</td>
<td>relevant for ( ttH )</td>
</tr>
<tr>
<td>6. ( pp \rightarrow VV ) b\bar{b}</td>
<td>relevant for VBF ( \rightarrow H \rightarrow VV, ttH )</td>
</tr>
<tr>
<td>7. ( pp \rightarrow VV+2jets )</td>
<td>relevant for VBF ( \rightarrow H \rightarrow VV )</td>
</tr>
<tr>
<td>8. ( pp \rightarrow V+3jets )</td>
<td>VBF contributions calculated by (Borzi)/Jüger/Oleari/Zeppenfeld and various new physics signatures</td>
</tr>
<tr>
<td>NLO calculations added to list in 2007</td>
<td>Higgs and new physics signatures</td>
</tr>
<tr>
<td>9. ( pp \rightarrow b\bar{b}b\bar{b} )</td>
<td>Calculations beyond NLO added in 2007</td>
</tr>
<tr>
<td>10. ( gg \rightarrow W^{+}W^{-}O(\alpha^{2}g) )</td>
<td>backgrounds to Higgs</td>
</tr>
<tr>
<td>11. ( NNLO pp \rightarrow tt )</td>
<td>normalization of a benchmark process</td>
</tr>
<tr>
<td>12. ( NNLO to VBF and Z/\gamma+jet )</td>
<td>Higgs couplings and SM benchmark</td>
</tr>
<tr>
<td>Calculations including electroweak effects</td>
<td></td>
</tr>
<tr>
<td>13. ( NNLO QCD+NLO EW for W/Z )</td>
<td>precision calculation of a SM benchmark</td>
</tr>
</tbody>
</table>

Table 1: The updated experimenter’s wishlist for LHC processes
In 2005 (and then continued in 2007), a priority list of processes to be calculated at NLO was created. Each of these calculations requires a tremendous amount of work, and thus the need for an experimental priority list:

- The calculation can be done
- The calculation needs to be done

The Les Houches wishlist from 2005/2007 is filling up slowly but progressively.

In 2009, we added \( pp \rightarrow t\bar{t}b\bar{b}, Wbbj, Z+3j, W+4j \) plus an extra column for each process indicating the level of precision required by the experiments:

- Relevant for \( t\bar{t}H \)
- Relevant for \( t\bar{t}H \)
- Relevant for VBF \( \rightarrow H \rightarrow VV \)
- Relevant for VBF \( \rightarrow H \rightarrow VV \)
- VBF contributions calculated by Bozzi/Jeger/Oleari/Zeppenfeld
- Various new physics signatures

### Table 1: The updated experimenter’s wishlist for LHC processes

<table>
<thead>
<tr>
<th>Process ((V \in {Z, W, \gamma}))</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Calculations completed since Les Houches 2005</strong></td>
<td></td>
</tr>
<tr>
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<td>( ZZ ) completed by Lazopoulos/Melnikov/Petriello and ( WWZZ ) by Hankele/Zeppenfeld</td>
</tr>
<tr>
<td><strong>Calculations remaining from Les Houches 2005</strong></td>
<td></td>
</tr>
<tr>
<td>4. ( pp \rightarrow t\bar{t}b\bar{b} )</td>
<td>Relevant for ( t\bar{t}H )</td>
</tr>
<tr>
<td>5. ( pp \rightarrow t\bar{t}+2\text{jets} )</td>
<td>Relevant for ( t\bar{t}H )</td>
</tr>
<tr>
<td>6. ( pp \rightarrow VV b\bar{b} )</td>
<td>Relevant for VBF ( \rightarrow H \rightarrow VV ) and ( t\bar{t}H )</td>
</tr>
<tr>
<td>7. ( pp \rightarrow VV+2\text{jets} )</td>
<td>VBF contributions calculated by Bozzi/Jeger/Oleari/Zeppenfeld</td>
</tr>
<tr>
<td>8. ( pp \rightarrow V+3\text{jets} )</td>
<td>Various new physics signatures</td>
</tr>
<tr>
<td><strong>NLO calculations added to list in 2007</strong></td>
<td></td>
</tr>
<tr>
<td>9. ( pp \rightarrow b\bar{b}b\bar{b} )</td>
<td>Higgs and new physics signatures</td>
</tr>
<tr>
<td><strong>Calculations beyond NLO added in 2007</strong></td>
<td></td>
</tr>
<tr>
<td>10. ( gg \rightarrow W^+W^- \ O(\alpha_s^4) )</td>
<td>Backgrounds to Higgs</td>
</tr>
<tr>
<td>11. NNLO ( pp \rightarrow t\bar{t} )</td>
<td>Normalization of a benchmark process</td>
</tr>
<tr>
<td>12. NNLO to VBF and ( Z/\gamma+\text{jet} )</td>
<td>Higgs couplings and SM benchmark</td>
</tr>
<tr>
<td><strong>Calculations including electroweak effects</strong></td>
<td></td>
</tr>
<tr>
<td>13. NNLO QCD+NLO EW for ( WZ )</td>
<td>Precision calculation of a SM benchmark</td>
</tr>
</tbody>
</table>

If all else fails...
Difficult calculations

I know that the multi-loop and multi-leg calculations are very difficult but just compare them to the complexity of the sentences that Sarah Palin used in her run for the vice-presidency.

Note: to her, you guys are just a bunch of Southerners.
New: W + 3 jets

Consider a scale of $m_W$ for W + 1,2,3 jets. We see the K-factors for W + 1,2 jets in the table below, and recently the NLO corrections for W + 3 jets have been calculated, allowing us to estimate the K-factors for that process. (Let's also use $m_{Higgs}$ for Higgs + jets.)

<table>
<thead>
<tr>
<th>Process</th>
<th>Typical scales</th>
<th>Tevatron $K$-factor</th>
<th>LHC $K$-factor</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\mu_0$</td>
<td>$\mu_1$</td>
<td>$K(\mu_0)$</td>
</tr>
<tr>
<td>W</td>
<td>$m_W$</td>
<td>$2m_W$</td>
<td>1.33</td>
</tr>
<tr>
<td>W+1jet</td>
<td>$m_W$</td>
<td>$p_T^{jet}$</td>
<td>1.42</td>
</tr>
<tr>
<td>W+2jets</td>
<td>$m_W$</td>
<td>$p_T^{jet}$</td>
<td>1.16</td>
</tr>
<tr>
<td>WW+jet</td>
<td>$m_W$</td>
<td>$2m_W$</td>
<td>1.19</td>
</tr>
<tr>
<td>$tt$</td>
<td>$m_t$</td>
<td>$2m_t$</td>
<td>1.08</td>
</tr>
<tr>
<td>$tt+1jet$</td>
<td>$m_t$</td>
<td>$2m_t$</td>
<td>1.13</td>
</tr>
<tr>
<td>$bb$</td>
<td>$m_b$</td>
<td>$2m_b$</td>
<td>1.20</td>
</tr>
<tr>
<td>Higgs</td>
<td>$m_H$</td>
<td>$p_T^{jet}$</td>
<td>2.33</td>
</tr>
<tr>
<td>Higgs via VBF</td>
<td>$m_H$</td>
<td>$p_T^{jet}$</td>
<td>1.07</td>
</tr>
<tr>
<td>Higgs+1jet</td>
<td>$m_H$</td>
<td>$p_T^{jet}$</td>
<td>2.02</td>
</tr>
<tr>
<td>Higgs+2jets</td>
<td>$m_H$</td>
<td>$p_T^{jet}$</td>
<td>–</td>
</tr>
</tbody>
</table>

Table 2: $K$-factors for various processes at the Tevatron and the LHC calculated using a selection of input parameters. In all cases, the CTEQ6M PDF is used at NLO, $K$ uses the CTEQ6L1 set at leading order, whilst $K'$ uses the same set, CTEQ6M, as at NLO. For most of the processes listed, jets satisfy the requirements $p_T > 15$ GeV/c and $|\eta| < 2.5$ (5.0) at the Tevatron (LHC). For Higgs+1,2jets, a jet cut of 40 GeV/c and $|\eta| < 4.5$ has been applied. A cut of $p_T^{jet} > 20$ GeV/c has been applied for the $tt+1jet$ process, and a cut of $p_T^{jet} > 50$ GeV/c for WW+jet. In the W(Higgs)+2jets process the jets are separated by $\Delta R > 0.52$, whilst the VBF calculations are performed for a Higgs boson of mass 120 GeV. In each case the value of the $K$-factor is compared at two often-used scale choices, where the scale indicated is used for both renormalization and factorization scales.

Is the K-factor (at $m_W$) at the LHC surprising?
Is the K-factor (at $m_W$) at the LHC surprising?

The K-factors for W + jets ($p_T > 30$ GeV/c) fall near a straight line, as do the K-factors for the Tevatron. By definition, the K-factors for Higgs + jets fall on a straight line, but it's interesting that the slope is similar.

Nothing special about $m_W$; just a typical choice.

The only way to know a cross section to NLO, say for W + 4 jets or Higgs + 3 jets, is to calculate it, but in lieu of the calculations, especially for observables that have been deemed important at Les Houches, can we make rules of thumb?

Related to this is:
- understanding the reduced scale dependences/pdf uncertainties for the cross section ratios we have been discussing
- scale choices at LO for cross sections uncalculated at NLO
Is the K-factor (at \( m_W \)) at the LHC surprising?

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Related to this is:
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- scale choices at LO for cross sections uncalculated at NLO

Will it be smaller still for \( W + 4 \) jets?
Scale choice for $W + 3$ jets

\[ H_T = \sum_j E_{T,j}^{\text{jet}} + E_T^e + \not{E}_T \]

\[ \sqrt{s} = 14 \text{ TeV} \]

\[ \mu = H_T \]

$H_T$ was the variable that gave a constant K-factor.
Scale choice: why is $E_T^W$ a bad one at the LHC?

If configuration a dominated, then as jet $E_T$ increased, $E_T^W$ would increase along with it. But configuration b is kinematically favored for high jet $E_T$'s (smaller partonic center-of-mass energy); $E_T^W$ remains small, and that scale does not describe the process very well.

Note that now split/merge can become important as the partonic jets can overlap and share partons.

Configuration b also tends to dominate in the tails of multi-jet distributions (such as $H_T$ or $M_{ij}$); for high jet $E_T$, $W$ behaves like a massless boson, and so there's a kinematic enhancement when it's soft.

FIG. 9: The $E_T$ distribution of the second jet at LO and NLO, for two dynamical scale choices, $\mu = E_T^W$ (left plot) and $\mu = H_T$ (right plot). The histograms and bands have the same meaning as in previous figures. The NLO distribution for $\mu = E_T^W$ turns negative beyond $E_T = 475$ GeV.

arXiv:0907.1984
Shape dependence of a K-factor

- Inclusive jet production probes very wide $x,Q^2$ range along with varying mixture of $gg,gq,$ and $qq$ subprocesses
- PDF uncertainties are significant at high $p_T$
- Over limited range of $p_T$ and $y,$ can approximate effect of NLO corrections by K-factor but not in general
  - in particular note that for forward rapidities, K-factor $<<1$
  - LO predictions will be large overestimates
  - this is true for both the Tevatron and for the LHC

Figure 105. The ratios of the jet cross section predictions for the LHC using the CTEQ6.1 error pdfs to the prediction using the central pdf. The extremes are produced by eigenvector 15.

Figure 106. The ratios of the NLO to LO jet cross section predictions for the LHC using the CTEQ6.1 pdfs for the three different rapidity regions $0-1$ (squares), $1-2$ (triangles), $2-3$ (circles).
Aside: Why K-factors < 1 for inclusive jet production?

- Write cross section indicating explicit scale-dependent terms
- First term (lowest order) in (3) leads to monotonically decreasing behavior as scale increases (the LO piece)
- Second term is negative for $\mu < p_T$, positive for $\mu > p_T$
- Third term is negative for factorization scale $M < p_T$
- Fourth term has same dependence as lowest order term
- Thus, lines one and four give contributions which decrease monotonically with increasing scale while lines two and three start out negative, reach zero when the scales are equal to $p_T$, and are positive for larger scales
- At NLO, result is a roughly parabolic behavior

Consider a large transverse momentum process such as the single jet inclusive cross section involving only massless partons. Furthermore, in order to simplify the notation, suppose that the transverse momentum is sufficiently large that only the quark distributions need be considered. In the following, a sum over quark flavors is implied. Schematically, one can write the lowest order cross section as

$$E \frac{d^3\sigma}{dp^3} = \sigma = a^2(\mu) \hat{\sigma}_B \otimes q(M) \otimes q(M)$$  (1)

where $a(\mu) = \alpha_s(\mu)/2\pi$ and the lowest order parton-parton scattering cross section is denoted by $\hat{\sigma}_B$. The renormalization and factorization scales are denoted by $\mu$ and $M$, respectively. In addition, various overall factors have been absorbed into the definition of $\hat{\sigma}_B$. The symbol $\otimes$ denotes a convolution defined as

$$f \otimes g = \int_0^1 \frac{dy}{y} f\left(\frac{x}{y}\right) g(y).$$  (2)

When one calculates the $O(\alpha_s^3)$ contributions to the inclusive cross section, the result can be written as

$$\begin{align*}
  (1) & \quad \sigma = a^2(\mu) \hat{\sigma}_B \otimes q(M) \otimes q(M) \\
  (2) & \quad + 2a^3(\mu) b \ln(\mu/p_T) \hat{\sigma}_B \otimes q(M) \otimes q(M) \\
  (3) & \quad + 2a^3(\mu) \ln(p_T/M) P_{ss} \otimes \hat{\sigma}_B \otimes q(M) \otimes q(M) \\
  (4) & \quad + a^2(\mu) K \otimes q(M) \otimes q(M). \\
\end{align*}$$  (3)

In writing Eq. (3), specific logarithms associated with the running coupling and the scale dependence of the parton distributions have been explicitly displayed; the remaining higher order corrections have been collected in the function $K$ in the last line of Eq. (3). The $\mu$
Why K-factor for inclusive jets < 1?

- First term (lowest order) in (3) leads to monotonically decreasing behavior as scale increases.
- Second term is negative for $\mu < p_T$, positive for $\mu > p_T$.
- Third term is negative for factorization scale $M < p_T$.
- Fourth term has same dependence as lowest order term.
- Thus, lines one and four give contributions which decrease monotonically with increasing scale while lines two and three start out negative, reach zero when the scales are equal to $p_T$, and are positive for larger scales.
- NLO parabola moves out towards higher scales for forward region.
- Scale of $E_T/2$ results in a K-factor of ~1 for low $E_T$, <<1 for high $E_T$ for forward rapidities at Tevatron, and at the LHC.
Why stop at NLO?

- NNLO is even better, but also more complicated
- Expect reduced scale dependence
- And in some cases, extra contributions, such as Higgs
- Have only been carried out for a few processes to date
- Would really like to have it for inclusive jet production, for example
Higgs: LO->NLO->NNLO


convergence in going:
LO → NLO → NNLO

Confirmed by the full
scale dependence:

\[ \downarrow \]
All-orders approaches

- Rather than systematically calculating to higher and higher orders in the perturbative expansion, can also use a number of all-orders approaches.
- In resummation, dominant contributions from each order in perturbation theory are singled out and resummed by use of an evolution equation.
- Near boundaries of phase space, fixed order calculations break down due to large logarithmic corrections.
- Consider W production:
  - One large logarithm associated with production of vector boson close to threshold.
  - Takes form of \( \alpha_s n \log^{2n-1}(1-z)/(1-z) \) where \( z = Q^2/s - 1 \).
  - Other large logarithm is associated with recoil of vector boson at very small \( p_T \).
  - Logarithms appear as \( \alpha_s n \log^{2n-1}(Q^2/p_T^2) \).

Expression for W boson transverse momentum in which leading logarithms have been resummed to all orders is given by

\[
\frac{d\sigma}{dp_T^2} = \sigma \frac{d}{dp_T^2} \exp \left( -\frac{\alpha_s C_F}{2\pi} \log^2 \frac{M_W^2}{p_T^2} \right)
\]

Note that distribution goes to zero as \( p_T \to 0 \).

Figure 20. The resummed (leading log) W boson transverse momentum distribution.

In both cases there is a restriction of phase space for gluon emission and thus large logs.
Parton showers

- A different, but related approach, is provided by parton showering.
- By the use of the parton showering process, a few partons produced in a hard interaction at a high-energy scale can be related to partons at an energy scale close to $\Lambda_{\text{QCD}}$.
- At this lower energy scale, a universal non-perturbative model can then be used to provide the transition to hadrons.
- Parton showering allows for evolution, using DGLAP formalism, of parton fragmentation function.
- Successive values of an evolution variable $t$, a momentum fraction $z$, and an azimuthal angle $\phi$ are generated, along with the flavors of the partons emitted during the parton shower.

---

Due to successive branching, parton cascade or shower develops. Each outgoing line is source of new cascade, until all outgoing lines have stopped branching. At this stage, which depends on cutoff scale $t_0$, outgoing partons have to be converted into hadrons via a hadronization model.
Parton shower evolution

- On average, emitted gluons have decreasing angles with respect to parent parton directions
  - angular ordering
- The evolution variable \( t \) can be the virtuality of the parent parton [old Pythia and Sherpa], \( E^2(1-\cos \theta) \) where \( E \) is the energy of the parent parton and \( \theta \) is the opening angle between the two partons [Herwig], or the square of the transverse momentum between the two partons [new Pythia]
Sudakov form factors

- Sudakov form factors form the basis for both resummation and parton showering.
- We can write an expression for the Sudakov form factor of an initial state parton in the form below, where $t$ is the hard scale, $t_0$ is the cutoff scale, and $P(z)$ is the splitting function:

$$\Delta(t) \equiv \exp \left[ -\int_{t_0}^{t} \frac{dt'}{t'} \int \frac{dz}{z} \frac{\alpha_S}{2\pi} P(z) \frac{f(x/z, t)}{f(x, t)} \right]$$

- Similar form for the final state but without the pdf weighting.
- Sudakov form factor resums all effects of soft and collinear gluon emission, but does not include non-singular regions that are due to large energy, wide angle gluon emission.
- Gives the probability **not** to radiate a gluon greater than some energy.

Figure 21. The Sudakov form factors for initial-state gluons at a hard scale of 100 GeV as a function of the transverse momentum of the emitted gluon. The form factors are for (top to bottom) parton $x$ values of 0.3, 0.1, 0.03, 0.01, 0.001 and 0.0001.

Figure 22. The Sudakov form factors for initial-state gluons at a hard scale of 500 GeV as a function of the transverse momentum of the emitted gluon. The form factors are for (top to bottom) parton $x$ values of 0.3, 0.1, 0.03, 0.01, 0.001, and 0.0001.
Sudakov form factors: quarks and gluons

so quarks don’t radiate as much
Merging ME and PS approaches

- Parton showers provide an excellent description in regions which are dominated by soft and collinear gluon emission.

- Matrix element calculations provide a good description of processes where the partons are energetic and widely separated and also take into account interference effects between amplitudes.
  - but do not take into account interference effects in soft and collinear emissions which cannot be resolved, and thus lead to Sudakov suppression of such emissions.

- Hey, I know, let’s put them together, but we have to be careful not to double-count.
  - parton shower producing same event configuration already described by matrix element.
  - Les Houches Accord (which I named) allows the ME program to talk to the PS program.

Figure 26. In the NLO formalism, the same scale, proportional to the hardness of the process, is used for each QCD vertex. For the case of the W+ 2 jet diagram shown above to the left, a scale related to the mass of the W boson, or to the average transverse momentum of the produced jets, is typically used. The figure to the right shows the results of a simulation using the CKKW formalism. Branchings occur at the vertices with resolution parameters $d_i$, where $d_1 > d_2 \gg d_3 > d_4 > d_5 > d_6$. Branchings at the vertices 1–2 are produced with matrix element information while the branchings at vertices 3–6 are produced by the parton shower.
Note

- We can only observe emissions above a certain resolution scale.
- Below this resolution scale, singularities cancel, leaving a finite remnant.
- (some of) the virtual corrections encountered in a full NLO calculation are included by the use of Sudakov suppression between vertices.
- So a parton shower Monte Carlo is not purely a fixed order calculation.
Merging ME and PS approaches

- A number of techniques to combine, with most popular/correct being CKKW
  - matrix element description used to describe parton branchings at large angle and/or energy
  - parton shower description is used for smaller angle, lower energy emissions

- Division into two regions of phase space provided by a resolution parameter \( d_{\text{ini}} \)

- Argument of \( \alpha_s \) at all of the vertices is taken to be equal to the resolution parameter \( d_i \) (showering variable) at which the branching has taken place

- Sudakov form factors are inserted on all of the quark and gluon lines to represent the lack of any emissions with a scale larger than \( d_{\text{ini}} \) between vertices
  - parton showering is used to produce additional emissions at scales less than \( d_i \)

- For typical matching scale, \(~10\% of the n-jet cross section is produced by parton showering from n-1 parton ME

---

Figure 26. In the NLO formalism, the same scale, proportional to the hardness of the process, is used for each QCD vertex. For the case of the \( W + 2 \) jet diagram shown above to the left, a scale related to the mass of the \( W \) boson, or to the average transverse momentum of the produced jets, is typically used. The figure to the right shows the results of a simulation using the CKKW formalism. Branchings occur at the vertices with resolution parameters \( d_i \), where \( d_1 > d_2 \gg d_{\text{ini}} > d_4 > d_5 > d_6 \). Branchings at the vertices 1–2 are produced with matrix element information while the branchings at vertices 3–6 are produced by the parton shower.

see Alpgen, Madgraph, Sherpa,…
Best of all worlds

Several groups have worked on the subject to consistently combine partonic NLO calculations with parton showers.

- Collins, Zu [95, 96].
- Frixione, Nason, Webber (MC@NLO) [97–99].
- Kurihara, Fujimoto, Ishikawa, Kato, Kawabata, Munehsia, Tanaka [100].
- Krämer, Soper [101–103].
- Nagy, Soper [104, 105].

MC@NLO is the only publicly available program that combines NLO calculations with parton showering and hadronization. The HERWIG Monte Carlo is used for the latter. The use of a different Monte Carlo, such as PYTHIA, would require a different subtraction scheme for the NLO matrix elements. The processes included to date are (W, Z, $\gamma^*$, $H$, $bb$, $tt$, $HW$, $HZ$, $WW$, $WZ$, $ZZ$). Recently, single top hadroproduction has been added to MC@NLO [106]. This is the first implementation of a process that has both initial- and final-state singularities. This allows a more general category of additional processes to be added in the future. Work is proceeding on the addition of inclusive jet production and the production of a Higgs boson via $WW$ fusion. Adding spin correlations to a process increases the level of difficulty but is important for processes such as single top production.

If, in addition, the CKKW formalism could be used for the description of hard parton emissions, the utility and accuracy of a NLO Monte Carlo could be greatly increased. The merger of these two techniques should be possible in Monte Carlos available by the time of the LHC turn-on.

Powheg is a program/technique that results in no negative weight events; many new processes have been added.

But it’s still very time(theorist)-consuming to add a new process, and most of the processes so far are 2->1, 2->2.
t-tbar in MC@NLO

- At low $p_T$ for the t-tbar system, the cross section is described (correctly) by the parton shower, which resums the large logs near $p_T \sim 0$
- At high $p_T$, the cross section is described (correctly) by the NLO matrix element
Parton distribution functions and global fits

- Calculation of production cross sections at the LHC relies upon knowledge of pdf's in the relevant kinematic region
- Pdf's are determined by global analyses of data from DIS, DY and jet production
- Two major groups that provide semi-regular updates to parton distributions when new data/theory becomes available
  - CTEQ->CTEQ5->CTEQ6 ->CTEQ6.1->CTEQ6.5 ->CTEQ6.6->CT09

Figure 27. The CTEQ6.1 parton distribution functions evaluated at a $Q$ of 10 GeV.
Hadrons

- The proton is a dynamical object; the structure observed depends on the time-scale \( Q^2 \) of the observation.
- But we know how to calculate this variation.

\[ f_i(x, Q^2) = \text{number density of partons } i \]

at momentum fraction \( x \) and probing scale \( Q^2 \).
Global fits

- With the DGLAP equations, we know how to evolve pdf’s from a starting scale $Q_0$ to any higher scale
- …but we can’t calculate what the pdf’s are ab initio
  - one of the goals of lattice QCD
- We have to determine them from a global fit to data
  - factorization theorem tells us that pdf’s determined for one process are applicable to another

- So what do we need
  - a value of $Q_0$ (1.3 GeV for CTEQ, 1 GeV for MSTW) lower than the data used in the fit (or any prediction)
  - a parametrization for the pdf’s
  - a scheme for the pdf’s
  - hard-scattering calculations at the order being considered in the fit
  - pdf evolution at the order being considered in the fit
  - a world average value for $\alpha_s$
  - a lot of data
    - with appropriate kinematic cuts
  - a treatment of the errors for the experimental data
  - MINUIT
Back to global fits

- **Parametrization: initial form**
  - $f(x) \sim x^\alpha (1-x)^\beta$
  - estimate $\beta$ from quark counting rules
    - $\beta = 2n_s - 1$ with $n_s$ being the minimum number of spectator quarks
    - so for valence quarks in a proton ($qqq$), $n_s = 2$, $\beta = 3$
    - for gluon in a proton ($qqg$), $n_s = 3$, $\beta = 5$
    - for anti-quarks in a proton ($qqqqq\bar{q}$), $n_s = 4$, $\beta = 7$
  - estimate $\alpha$ from Regge arguments
    - gluons and anti-quarks have $\alpha \sim -1$ while valence quarks have $\alpha \sim 1/2$
  - but at what $Q$ value are these arguments valid?

- **What do we know?**
  1. we know that the sum of the momentum of all partons in the proton is 1 (but see later for modified LO fits)
  2. we know the sum of valence quarks is 3
    - and 2 of them are up quarks and 1 of them is a down quark
    - we know that the net number of anti-quarks is 0, but what about $\bar{d} = u$?
  3. we know that the net number of strange quarks (charm quarks/bottom quarks) in the proton is 0
    - but we don’t know if $s = \bar{s}$ locally

This already puts a lot of restrictions on the pdf’s
Parametrizations

- That simple parametrization worked for early fits, where the data was not very precise (nor very abundant), but it does not work for modern global fits, where a more flexible form is needed
  - the simple ansatz can be dangerous in that it can (falsely) tie together low $x$ and high $x$ behavior (other than by momentum sum rule)
- In order to more finely tune parametrization, usually multiply simple form by a polynomial in $x$ or some more complicated function
- Currently CTEQ uses for the quark and gluon distributions
  \[ f(x) = x^{(a_1-1)}(1-x)^{a_2} e^{a_3 x} [1 + e^{a_4 x}]^{a_5} \]
- For the ratio of $d\bar{b}/u\bar{b}$
  \[ \frac{d\bar{b}}{u\bar{b}} = e^{a_1} x^{(a_2-1)}(1-x)^{a_3} + (1 + a_4 x)(1-x)^{a_5} \]
- How do we know this is flexible enough?
  - data is well-described ($\chi^2$/dof $\sim$1 for a NLO fit)
  - adding more parameters just results in those parameters being unconstrained
  - note that with this form, the pdf's are positive definite (they don't have to be)
Orders and Schemes

- Fits are available at
  - LO
    - CTEQ6L or CTEQ6L1
      - 1 loop or 2 loop $\alpha_s$
    - in common use with parton shower Monte Carlos
    - poor fit to data due to deficiencies of LO ME's
  - LO*
    - better for parton shower Monte Carlos (see later)
  - NLO
    - CTEQ6.1 or CTEQ6.6
    - precision level: error pdf's defined at this order
  - NNLO
    - more accurate but not all processes known
- At NLO and NNLO, one needs to specify a scheme or convention for subtracting the divergent terms
- Basically the scheme specifies how much of the finite corrections to subtract along with the divergent pieces
  - most widely used is the modified minimal subtraction scheme (or MSbar)
  - used with dimensional regularization: subtract the pole terms and accompanying $\log 4\pi$ and Euler constant terms
  - also may find pdf's in DIS scheme, where full order $\alpha_s$ correction for $F_2$ in DIS absorbed into quark pdf's
Scales and Masses

- Processes used in global fits are characterized by a single large scale
  - DIS-$Q^2$
  - lepton pair production-$M^2$
  - vector boson production-$M_V^2$
  - jet production-$p_T^{\text{jet}}$

- By choosing the factorization and renormalization scales to be of the same order as the characteristic scale
  - can avoid some large logarithms in the hard scattering cross section
  - some large logarithms in running coupling and pdf's are resummed

- Different treatment of quark masses and thresholds
  - zero mass variable flavor number scheme (ZM-VFNS)
  - fixed flavor number scheme (FFNS)
  - variable flavor number scheme (VFNS)
Zero mass variable flavor number scheme (ZM-VFNS)

- Start pdf evolution at charm threshold \( Q=m_c=1.3 \text{ GeV} \)
  - set c and b distributions to zero at this scale (although can allow for possibility of intrinsic charm/bottom)
    - start b evolution at \( Q=m_b \)
  - all heavy quarks treated as massless
  - c and b pairs created by gluon splitting
  - adjust running coupling \( \alpha_s \) as each flavor threshold is crossed since QCD \( \beta \) function depends on \# of active flavors
  - in this approach, only mass effects are due to flavor thresholds and changing of \( \beta \) function

- Most commonly used CTEQ NLO pdf's prior to CTEQ6.5 (such as CTEQ6M, CTEQ6.1) are of this type

- Advantages
  - easy to implement
  - sums large logs of \( Q^2/m_Q^2 \) via DLGAP equation
  - asymptotically correct when \( Q^2 >> m_Q^2 \)

- Disadvantages
  - does not treat heavy quark threshold correctly
Fixed flavor number scheme

- Calculate heavy quark production from relevant subprocesses such as $\gamma^*g\rightarrow Q\bar{Q}$ keeping only light quarks in DGLAP equations
- Only light quarks have pdf’s
  - no charm or bottom quark pdf’s

- Advantage
  - gets threshold behavior correct

- Disadvantage
  - does not resum potentially large logs of $Q^2/m_Q^2$
Variable flavor number scheme (VFNS)

- This is the “just right” scheme
- It combines the ZM-VFNS and FFNS by interpolating between the FFNS (correct near threshold) and the ZM-FFNS (resums large logs)
- But it’s technically more complicated than the ZM-VFNS since there must be subtraction terms in order to avoid large logarithms
- All current (CTEQ6.6) and future NLO CTEQ pdf’s will be of this type
- Its use has an impact on predictions for the LHC
Data sets used in global fits (CTEQ6.6)

1. BCDMS $F_2^{\text{proton}}$ (339 data points)
2. BCDMS $F_2^{\text{deuteron}}$ (251 data points)
3. NMC $F_2$ (201 data points)
4. NMC $F_2^d/F_2^p$ (123 data points)
5. $F_2^{\text{(CDHSW)}}$ (85 data points)
6. $F_3^{\text{(CDHSW)}}$ (96 data points)
7. CCFR $F_2$ (69 data points)
8. CCFR $F_3$ (86 data points)
9. H1 NC $e^+p$ (126 data points; 1998-98 reduced cross section)
10. H1 NC $e^+p$ (13 data points; high $y$ analysis)
11. H1 NC $e^+p$ (115 data points; reduced cross section 1996-97)
12. H1 NC $e^+p$ (147 data points; reduced cross section; 1999-00)
13. ZEUS NC $e^+p$ (92 data points; 1998-99)
14. ZEUS NC $e^+p$ (227 data points; 1996-97)
15. ZEUS NC $e^+p$ (90 data points; 1999-00)
16. H1 $F_2^c$ $e^+p$ (8 data points; 1996-97)
17. H1 $R_{cb}^d$ for $cc\overline{b}$ $e^+p$ (10 data points; 1996-97)
18. H1 $R_{bb}^c$ for $bb\overline{b}$ $e^+p$ (10 data points; 1999-00)
19. ZEUS $F_2^c$ $e^+p$ (18 data points; 1996/97)
20. ZEUS $F_2^c$ $e^+p$ (27 data points; 1998/00)
21. H1 CC $e^+p$ (28 data points; 1998-99)
22. H1 CC $e^+p$ (25 data points; 1994-97)
23. H1 CC $e^+p$ (28 data points; 1999-00)
24. ZEUS CC $e^+p$ (26 data points; 1998-99)
25. ZEUS CC $e^+p$ (29 data points; 1994-97)
26. ZEUS CC $e^+p$ (30 data points; 1999-00)
27. NuTev neutrino dimuon cross section (38 data points)
28. NuTev anti-neutrino dimuon cross section (33 data points)
29. CCFR neutrino dimuon cross section (40 data points)
30. CCFR anti-neutrino cross section (38 data points)
31. E605 dimuon (199 data points)
32. E866 dimuon (13 data points)
33. Lepton asymmetry from CDF (11 data points)
34. CDF Run 1B jet cross section (33 data points)
35. D0 Run 1B jet cross section (90 data points)

- 2794 data points from DIS, DY, jet production
- All with (correlated) systematic errors that must be treated correctly in the fit
- Note that DIS is the 800 pound gorilla of the global fit with many data points and small statistical and systematic errors
  - and fixed target DIS data still have a significant impact on the global fitting, even with an abundance of HERA data
- To avoid non-perturbative effects, kinematic cuts on placed on the DIS data
  - $Q^2>5 \text{ GeV}^2$
  - $W^2(=m^2+Q^2(1-x)/x)>12.25 \text{ GeV}^2$
Influence of data in global fit

- **Charged lepton DIS**
  \[ F_2(x,Q^2) = x \sum_i e_i^2 [q_i(x,Q^2) + \bar{q}_i(x,Q^2)] \]
  - each flavor weighted by its squared charge
  - quarks and anti-quarks enter together
  - gluon doesn’t enter, in lowest order, but does enter into the structure functions at NLO
  - also enters through mixing in evolution equations so gluon contributes to the change of the structure functions as \( Q^2 \) increases
  - at low values of \( x \)

\[ Q^2 \frac{dF_2}{dQ^2} \approx \frac{\alpha_s}{2\pi} \sum_i e_i^2 \int y \frac{dy}{x} P_{qg}(y) G\left(\frac{x}{y}, Q^2\right) \]

- \( Q^2 \) dependence at small \( x \) is driven directly by gluon pdf

- **At low \( x \), structure functions increase with \( Q^2 \); at high \( x \) decrease**
Some history

What caused the changes from CTEQ1 to CTEQ2 (and up)?
Infusion of low x HERA data. Before was just an extrapolation and guess was wrong.
Influence of data in global fit

**Neutrino DIS**

\[
F_2(x, Q^2) = x \sum_i [q_i(x, Q^2) + \bar{q}_i(x, Q^2)]
\]

\[
xF_3(x, Q^2) = x \sum_i [q_i(x, Q^2) - \bar{q}_i(x, Q^2)]
\]

- additional structure function allows the separation of quarks and antiquarks but not a complete flavor separation
- caveat: neutrino observables usually obtained using nuclear targets so there is added question of nuclear corrections
Some observations from DIS

- DIS data provide strong constraints on the u and d distributions over the full range of x covered by the data.
- The combination $4\times u_{\bar{u}} + d_{\bar{d}}$ is well-constrained at small x.
- The gluon is constrained at low values of x by the slope of the $Q^2$ dependence of $F_2$.
  - momentum sum rule connects low x and high x behavior, but loosely.
dbar/ubar and Gottfried sum rule

\[ S_G = \int_0^1 \frac{dx}{x} \left[ F_2^p(x) - F_2^n(x) \right] \]

\[ = \frac{1}{3} + \frac{2}{3} \int_0^1 dx \left( \bar{u}(x) - \bar{d}(x) \right) \]

\[ = 0.235 \pm 0.026 \]

\[ \bar{d} \neq \bar{u} \]

Why is \( \bar{u} \neq \bar{d} \)?

Pion cloud argument

- proton can fluctuate into a neutron and a positive pion
- \( p \rightarrow n \pi^+ \rightarrow p \)
- \( \ldots \)or \( uud \rightarrow (udd)(ud) \)
- \( \ldots \)or \( d > u \)
- \( \ldots \)so SU(2) symmetry of sea quarks is broken

Doesn’t tell us the \( x \) dependence though
Information from Drell-Yan

Note NA-51: only one data point but provided crucial information on dbar/ubar before E866
What about $s$ and $\bar{s}$?

- Can get information from
  
  $$W^+ s \rightarrow c$$
  
  $$W^- \bar{s} \rightarrow \bar{c}$$

- Look from muon pairs in final state due to charm hadrons decaying semi-leptonically
  
  $$c \rightarrow s \mu^+ \nu$$
  
  $$\bar{c} \rightarrow \bar{s} \mu^- \bar{\nu}$$

- Information from dimuon production in neutrino interactions
  
  $$\nu N \rightarrow \mu^- c + X' \rightarrow \mu^- \mu^+ + X$$
  
  $$\bar{\nu} N \rightarrow \mu^+ \bar{c} + X' \rightarrow \mu^+ \mu^- + X$$

- So $s$ carries somewhat more momentum than $\bar{s}$

- In previous fits, assumption was that $s = \bar{s}$; in CTEQ6.6 fit remove that assumption $\rightarrow 2$ new free parameters the fit
Inclusive jets and global fits

- We don’t have many handles on the high x gluon distribution in the global pdf fits
- Best handle is provided by the inclusive jet cross section from the Tevatron

At high $E_T$ (high x), gq is subdominant, but there’s a great deal of freedom/uncertainty on the high x gluon distribution

- about 42% of the proton’s momentum is carried by gluons, and most of that momentum is at low x

<table>
<thead>
<tr>
<th>X Bin</th>
<th>Momentum fraction</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-1}$ to $10^{-3}$</td>
<td>0.6%</td>
</tr>
<tr>
<td>$10^{-3}$ to 0.01</td>
<td>3%</td>
</tr>
<tr>
<td>0.01 to 0.1</td>
<td>16%</td>
</tr>
<tr>
<td>0.1 to 0.2</td>
<td>10%</td>
</tr>
<tr>
<td>0.2 to 0.3</td>
<td>6%</td>
</tr>
<tr>
<td>0.3 to 0.5</td>
<td>5%</td>
</tr>
<tr>
<td>0.5 to 1.0</td>
<td>1%</td>
</tr>
</tbody>
</table>

**TABLE 1.** The momentum fraction carried by gluons in a given x bin at a Q value of 5 GeV.

- The inclusion of the CDF/D0 inclusive jet cross sections from Run 1 boosted the high x gluon distribution and thus the predictions for the high $E_T$ jet cross sections

**Figure 56.** The subprocess contributions to inclusive jet production at the Tevatron for the CTEQ5M and CTEQ6M pdfs. The impact of the larger larger gluon at high x for CTEQ6 is evident.
Some more history

- Note that the high x gluon for CTEQ6.1 is much larger than that for either CTEQ4M or CTEQ5M. Why?
- Full inclusion of Run 1 jet data (especially from D0) which preferred to have a larger gluon
  - similar to the hypothesis of CTEQ4HJ
- Caveat: high x gluon is decreasing somewhat with the inclusion of the Run 2 jet data, which don’t prefer as large of a high x gluon
Inclusive jets at the Tevatron

41 curves corresponding to CTEQ6.1 and 40 error pdf's

predictions using CTEQ6.1
Global fitting: best fit

- Using our 2794 data points, we do our global fit by performing a $\chi^2$ minimization
  - where $D_i$ are the data points and $T_i$ are the theoretical predictions; we allow for a normalization shift $f_N$ for each experimental data set
    - but we provide a quadratic penalty for any normalization shift
  - where there are $k$ systematic errors $\beta$ for each data point in a particular data set
    - and where we allow the data points to be shifted by the systematic errors with the shifts given by the $s_j$ parameters
    - but we give a quadratic penalty for non-zero values of the shifts $s_j$
  - where $\sigma_i$ is the statistical error for data point $i$

- For each data set, we calculate
  \[
  \chi^2 = \sum_i \left( \frac{\left( f_N D_i - \sum_{j=1}^{k} \beta_{ij} s_j \right) - T_i}{\sigma_i^2} \right)^2 + \sum_{j=1}^{k} s_j^2
  \]

- For a set of theory parameters it is possible to analytically solve for the shifts $s_j$, and therefore, continually update them as the fit proceeds

- To make matters more complicated, we may give additional weights to some experiments due to the utility of the data in those experiments (i.e. NA-51), so we adjust the $\chi^2$ to be
  \[
  \chi^2 = \sum_k w_k \chi_k^2 + \sum_k w_{N,k} \left[ \frac{1 - f_N}{\sigma_N^{\text{norm}}} \right]^2
  \]

- where $w_k$ is a weight given to the experimental data and $w_{N,k}$ is a weight given to the normalization
Minimization and errors

- Free parameters in the fit are parameters for quark and gluon distributions
  
  \[ f(x) = x^{(a_1-1)}(1-x)^a_2 e^{a_3 x} [1 + e^{a_4 x}]^{a_5} \]

- Too many parameters to allow all to remain free
  - some are fixed at reasonable values or determined by sum rules

- 20 free parameters for CTEQ6.1, 22 for CTEQ6.6
  - 2 additional parameters for strange quark distributions

- Result is a global $\chi^2$/dof on the order of 1
  - for a NLO fit
  - worse for a LO fit, since the LO pdf’s can not make up for the deficiencies in the LO matrix elements
PDF Errors: old way

- Make plots of lots of pdf’s (no matter how old) and take spread as a measure of the error
- Can either underestimate or overestimate the error
- Review sources of uncertainty on pdf’s
  - data set choice
  - kinematic cuts
  - parametrization choices
  - treatment of heavy quarks
  - order of perturbation theory
  - errors on the data
- There are now more sophisticated techniques to deal with at least the errors due to the experimental data uncertainties
PDF Errors: new way

- So we have optimal values (minimum $\chi^2$) for the $d=20$ (22) free pdf parameters in the global fit
  - $\{a_\mu\}, \mu=1,...,d$
- Varying any of the free parameters from its optimal value will increase the $\chi^2$
- It’s much easier to work in an orthonormal eigenvector space determined by diagonalizing the Hessian matrix, determined in the fitting process

$$H_{\mu\nu} = \frac{1}{2} \frac{\partial \chi^2}{\partial a_\mu \partial a_\nu}$$

To estimate the error on an observable $X(a)$, due to the experimental uncertainties of the data used in the fit, we use the Master Formula

$$(\Delta X)^2 = \Delta \chi^2 \sum_{\mu,\nu} \frac{\partial X}{\partial a_\mu} (H^{-1})_{\mu\nu} \frac{\partial X}{\partial a_\nu}$$
PDF Errors: new way

- Recap: 20 (22) eigenvectors with the eigenvalues having a range of >1E6
- Largest eigenvalues (low number eigenvectors) correspond to best determined directions; smallest eigenvalues (high number eigenvectors) correspond to worst determined directions
- Easiest to use Master Formula in eigenvector basis

To estimate the error on an observable $X(a)$, from the experimental errors, we use the Master Formula

$$(\Delta X)^2 = \Delta \chi^2 \sum_{\mu,\nu} \frac{\partial X}{\partial a_\mu} (H^{-1})_{\mu\nu} \frac{\partial X}{\partial a_\nu}$$

where $X_i^+$ and $X_i^-$ are the values for the observable $X$ when traversing a distance corresponding to the tolerance $T(=\sqrt{\Delta \chi^2})$ along the $i^{th}$ direction.
What is the tolerance T?

This is one of the most controversial questions in global PDF fitting?

We have 2794 data points in the CTEQ6.6 data set (on order of 2000 for CTEQ6.1)

Technically speaking, a 1-sigma error corresponds to a tolerance

\[ T = \sqrt{\Delta \chi^2} = 1 \]

This results in far too small an uncertainty from the global fit

- with data from a variety of processes from a variety of experiments from a variety of accelerators

For CTEQ6.1, we chose a \( \Delta \chi^2 \) of 100 to correspond to a 90% CL limit

- with an appropriate scaling for the larger data set for CTEQ6.6

MSTW has chosen a \( \Delta \chi^2 \) of 50 for the same limit so CTEQ errors will be larger than MSTW errors

**Figure 29.** The pdf errors for the CDF inclusive jet cross section in Run 1 for the 20 different eigenvector directions. The vertical axes show the fractional deviation from the central prediction and the horizontal axes the jet transverse momentum in GeV.
What do the eigenvectors mean?

- Each eigenvector corresponds to a linear combination of all 20 (22) pdf parameters, so in general each eigenvector doesn’t mean anything?

- However, with 20 (22) dimensions, often eigenvectors will have a large component from a particular direction.

- Take eigenvector 1 (for CTEQ6.1); error pdf’s 1 and 2

- It has a large component sensitive to the small x behavior of the u quark valence distribution.

- Not surprising since this is the best determined direction.
What do the eigenvectors mean?

- Take eigenvector 8 (for CTEQ6.1); error pdf's 15 and 16
- No particular direction stands out
What do the eigenvectors mean?

- Take eigenvector 15 (for CTEQ6.1); error pdf's 29 and 30
- Probes high x gluon distribution

| 29, 30 | BP( 2, 1) | 0.012701 |
| 29, 30 | BP( 2, 2) | -0.162018 |
| 29, 30 | BP( 2, 3) | 0.018666 |
| 29, 30 | BP( 2, 4) | -0.111238 |
| 29, 30 | BP( 2, 5) | frozen |
| 29, 30 | BP( 1, 1) | -0.003049 |
| 29, 30 | BP( 1, 2) | -0.001074 |
| 29, 30 | BP( 1, 3) | -0.034151 |
| 29, 30 | BP( 1, 4) | -0.005735 |
| 29, 30 | BP( 1, 5) | 0.032812 |
| 29, 30 | BP( 0, 1) | -0.045923 |
| 29, 30 | BP( 0, 2) | 0.873418 |
| 29, 30 | BP( 0, 3) | frozen |
| 29, 30 | BP( 0, 4) | -0.241822 |
| 29, 30 | BP( 0, 5) | frozen |
| 29, 30 | BP( -1, 1) | -0.071419 |
| 29, 30 | BP( -1, 2) | -0.067488 |
| 29, 30 | BP( -1, 3) | 0.100283 |
| 29, 30 | BP( -1, 4) | frozen |
| 29, 30 | BP( -1, 5) | 0.179551 |
| 29, 30 | BP( -2, 1) | -0.009441 |
| 29, 30 | BP( -2, 2) | -0.196100 |
| 29, 30 | BP( -2, 3) | 0.211281 |
| 29, 30 | BP( -2, 4) | frozen |
| 29, 30 | BP( -2, 5) | frozen |

Figure 29. The pdf errors for the CDF inclusive jet cross section in Run 1 for the 20 different eigenvector directions. The vertical axes show the fractional deviation from the central prediction and the horizontal axes the jet transverse momentum in GeV.
Aside: PDF re-weighting

- Any physical cross section at a hadron-hadron collider depends on the product of the two pdf's for the partons participating in the collision convoluted with the hard partonic cross section.

- Nominally, if one wants to evaluate the pdf uncertainty for a cross section, this convolution should be carried out 41 times (for CTEQ6.1); once for the central pdf and 40 times for the error pdf's.

- However, the partonic cross section is not changing, only the product of the pdf's.

- So one can evaluate the full cross section for one pdf (the central pdf) and then evaluate the pdf uncertainty for a particular cross section by taking the ratio of the product of the pdf's (the pdf luminosity) for each of the error pdf's compared to the central pdf's.

\[
\sigma_{AB} = \int dx_a dx_b \ f_{a/A}(x_a, Q^2) f_{b/B}(x_b, Q^2) \hat{\sigma}_{ab \to X}
\]

\[
\frac{f^i_{a/A}(x_a, Q^2) f^i_{b/B}(x_b, Q^2)}{f^0_{a/A}(x_a, Q^2) f^0_{b/B}(x_b, Q^2)}
\]

- \(f^i\) is the error pdf and \(f^0\) the central pdf.

This works exactly for fixed order calculations and works well enough (see later) for parton shower Monte Carlo calculations.

Most experiments now have code to easily do this...
and many programs will do it for you (MCFM)
New CTEQ technique

- With Hessian method, diagonalize the Hessian matrix to determine orthonormal eigenvector directions; 1 eigenvector for each free parameter in the fit
  - CTEQ6.6 has 22 free parameters, so 22 eigenvectors and 44 error pdf’s
  - CT09 pdf’s have 24 free parameters
- Each eigenvector/error pdf has components from each of the free parameters
- Sum over all error pdf’s to determine the error for any observable
- But, we are free to make an additional orthogonal transformation that diagonalizes one additional quantity $G$

![Diagram](image.png)

Figure 28. A schematic representation of the transformation from the pdf parameter basis to the orthonormal eigenvector basis.

- In these new coordinates, variation in a given quantity is now given by one or a few eigenvectors, rather than by all 44 (or however many)
- $G$ may be the W cross section, or the $W^+$ rapidity distribution or a Higgs cross section, depending on how clever one wants to be
- Still some technology to be developed, but in principle these principal error pdf’s could be included in the ntuples
A very useful tool

Allows easy calculation and comparison of pdf’s
Let’s try it out

Up and down quarks dominate at high x, gluon at low x. As $Q^2$ increases, note the growth of the gluon distribution, and to a lesser extent the sea quark distributions.
Uncertainties

Uncertainties get large at high $x$

Uncertainty for gluon larger than that for quarks

PDF's from one group don't necessarily fall into uncertainty band of another … would be nice if they did
Uncertainties and parametrizations

- Beware of extrapolations to $x$ values smaller than data available in the fits, especially at low $Q^2$
- Parameterization may artificially reduce the apparent size of the uncertainties
- Compare for example uncertainty for the gluon at low $x$ from the recent neural net global fit to global fits using a parametrization

$Q^2 = 2 \text{ GeV}^2$

Note: gluon can range negative at low $x$
Correlations

- Consider a cross section $X(a)$
- $i^{th}$ component of gradient of $X$ is
  \[ \frac{\partial X}{\partial a_i} \equiv \partial_i X = \frac{1}{2}(X_i^+ - X_i^-) \]
- Now take 2 cross sections $X$ and $Y$
  - or one or both can be pdf's
- Consider the projection of gradients of $X$ and $Y$ onto a circle of radius 1 in the plane of the gradients in the parton parameter space
- The circle maps onto an ellipse in the $XY$ plane
- The angle $\phi$ between the gradients of $X$ and $Y$ is given by
  \[ \cos \phi = \frac{\bar{\nabla} X \cdot \bar{\nabla} Y}{\Delta X \Delta Y} = \frac{1}{4\Delta X \Delta Y} \sum_{i=1}^{N} \left( X_i^+ - X_i^- \right) \left( Y_i^+ - Y_i^- \right) \]
- The ellipse itself is given by
  \[ \left( \frac{\delta X}{\Delta X} \right)^2 + \left( \frac{\delta Y}{\Delta Y} \right)^2 - 2 \left( \frac{\delta X}{\Delta X} \right) \left( \frac{\delta Y}{\Delta Y} \right) \cos \varphi = \sin^2 \varphi \]

- If two cross sections/pdf's are very correlated, then $\cos \phi \sim 1$
- ...uncorrelated, then $\cos \phi \sim 0$
- ...anti-correlated, then $\cos \phi \sim -1$
Correlations

- Consider a cross section $X(a)$
- $i^{th}$ component of gradient of $X$ is
  \[ \frac{\partial X}{\partial a_i} \equiv \partial_i X = \frac{1}{2} (X_i^+ - X_i^-) \]
- Now take 2 cross sections $X$ and $Y$
  - or one or both can be pdf’s
- Consider the projection of gradients of $X$ and $Y$ onto a circle of radius 1 in the plane of the gradients in the parton parameter space
- The circle maps onto an ellipse in the $XY$ plane
- The angle $\varphi$ between the gradients of $X$ and $Y$ is given by
  \[ \cos \varphi = \frac{\vec{V}_X \cdot \vec{V}_Y}{\Delta X \Delta Y} = \frac{1}{4\Delta X \Delta Y} \sum_{i=1}^{N} \left( X_i^+ - X_i^- \right) \left( Y_i^+ - Y_i^- \right) \]
- The ellipse itself is given by
  \[ \left( \frac{\delta X}{\Delta X} \right)^2 + \left( \frac{\delta Y}{\Delta Y} \right)^2 - 2 \left( \frac{\delta X}{\Delta X} \right) \left( \frac{\delta Y}{\Delta Y} \right) \cos \varphi = \sin^2 \varphi \]

- If two cross sections/pdf’s are very correlated, then $\cos \varphi \approx 1$
- …uncorrelated, then $\cos \varphi \approx 0$
- …anti-correlated, then $\cos \varphi \approx -1$

Figure 28. A schematic representation of the transformation from the pdf parameter basis to the orthonormal eigenvector basis.

Figure 1. Dependence on the correlation ellipse formed in the $\Delta X - \Delta Y$ plane on the value of the correlation cosine $\cos \varphi$. 
Correlations between pdf’s

Correlations between $f(x_1, Q)$ and $f(x_2, Q)$ at $Q = 85$ GeV

$f_1(x_1, Q)$ VS. $f_1(x_2, Q)$

$f_2(x_1, Q)$ VS. $f_2(x_2, Q)$

Can you guess which PDF’s these are?

Homework assignment: which pdf’s and why?
Correlations between $f_1(x_1, Q)$ and $f_2(x_2, Q)$ at $Q = 85$ GeV

$d$ vs $u$

$s$ vs $\bar{u}$ at $Q=2$

$s$ vs. $g$

Sometimes there is a clear physics reason behind the correlation (e.g., sum rules or assumed Regge-like behavior); sometimes not