PHY 410 – Spring 2009 Exam #2

(1 Hour)

PLEASE WAIT UNTIL YOU ARE TOLD TO BEGIN THE EXAM

While waiting, carefully fill in the information requested below

Your Name:…………………………………………………………….

Your Student Number:…………………………………………………

There are 4 problems. Please answer them all showing your work clearly (for partial credit).

USEFUL CONSTANTS AND INTEGRALS

$$\textit{Thermal wavelength } \lambda_{th} = \sqrt{\frac{2\pi\hbar^2}{M\tau}}$$

$$\textit{Quantum concentration } n_Q = \left(\frac{M\tau}{2\pi\hbar^2}\right)^{3/2} = \frac{1}{\lambda_{th}^3}$$

Boltzmann constant $k_B = 1.38066\,x10^{-23}\,JK^{-1}$

Planck's constant $\hbar = 1.05459\,x10^{-34}\,Js$

Energy of a photon in mode $\vec{n}$ inside a cubic box of volume $L^3 : \hbar\omega_{\vec{n}} = \hbar n\pi c / L$

$$\int_0^\infty \frac{x^3}{e^x - 1} dx = \frac{\pi^4}{15}$$

Problem 1 (10 points)

A hemoglobin molecule can bind 4 O_2 molecules at 4 different sites. The energy of the bound O_2 molecule at a given site is ε. Assume that the energy to bind N molecules is $N\varepsilon$. The hemoglobin molecule is in contact with a bath of oxygen molecules at temperature τ and chemical potential μ. Let $\lambda = e^{\mu/\tau}$ be the absolute activity of the O_2 molecules. What is the probability of finding 2 O_2 molecules bound to a hemoglobin molecule? What is the probability that there is no O_2 molecule bound to a hemoglobin?

Let's denote the Gibb's sum as $\overline{Z}$. Note the multiplicity factors 4, 6, 4, and 1 associated with the number of ways 1, 2, 3, and 4 particles can occupy 4 sites. This is very important.

$$\overline{Z} = 1 + \lambda\, 4 e^{-\varepsilon/\tau} + \lambda^2\, 6 e^{-2\varepsilon/\tau} + \lambda^3\, 4\, e^{-3\varepsilon/\tau} + \lambda^4 1 e^{-4\varepsilon/\tau}$$

$$P(2) = \frac{\lambda^2\, 6 e^{-2\varepsilon/\tau}}{1 + \lambda\, 4 e^{-\varepsilon/\tau} + \lambda^2\, 6 e^{-2\varepsilon/\tau} + \lambda^3\, 4\, e^{-3\varepsilon/\tau} + \lambda^4 e^{-4\varepsilon/\tau}}$$

$$P(0) = \frac{1}{1 + \lambda\, 4 e^{-\varepsilon/\tau} + \lambda^2\, 6 e^{-2\varepsilon/\tau} + \lambda^3\, 4\, e^{-3\varepsilon/\tau} + \lambda^4 e^{-4\varepsilon/\tau}}$$

Problem 2 (10 points)

The two possible states (orbitals) of an electron in the presence of an external magnetic field have energies $\varepsilon_\uparrow$ *and* $\varepsilon_\downarrow$ for the up and down spin orientations respectively. Derive an expression for the average number of electrons occupying these two states as a function of the chemical potential μ? What is the ratio of the probabilities of finding the electron with up and down spin orientations respectively.

$$\overline{Z} = 1+\lambda\ e^{-\varepsilon_{\uparrow=}/\tau} + \lambda e^{-\varepsilon_\downarrow/\tau} + \lambda^2\ e^{-(\varepsilon_\uparrow+\varepsilon_\downarrow)/\tau}$$

$$< N(\varepsilon_\uparrow) >= \frac{\lambda\ e^{-\varepsilon_\uparrow/\tau} + \lambda^2\ e^{-(\varepsilon_\uparrow+\varepsilon_\downarrow)/\tau}}{1+\lambda\ e^{-\varepsilon_\uparrow/\tau} + \lambda e^{-\varepsilon_\downarrow/\tau} + \lambda^2\ e^{-(\varepsilon_\uparrow+\varepsilon_\downarrow)/\tau}}$$

$$< N(\varepsilon_\downarrow) >= \frac{\lambda\ e^{-\varepsilon_\downarrow/\tau} + \lambda^2\ e^{-(\varepsilon_\uparrow+\varepsilon_\downarrow)/\tau}}{1+\lambda\ e^{-\varepsilon_\uparrow/\tau} + \lambda e^{-\varepsilon_\downarrow/\tau} + \lambda^2\ e^{-(\varepsilon_\uparrow+\varepsilon_\downarrow)/\tau}}$$

$$\frac{< N(\varepsilon_\uparrow) >}{< N(\varepsilon_\downarrow) >} = \frac{\lambda\ e^{-\varepsilon_\uparrow/\tau} + \lambda^2\ e^{-(\varepsilon_\uparrow+\varepsilon_\downarrow)/\tau}}{\lambda\ e^{-\varepsilon_\downarrow/\tau} + \lambda^2\ e^{-(\varepsilon_\uparrow+\varepsilon_\downarrow)/\tau}}$$

$$= \frac{e^{-\varepsilon_\uparrow/\tau}(1+\lambda\ e^{-\varepsilon_\downarrow/\tau})}{e^{-\varepsilon_\downarrow/\tau}(1+\lambda\ e^{-\varepsilon_\uparrow/\tau})}$$

Note: The state where two electrons occupy the two states with energies $\varepsilon_\uparrow$ *and* $\varepsilon_\downarrow$ respectively we will have 1 electron contributing to $< N(\varepsilon_\uparrow) >$ and 1 electron contributing to $< N(\varepsilon_\downarrow) >$.

Problem 3 (15 points)

Starting from the Plank's distribution function for the photons show that the energy density $u = U/V$ of the black body radiation is given by

$$u = \frac{U}{V} = \frac{\pi^2}{15c^3\hbar^3}\tau^4 .$$

(Hint: After writing down the expression for the U energy in terms of integration over modes isolate the V and τ dependence by a suitable change of variables and then use the integral given on the first page)(6 points)

What is the heat capacity per unit volume of the photon gas?(4 points)

Starting from the above expression for the heat capacity derive an expression for the entropy σ as a function of τ using the thermodynamic identity. (5 points)

(a)

$$U = 2\sum_{\vec{n}} \hbar\omega_{\vec{n}} \frac{1}{e^{\frac{\hbar\omega_{\vec{n}}}{\tau}} - 1} = 2.\frac{1}{8}\int_0^\infty 4\pi n^2 dn \left(\frac{\hbar n \pi c}{L}\right) \frac{1}{e^{\frac{\hbar n \pi c}{L\tau}} - 1}$$

$$Define\ a\ new\,\mathrm{var}iable\ x = \frac{\hbar n \pi c}{L\tau};\ n = \frac{L\tau}{\pi c\hbar}x;\ dn = \frac{L\tau}{\pi c\hbar}dx$$

$$U = 2.\frac{1}{8}.4\pi.\left(\frac{L\tau}{\pi c\hbar}\right)^3 \tau \int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{V}{\pi^2 c^3 \hbar^3}\tau^4 \frac{\pi^4}{15}$$

$$u = \frac{U}{V} = \frac{\pi^2}{15c^3\hbar^3}\tau^4$$

(b)

$$\frac{C_V}{V} = \frac{du}{d\tau} = \frac{4\pi^2}{15c^3\hbar^3}\tau^3$$

(c)

$$C_V = \tau\left(\frac{\partial\sigma}{\partial\tau}\right)_V;\ d\sigma = \frac{C_V}{\tau}d\tau$$

$$\sigma(\tau) - \sigma(0) = \int_0^\tau \frac{C_V(\tau')}{\tau'}d\tau' = \frac{4\pi^2}{45c^3\hbar^3}\tau^3$$

$$Since\ \sigma(0) = 0$$

$$\sigma(\tau) = \frac{4\pi^2}{45c^3\hbar^3}\tau^3$$

Problem 4 (15 points)

Write down the Fermi-Dirac and Bose-Einstein distribution functions for the occupation of a single orbital of energy ε for fermions and bosons respectively. What is the distribution function in the classical regime? When is the classical regime applicable? (4 points)

Using the above classical limit of the distribution function derive an expression for the chemical potential of a system of classical particles as a function of the average number of particles <*N*>, volume *V*, and the quantum concentration n_Q.(5 points)

Using the FD distribution function calculate the chemical potential at $\tau = 0$ (ε_F) for an ideal gas of fermions as a function of the number density $N/V = <N>/V$. If the volume of an electron gas is compressed by a factor of 2 without changing *N*, what is the change in the Fermi energy? (6 points)

(a)

$$f_{FD}(\varepsilon) = \frac{1}{e^{\frac{(\varepsilon-\mu)}{\tau}} + 1}; \; f_{BE}(\varepsilon) = \frac{1}{e^{\frac{(\varepsilon-\mu)}{\tau}} - 1}$$

$$Classical \;\; regime : e^{\frac{(\varepsilon-\mu)}{\tau}} >> 1; \; f_{class}(\varepsilon) = e^{-\frac{(\varepsilon-\mu)}{\tau}}$$

(b)

$$In \; general < N >= \sum_{\vec{n}} \frac{1}{e^{\frac{(\varepsilon_{\vec{n}}-\mu)}{\tau}} \pm 1};$$

$$Classical \; \lim it < N >= \sum_{\vec{n}} e^{-\frac{(\varepsilon_{\vec{n}}-\mu)}{\tau}} = e^{\mu/\tau} \sum_{\vec{n}} e^{-\frac{\varepsilon_{\vec{n}}}{\tau}} = e^{\mu/\tau} V n_Q = e^{\mu/\tau} \frac{V}{\lambda_{th}^3}$$

$$\mu = \tau \ln \frac{n}{n_Q}; \; where \; the \; density \; n = \frac{< N >}{V}$$

(c)

Fermions at $\tau = 0$

$$< N >= 2.\sum_{\vec{n}} \frac{1}{e^{\frac{(\varepsilon-\mu)}{\tau}} + 1} \Rightarrow 2.\sum_{\vec{n},n\le n_F} 1 \;\; as \; \tau \to 0;$$

where the Fermi energy $\varepsilon_F = \mu(\tau = 0)$ *is related to* n_F *by the relation* $\varepsilon_F = \dfrac{\hbar^2}{2m}\left(\dfrac{\pi n_F}{L}\right)^2$

$$< N >= 2.\frac{1}{8}.\int_0^{n_F} 4\pi\, n^2\, dn = \frac{\pi\, n_F^3}{3};\; n_F = \left(\frac{3 < N >}{\pi}\right)^{1/3} = \left(\frac{3N}{\pi}\right)^{1/3}$$

$$\varepsilon_F = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2\left(\frac{3N}{\pi}\right)^{2/3} = \frac{\hbar^2}{2m}\left(\frac{\pi^3}{L^3}\frac{3N}{\pi}\right)^{2/3} = \frac{\hbar^2}{2m}\left(3\pi^2\frac{N}{V}\right)^{2/3}$$

If the volume V decreases by a factor of 2 without changing N then the Fermi energy increases by a factor of $2^{2/3}$