PHY410
Home Work #2

Problem 2.1

$$g(N,s) = \frac{N!}{\left(\frac{N}{2}+s\right)!\left(\frac{N}{2}-s\right)!}$$

$$\ln g(N,s) = \ln N! - \ln\left(\frac{N}{2}+s\right)! - \ln\left(\frac{N}{2}-s\right)!$$

$$= \ln N! - \ln N_\uparrow! - \ln N_\downarrow! \quad (1)$$

Here, $N_\uparrow = \frac{N}{2}+s$, $N_\downarrow = \frac{N}{2}-s \Rightarrow N = N_\uparrow + N_\downarrow$

Using stirling's approximation

$$\ln N! \simeq \frac{1}{2}\ln 2\pi + \left(N+\frac{1}{2}\right)\ln N - N \quad (2)$$

$$\ln N_\uparrow! \cong \frac{1}{2}\ln 2\pi + \left(N_\uparrow+\frac{1}{2}\right)\ln N_\uparrow - N_\uparrow \quad (3)$$

$$\ln N_\downarrow! \cong \frac{1}{2}\ln 2\pi + \left(N_\downarrow+\frac{1}{2}\right)\ln N_\downarrow - N_\downarrow \quad (4)$$

Substitute (2, 3, 4) into (1):

$$\ln g(N,s) \cong -\frac{1}{2}\ln 2\pi + \left(\overbrace{N_\uparrow+N_\downarrow}^{N}+\frac{1}{2}\right)\ln N - \left(N_\uparrow+\frac{1}{2}\right)\ln N_\uparrow - \left(N_\downarrow+\frac{1}{2}\right)\ln N_\downarrow$$

$$\Rightarrow \ln g(N,s) \cong -\frac{1}{2}\ln 2\pi - \frac{1}{2}\ln N - \left(N_\uparrow+\frac{1}{2}\right)\left(\ln N_\uparrow - \ln N\right) - \left(N_\downarrow+\frac{1}{2}\right)\left(\ln N_\downarrow - \ln N\right)$$

$$\Rightarrow \ln g(N,s) \cong \frac{1}{2}\ln\frac{1}{2\pi N} - \left(N_\uparrow+\frac{1}{2}\right)\ln\frac{N_\uparrow}{N} - \left(N_\downarrow+\frac{1}{2}\right)\ln\frac{N_\downarrow}{N} \quad (5)$$

(4) We have

$$\ln \frac{N_\uparrow}{N} = \ln \frac{\frac{N}{2}+s}{N} = \ln \frac{1}{2}\left(1+\frac{2s}{N}\right) = -\ln 2 + \ln\left(1+\frac{2s}{N}\right).$$

Here, using the expansion

$$\ln(1+x) = x - \frac{1}{2}x^2 \ldots \qquad x \ll 1$$

$$\Rightarrow \ln\left(1+\frac{2s}{N}\right) \cong \frac{2s}{N} - \frac{1}{2}\left(\frac{2s}{N}\right)^2 = \frac{2s}{N} - \frac{2s^2}{N^2}$$

$$\Rightarrow \ln\frac{N_\uparrow}{N} \cong -\ln 2 + \frac{2s}{N} - \frac{2s^2}{N^2} \qquad (6)$$

Similarly, $\ln \frac{N_\downarrow}{N} = \ln \frac{\frac{N}{2}-s}{N} = \ln \frac{1}{2}\left(1-\frac{2s}{N}\right)$

$$\Rightarrow \ln\frac{N_\downarrow}{N} \cong -\ln 2 - \frac{2s}{N} - \frac{2s^2}{N^2} \qquad (7)$$

Substitute (6) & (7) into (5)

$$\ln g(N,s) \cong \frac{1}{2}\ln\frac{1}{2\pi N} - \left(N_\uparrow + \frac{1}{2}\right)\left(-\ln 2 + \frac{2s}{N} - \frac{2s^2}{N^2}\right)$$
$$- \left(N_\downarrow + \frac{1}{2}\right)\left(-\ln 2 - \frac{2s}{N} - \frac{2s^2}{N^2}\right)$$

Using $N_\uparrow + N_\downarrow = N$

$$\Rightarrow \ln g(N,s) \cong \frac{1}{2}\ln\frac{1}{2\pi N} + (N+1)\ln 2 - \underbrace{(N_\uparrow - N_\downarrow)}_{2s}\frac{2s}{N} + (N+1)\frac{2}{N}$$

$$\cong \frac{1}{2}\ln\frac{2}{\pi N} + N\ln 2 - \frac{2s^2}{N} + \frac{2s^2}{N^2}$$

$$\boxed{\ln g(N,s) \cong \frac{1}{2}\ln\frac{2}{\pi N} + N\ln 2 - \frac{2s^2}{N}}$$

(Neglect term $\frac{2s^2}{N^2}$ because $\frac{2s^2}{N^2} \ll \frac{2}{N}$

$$\Rightarrow \ln g(N,s) \cong \ln\sqrt{\frac{2}{\pi N}} + \ln 2^N + \ln e^{-\frac{2s^2}{N}}$$

$$\cong \ln\left(\sqrt{\frac{2}{\pi N}} \cdot 2^N \cdot e^{-2s^2/N}\right)$$

$$\Rightarrow \quad g(N,s) \cong \sqrt{\frac{2}{\pi N}}\; 2^N \; e^{-2s^2/N}$$

$$\boxed{g(N,s) \cong g(N,0)\, e^{-2s^2/N}}, \boxed{g(N,0) = \sqrt{\frac{2}{\pi N}}\; 2^N}$$

Problem 2.2

(a) Condition: $s_1 + s_2 + s_3 = n = 2$

$\Rightarrow$ there're 6 microstates

$$(N; s_1, s_2, s_3) = (3; 0, 0, 2)$$
$$(3; 0, 2, 0)$$
$$(3; 2, 0, 0)$$
$$(3; 0, 1, 1)$$
$$(3; 1, 0, 1)$$
$$(3; 1, 1, 0)$$

(b) $$g(N, n) = \frac{(N+n-1)!}{n!\,(N-1)!}$$

Here $N = 3$, $n = 2$ $\Rightarrow g(3, 2) = \dfrac{(3+2-1)!}{2!\,(3-1)!} = 6$ ✓

Problem 2.3

Using $g(N,s) = \dfrac{N!}{(\frac{N}{2}+s)!\,(\frac{N}{2}-s)!}$

$N = 4 \Rightarrow s = -2, -1, 0, 1, 2$

$N_1 = 2,\ N_2 = 2 \Rightarrow s_1, s_2 = -1, 0, 1$

Case 1 $s = -2 \Rightarrow g(4,-2) = \dfrac{4!}{(2-2)!\,(2+2)!} = 1$

$s_1 + s_2 = s = -2 \Rightarrow s_1 = -1,\ s_2 = -1$

$g_1(2,-1) = \dfrac{2!}{(1-1)!\,(1+1)!} = 1$

$g_2(2,-1) = \dfrac{2!}{(1-1)!\,(1+1)!} = 1$

$\Rightarrow g_1(2,-1)\cdot g_2(2,-1) = 1 = g(4,-2)$

$\Rightarrow g(4,s) = \sum_{s_1}\sum_{s_2} g_1(2,s_1)\cdot g_2(2,s_2)$ ✓

Case 2 $s = -1 \Rightarrow g(4,-1) = \dfrac{4!}{(2-1)!\,(2+1)!} = 4$

$s_1 + s_2 = s = -1 \Rightarrow \begin{cases} s_1 = 0, & s_2 = -1 \\ s_1 = -1, & s_2 = 0 \end{cases}$

$$g_1(2,0) = g_2(2,0) = \frac{2!}{(1+0)!\,(1-0)!} = 2$$

$$g_1(2,-1) = g_2(2,-1) = \frac{2!}{(1-1)!\,(1+1)!} = 1$$

$$\Rightarrow \quad g_1(2,0)\cdot g_2(2,-1) + g_1(2,-1)\cdot g_2(2,0) = 2+2 = 4 = g(4,$$

Case 3

$$S = 0 \Rightarrow g(4,0) = \frac{4!}{(2+0)!\,(2-0)!} = 6$$

$$S_1 + S_2 = S = 0 \rightarrow \begin{cases} S_1 = S_2 = 0 \\ S_1 = -1,\ S_2 = 1 \\ S_1 = 1,\ S_2 = -1 \end{cases}$$

$$g_1(2,0) = g_2(2,0) = 2$$

$$g_1(2,-1) = g_2(2,-1) = 1$$

(see case 2)

$$g_1(2,1) = g_2(2,1) = \frac{2!}{(1+1)!\,(1-1)!} = 1$$

$$\Rightarrow g_1(2,0)\cdot g_2(2,0) + g_1(2,-1)\cdot g_2(2,1) + g_1(2,1)\cdot g_2(2,-1)$$

$$= 2\cdot 2 + 1\cdot 1 + 1\cdot 1 = 6 = g(4,0) \checkmark$$

Case 4

$s = 1 \Rightarrow s_1 + s_2 = 1 \Rightarrow \begin{cases} s_1 = 0, \; s_2 = 1 \\ s_1 = 1, \; s_2 = 0 \end{cases}$ $\Rightarrow$ The same situat as case 2

$$g(4,1) = g_1(2,0) \cdot g_2(2,1) + g_1(2,1) \cdot g_2(2,0) = 4 \checkmark$$

Case 5

$s = 2 \Rightarrow s_1 + s_2 = 2 \rightarrow s_1 = s_2 = 1$

The same situation as case 1

$$g(4,2) = g_1(2,1) \cdot g_2(2,1) = 1$$

(b) We can't use Stirling's approximation because N_1, N_2, N are small.