PHY 410 – Spring 2009 Exam #1

(1 Hour)

PLEASE WAIT UNTIL YOU ARE TOLD TO BEGIN THE EXAM

While waiting, carefully fill in the information requested below

Your Name:…………………………………………………………….

Your Student Number:…………………………………………………

There are 4 problems. Please answer them all.

USEFUL CONSTANTS AND EQUATIONS

Stirling's formula: ln N! ~ N lnN – N when N >> 1

$$\text{Thermal wavelength } \lambda_{th} = \sqrt{\frac{2\pi\hbar^2}{M\tau}}$$

$$\text{Quantum concentration } n_Q = \left(\frac{M\tau}{2\pi\hbar^2}\right)^{3/2} = \frac{1}{\lambda_{th}^3}$$

Boltzmann constant $k_B = 1.38066 x 10^{-23} JK^{-1}$

Planck's constant $\hbar = 1.05459 x 10^{-34} Js$

Problem 1 (10 points)

Consider a system containing 2 magnetic dipoles. Each dipole can point either up or down. Magnetic moment of each dipole is m. There is no external magnetic field.
(4 points)

(i) Write down all the microstates for this system.
(ii) If all the microstates are accessible what is the probability of observing the system in any one microstate?
(iii) What is the probability of finding the system in the macrostate $N_{\uparrow} - N_{\downarrow} = 0$?

(i) There are 4 microstates : ↑↑,↑↓,↓↑,↓↓
(ii) ¼
(iii) There are two microstates associated with the macrostate $N_{\uparrow} - N_{\downarrow} = 0$.
There for the probability of seeing this macrostate is 2x1/4=1/2

Now apply an external magnetic field B. Energy of each magnet is either –mB (when ↑) or +mB (when ↓). The system is in equilibrium with a reservoir at temperature τ.
(6 points)
(i) What is the probability of finding the system in the macrostate $N_{\uparrow} - N_{\downarrow} = 0$?
(ii) What is the probability of finding the system in the macrostate $N_{\uparrow} - N_{\downarrow} = -2$?

$$P_s = P(\varepsilon_s) = \frac{e^{-\varepsilon_s/\tau}}{\sum_s e^{-\varepsilon_s/\tau}}; \textit{ Boltzmann distribution for a microstate } s \textit{ with energy } \varepsilon_s$$

The microstates and their enrgies are given by

s	ε_s	$N_{\uparrow} - N_{\downarrow}$
↑↑	$-2mB$	2
↑↓	0	0
↓↑	0	0
↓↓	$+2mB$	-2

Probaility of finding the system in macrostate $N_{\uparrow} - N_{\downarrow} = 0$

$$\frac{2e^{-0/\tau}}{\left(e^{2mB/\tau} + e^{0/\tau} + e^{0/\tau} + e^{-2mB/\tau}\right)} = \frac{2}{\left(e^{2mB/\tau} + 2 + e^{-2mB/\tau}\right)}$$

Probaility of finding the system in macrostate $N_{\uparrow} - N_{\downarrow} = -2$

$$\frac{e^{-2mB/\tau}}{\left(e^{2mB/\tau} + 2 + e^{-2mB/\tau}\right)}$$

Problem 2 (15 points)

A defect in a solid has three possible states with energies $0, \varepsilon, \varepsilon$. The defect is in equilibrium with the rest of the solid at temperature τ.

(i) What is the partition function for the defect?(2 points)
(ii) What is the average energy U?(4 points)
(iii) What is the average energy of the system as $\tau \to 0$ and as $\tau \to \infty$?(3 points)
(iv) What is the heat capacity associated with N non-interacting and distinguishable defects?(6 points)

$(i)\ Z = 1 + 2e^{-\varepsilon/\tau}$

$(ii)\ U = \tau^2 \dfrac{\partial}{\partial \tau} \ln Z = \dfrac{2\varepsilon e^{-\varepsilon/\tau}}{1+2e^{-\varepsilon/\tau}}; \textit{Also } U = \sum_s \varepsilon_s P(\varepsilon_s) = \sum_s \varepsilon_s e^{-\varepsilon_s/\tau} / \sum_s e^{-\varepsilon_s/\tau} \textit{ gives}$

the same answer.

$(iii)\ U \to 0 \textit{ as } \tau \to 0;\ U \to \dfrac{2\varepsilon}{3} \textit{ as } \tau \to \infty$

The heat capacity for N distinguishable and non-interacting defects is given by $C = N \dfrac{\partial U}{\partial \tau}$, keeping N and V fixed. Since there is no V dependence we ignore it.

$$C = N \frac{\partial}{\partial \tau} \frac{2\varepsilon e^{-\varepsilon/\tau}}{1+2e^{-\varepsilon/\tau}} = N(2\varepsilon)\left[\frac{e^{-\varepsilon/\tau}(\varepsilon/\tau^2)}{1+2e^{-\varepsilon/\tau}} - \frac{e^{-\varepsilon/\tau}(2\varepsilon e^{-\varepsilon/\tau})(\varepsilon/\tau^2)}{(1+2e^{-\varepsilon/\tau})^2}\right]$$

$$= N(2\varepsilon)(\varepsilon/\tau^2)\frac{e^{-\varepsilon/\tau}}{1+2e^{-\varepsilon/\tau}}\left[1 - \frac{2e^{-\varepsilon/\tau}}{1+2e^{-\varepsilon/\tau}}\right] = N2(\varepsilon/\tau)^2 \frac{e^{-\varepsilon/\tau}}{(1+2e^{-\varepsilon/\tau})^2}$$

Problem 3 (10 points)

An ideal gas of N_{Ar} argon atoms is inside a cubic box of volume V at temperature τ.

(A) (6 points)

(i)What is the partition function of this gas Z_{Ar} if the partition function of 1 argon atom is $Z_{1,Ar} = \left(\frac{V}{\lambda_{th,Ar}^3}\right); \lambda_{th,Ar} = \sqrt{\frac{2\pi\hbar^2}{M_{Ar}\tau}}$. (Treat the atoms as indistinguishable).

(ii)What is the Helmholtz Free energy for the system?

(iii)What is the pressure exerted by this gas?

$$Z_{Ar} = \frac{(Z_{1,Ar})^{N_{Ar}}}{N_{Ar}!} = \frac{\left(\frac{V}{\lambda_{th,Ar}}\right)^{N_{Ar}}}{N_{Ar}!}$$

$$F = -\tau \ln Z_{Ar} = -\tau N_{Ar} \ln\left[\frac{V}{N_{Ar}\lambda_{th,Ar}} + 1\right]$$

$$p = -\frac{\partial F}{\partial V} = \frac{\tau N_{Ar}}{V} = \tau\left(\frac{N_{Ar}}{V}\right)$$

(B) Now add an ideal gas of N_{Xe} indistinguishable atoms into the same volume at the same temperature τ. (4 points)

(i) Express the partition function of the total system in terms of Z_{Ar} and Z_{Xe}, where Z_{Xe} is the partition function of the xenon gas.

(ii) What is the total Helmholtz Free energy, express your answer in terms of $\lambda_{th,Ar}$, $\lambda_{th,Xe}$, N_{Ar}, N_{Xe}?

$$Z_{Ar+Xe} = Z_{Ar} \bullet Z_{Xe}$$

$$F_{Ar+Ze} = -\tau \ln Z_{Ar+Xe} = -\tau \ln Z_{Ar} - \tau \ln Z_{Xe} = F_{Ar} + F_{Xe}$$

$$= -\tau N_{Ar} \ln\left[\frac{V}{N_{Ar}\lambda_{th,Ar}} + 1\right] - -\tau N_{Xe} \ln\left[\frac{V}{N_{Xe}\lambda_{th,Xe}} + 1\right]$$

Problem 4 (15 points)

A.

Consider a system consisting of 4 quantum harmonic oscillators, all with same frequency ω. Microstates of each oscillator is characterized by an integer s=0,1,2,… with energy $\varepsilon_s = s\hbar\omega$.

What is the multiplicity of this system when the total energy of the oscillators $U = n\hbar\omega, n = 3$.? (3points)

The multiplicity associated with n quanta distributed oven N oscillators

$$g(N,n) = \frac{(N-1+n)!}{(N-1)!n!}$$

$$g(4,3) = \frac{(4-1+3)!}{(4-1)!3!} = 20$$

B.

Now consider two systems of oscillators A and B. A has 3 oscillators with total energy $2\hbar\omega$ and B has 2 oscillators with energy $1\hbar\omega$. Initially they are not in thermal contact. What is the total number of microstates accessible to the combined system (multiplicity)? (4 points)

$$g_{A+B} = g_A(3,2) \bullet g_B(2,1)$$

$$g_A(3,2) = \frac{(3-1+2)!}{(3-1)!2!} = 6$$

$$g_B(2,1) = \frac{(2-1+1)!}{(2-1)!1!} = 2$$

$$g_{A+B} = 12$$

C.

The two systems are brought into thermal contact. They can exchange energy but the total energy is kept constant at $3\hbar\omega$. What are the multiplicities of the combined system corresponding to different possible energy partitions between A and B.(8 points)

$$N_A = 3, N_B = 2;\ g = g_A(N_A, n_A) \bullet g_B(N_B, n_B)$$

$$n_A = 3, n_B = 0;\ g = \frac{(3-1+3)!}{(3-1)!3!} \bullet \frac{(2-1+0)!}{(2-1)!0!} = 10$$

$$n_A = 2, n_B = 1;\ g = \frac{(3-1+2)!}{(3-1)!2!} \bullet \frac{(2-1+1)!}{(2-1)!1!} = 12$$

$$n_A = 1, n_B = 2;\ g = \frac{(3-1+2)!}{(3-1)!2!} \bullet \frac{(2-1+2)!}{(2-1)!2!} = 9$$

$$n_A = 0, n_B = 3;\ g = \frac{(3-1+0)!}{(3-1)!0!} \bullet \frac{(2-1+3)!}{(2-1)!3!} = 4$$

The total number of microstates for the combined system of 5 oscillators with 3 energy quanta is10+12+9+4=35=$\frac{(5-1+3)!}{(5-1)!3!}$