PHY 410

Homework 10

10.1

(i) H atom is combined by 1 proton and 1 electron which are elementary particles spin $\frac{1}{2}$ $\Rightarrow$ H can be consider as boson

(ii)

average volume confined one atom:

$$V_0 = \frac{1}{\rho} = \frac{1}{1.8\times 10^{14}\,\text{atom}/\text{cm}^3} = 0.56\times 10^{-14}\,\text{cm}^3$$

$$\rightarrow \frac{4}{3}\pi r_0^3 = V_0 \Rightarrow r_0 = \left(\frac{3V_0}{4\pi}\right)^{1/3} = 1.1\times 10^{-5}\,\text{cm}$$

$\rightarrow$ average distance:

$$d_0 \simeq 2r_0 = 2.2\times 10^{-5}\,\text{cm}$$

(iii)

$$T_E = \frac{2\pi\hbar^2}{M k_B}\left(\frac{N}{2.612\,V}\right)^{2/3}$$

$$T_E = \frac{2\pi\times(1.054\times 10^{-34})^2}{0.167\times 10^{-26}\,\text{kg}\times 1.38\times 10^{-23}}\left(\frac{1.8\times 10^{20}}{2.612}\right)^{2/3}$$

$$T_E = 5.1\times 10^{-5}\,\text{K} = 51\,\mu\text{K}$$ ~ experiment result.

$\boxed{10.2}$

$$T_E = \frac{2\pi\hbar^2}{Mk_B}\left(\frac{N}{2.612V}\right)^{2/3}$$

$$T_E = \frac{2\pi \times (1.054\times10^{-34})^2}{0.66\times10^{-26}\times 1.38\times10^{-23}}\left(\frac{1}{2.612}\times\frac{0.145\times10^{-3}}{10^{-6}}\times\frac{1}{0.66\cdot10^{-26}}\right)^{2/3}$$

$T_E = 3.17\,k >$ experiment result $(2.17\,k)$

$\boxed{10.3}$ $$U = \int_0^\infty \varepsilon D(\varepsilon)f(\varepsilon) = \frac{V}{(4\pi)^2}\left(\frac{2M}{\hbar^2}\right)^{3/2}\int_0^\infty d\varepsilon\,\frac{\varepsilon^{3/2}}{\lambda^{-1}e^{\varepsilon/\tau}-1}$$

$$x = \frac{\varepsilon}{\tau} \rightarrow d\varepsilon = \tau\,dx$$

$$U = \frac{V}{(4\pi)^2}\left(\frac{2M}{\hbar^2}\right)^{3/2}\tau^{5/2}\int_0^\infty dx\,\frac{x^{3/2}}{\lambda^{-1}e^{x}-1}$$

Using approximation, set $\lambda = 1$

$$U = \frac{V}{(4\pi)^2}\left(\frac{2M}{\hbar^2}\right)^{3/2}\tau^{5/2}\underbrace{\int_0^\infty dx\,\frac{x^{3/2}}{e^{x}-1}}_{I}$$

$$U = \frac{V}{4\pi}\left(\frac{2M}{\hbar^2}\right)^{3/2}\tau^{5/2}\cdot I$$

$$C_V = \frac{dU}{d\tau} = \frac{5}{2}\cdot\frac{V}{4\pi}\left(\frac{2M}{\hbar^2}\right)^{3/2} I\,\tau^{3/2}$$

$$C_V = \tau\left(\frac{\partial\sigma}{\partial\tau}\right)_V \rightarrow \sigma = \frac{5}{2}\cdot\frac{V}{4\pi}\left(\frac{2M}{\hbar^2}\right)^{3/2} I\int \tau^{1/2}\,d\tau$$

$$\sigma = \frac{5}{3}\cdot\frac{V}{4\pi}\left(\frac{2M}{\hbar^2}\right)^{3/2} I\cdot\tau^{3/2}$$

10.4

③

Bose-Einstein distribution function $(N \gg 1)$

$$f(\varepsilon) = \frac{1}{e^{\frac{\varepsilon - \mu}{\tau}} - 1}$$

$$\langle N(\varepsilon = 0)\rangle = f(0) = \frac{1}{e^{-\mu/\tau} - 1}$$

$$\langle N(\varepsilon)\rangle = f(\varepsilon) = \frac{1}{e^{\frac{\varepsilon-\mu}{\tau}} - 1}$$

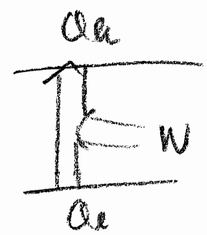
$$* \quad \langle N(0)\rangle = 2\langle N(\varepsilon)\rangle \Leftrightarrow \frac{1}{e^{-\mu/\tau} - 1} = \frac{2}{e^{\frac{\varepsilon-\mu}{\tau}} - 1}$$

$$N = \langle N(0)\rangle + \langle N(\varepsilon)\rangle \rightarrow \begin{cases} \langle N(0)\rangle = \frac{2}{3}N \\ \langle N(\varepsilon)\rangle = \frac{1}{3}N \end{cases}$$

$$\Leftrightarrow \begin{cases} \frac{1}{e^{-\mu/\tau} - 1} = \frac{2}{3}N \Rightarrow e^{-\mu/\tau} = 1 + \frac{3}{2N} \Rightarrow \tau = -\frac{\mu}{\log\left(1 + \frac{3}{2N}\right)} \\ \frac{1}{e^{\frac{\varepsilon-\mu}{\tau}} - 1} = \frac{1}{3}N \rightarrow e^{\frac{\varepsilon-\mu}{\tau}} = 1 + \frac{3}{N} \Rightarrow \tau = \frac{\varepsilon - \mu}{\log\left(1 + \frac{3}{N}\right)} \end{cases}$$

$$\Rightarrow \begin{cases} \tau = -\frac{\mu}{\log\left(1 + \frac{3}{2N}\right)} = -\frac{\varepsilon}{\ln\left(1+\frac{3}{2N}\right) - \ln\left(1 + \frac{3}{N}\right)} \\ \mu = \frac{\varepsilon}{1 - \frac{\log\left(1+\frac{3}{N}\right)}{\log\left(1 + \frac{3}{2N}\right)}} \end{cases}$$

10.5



(a) Energy require W : $Q_h = Q_l + W \rightarrow$

$$W = Q_h - Q_l = Q_h\left(1 - \frac{\tau_l}{\tau_h}\right)$$

$$\Rightarrow \frac{W}{Q_h} = \frac{Q_h}{Q_h}\left(1 - \frac{\tau_l}{\tau_h}\right) = \frac{\tau_h - \tau_l}{\tau_h} = \eta_C$$

(+) If heat pump is not reversible

$$\sigma_h \geq \sigma_l \rightarrow \frac{Q_h}{\tau_h} \geq \frac{Q_l}{\tau_l} \rightarrow Q_h \geq Q_l\left(\frac{\tau_h}{\tau_l}\right)$$

$$W = Q_h - Q_l \geq Q_h\left(\frac{\tau_h - \tau_l}{\tau_h}\right)$$

$$\Rightarrow \frac{W}{Q_h} \geq \frac{\tau_h - \tau_l}{\tau_h} = \eta_C$$

(b)

$$\left.\begin{aligned} \frac{W'}{Q_{hh}} &= \frac{\tau_{hh} - \tau_l}{\tau_{hh}} \\ \frac{W}{Q_h} &= \frac{\tau_h - \tau_l}{\tau_h} \end{aligned}\right\} \xrightarrow{W = W'} \frac{Q_{hh}}{Q_h} = \frac{\tau_h - \tau_l}{\tau_{hh} - \tau_l} \times \frac{\tau_{hh}}{\tau_h} = \boxed{\frac{2}{11}}$$