PHY 410

HOMEWORK # 3

3.1

$$g(U) = C U^{3N/2}$$

(a)

$$\sigma = \log g(U) = \frac{3N}{2} \log U + \log C$$

$$\frac{1}{\tau} = \left(\frac{\partial \sigma}{\partial U}\right)_N = \frac{3N}{2} \frac{1}{U} \rightarrow \boxed{U = \frac{3}{2} N \tau}$$

(b)

$$\left(\frac{\partial \sigma}{\partial U}\right)_N = \frac{3N}{2U} \Rightarrow \left(\frac{\partial^2 \sigma}{\partial U^2}\right)_N = -\frac{3N}{2U^2} < 0$$

3.2

(a) Suppose $\tau_1 < \tau_2$ then :

$$\frac{1}{k_B} m C_v (\tau_F - \tau_1) = \frac{1}{k_B} m C_v (\tau_2 - \tau_F)$$

$$\Rightarrow \boxed{\tau_F = \frac{\tau_1 + \tau_2}{2}}$$

(b) from $\frac{1}{\tau} = \left(\frac{\partial \sigma}{\partial U}\right)_N \Rightarrow d\sigma = \frac{dU}{\tau} = \frac{m C_v}{k_B} \frac{d\tau}{\tau}$

$$\Rightarrow \Delta \sigma_1 = \frac{m C_v}{k_B} \int_{\tau_1}^{\tau_F} \frac{d\tau}{\tau} = \frac{m C_v}{k_B} \ln \frac{\tau_F}{\tau_1}$$

$$\Delta \sigma_2 = \frac{m C_v}{k_B} \int_{\tau_2}^{\tau_F} \frac{d\tau}{\tau} = \frac{m C_v}{k_B} \ln \frac{\tau_F}{\tau_2}$$

$$\Rightarrow \Delta \sigma = \Delta \sigma_1 + \Delta \sigma_2 = \frac{m C_v}{k_B} \left(\ln \frac{\tau_F}{\tau_1} + \ln \frac{\tau_F}{\tau_2} \right)$$

$$= \frac{m C_v}{k_B} \ln \frac{\tau_F^2}{\tau_1 \tau_2}$$

The increase in number of accessible microstates

$$g = e^{\Delta\sigma} = \exp\left[\frac{mC_V}{k_B} \ln \frac{\tau_F^2}{\tau_1 \tau_2}\right]$$

$$\boxed{g = \left(\frac{\tau_F^2}{\tau_1 \tau_2}\right)^{\frac{mC_V}{k_B}}}$$

$\boxed{3.3}$

Energy : $U = -2msB \longrightarrow s = -\dfrac{U}{2mB}$

$$g(N,s) = g(N,0)\, e^{-2s^2/N}$$

$$\Rightarrow g(N,U) = g(N,0)\, e^{-\frac{U^2}{2m^2B^2N}}$$

$$\Rightarrow \sigma(U) = \log g(N,U) = \log g(N,0) - \frac{U^2}{2m^2B^2N}$$

$$\Rightarrow \sigma(U) = \sigma_0 - \frac{U^2}{2m^2B^2N}$$

$$\frac{1}{\tau} = \frac{\partial\sigma}{\partial U} = -\frac{U}{m^2B^2N} = \frac{2s}{mBN}$$

$$\Rightarrow s = \frac{mBN}{2\tau} \Rightarrow \frac{M}{Nm} = \frac{2s}{N} = \boxed{\frac{mB}{\tau}}$$

3.4

(a) $$g(N,n) = \frac{(N+n-1)!}{n!\,(N-1)!}$$

$$\Rightarrow \sigma(N,n) = \log g(N,n) = \log(N+n-1)! - \log n! - \log(N-1)!$$

+ Using stirling approximation:

$$\log(N+n-1)! \simeq (N+n-1)\log(N+n-1) - (N+n-1)$$

$$\log n! = n\log n - n$$

$$\log(N-1)! = (N-1)\log(N-1) - (N-1)$$

$$\Rightarrow \sigma(N,n) = (N+n-1)\log(N+n-1) - n\log n - (N-1)\log(N-1)$$

Because $N \gg 1$ so we can replace $N-1$ by N

$$\sigma(N,n) \simeq (N+n)\log(N+n) - n\log n - N\log N$$

$$\sigma(N,n) \cong N\log\frac{N+n}{N} + n\log\frac{N+n}{n}$$

$$\boxed{\sigma(N,n) \cong N\log\left(1+\frac{n}{N}\right) + n\log\left(\frac{N}{n}+1\right)}$$

(b) $$U = n\hbar\omega \Rightarrow n = \frac{U}{\hbar\omega}$$

$$\sigma(N,U) \cong N\log\left(1+\frac{U}{N\hbar\omega}\right) + \frac{U}{\hbar\omega}\log\left(\frac{N\hbar\omega}{U}+1\right)$$

$$\frac{1}{\tau} = \frac{\partial\sigma}{\partial U} = \frac{N}{1+\frac{U}{N\hbar\omega}}\cdot\frac{1}{N\hbar\omega} + \frac{1}{\hbar\omega}\log\left(\frac{N\hbar\omega}{U}+1\right)$$

$$+ \frac{U}{\hbar\omega}\cdot\frac{1}{\frac{N\hbar\omega}{U}+1}\left(-\frac{N\hbar\omega}{U^2}\right)$$

$$\Rightarrow \frac{1}{\tau} = \frac{N}{N\hbar\omega + U} + \frac{1}{\hbar\omega}\log\left(\frac{N\hbar\omega}{U}+1\right) - \frac{N}{N\hbar\omega+U}$$

$$\frac{1}{\tau} = \frac{1}{\hbar\omega}\log\left(\frac{N\hbar\omega}{U}+1\right)$$

$$\Rightarrow \log\left(\frac{N\hbar\omega}{U}+1\right) = \frac{\hbar\omega}{\tau}$$

$$\Rightarrow \frac{N\hbar\omega}{U}+1 = e^{\frac{\hbar\omega}{\tau}} \Rightarrow \boxed{U = \frac{N\hbar\omega}{e^{\frac{\hbar\omega}{\tau}}-1}}$$

$\boxed{3.5}$

$$N_1 \simeq N_2 = 10^{22}$$

$$s_1 = \hat{s}_1 + 10^{11} \rightarrow \delta = 10^{11}$$

$$s = 0$$

We have: $\frac{\hat{s}_1}{N_1} = \frac{\hat{s}_2}{N_2} \Rightarrow \hat{s}_1 = \hat{s}_2$ & $\hat{s}_1 + \hat{s}_2 = s = 0 \Rightarrow \hat{s}_1 = \hat{s}_2 = 0$

$$s = s_1 + s_2 = 0 \Rightarrow s_2 = -s_1 = -\hat{s}_1 - 10^{11} = \hat{s}_2 - \delta$$

Using (17) $g_1(N_1, \hat{s}_1+\delta)\, g_2(N_2, \hat{s}_2-\delta) = (g_1 g_2)_{max} \exp\left(-\frac{2\delta^2}{N_1} - \frac{2\delta^2}{N_2}\right)$

$$\Rightarrow \frac{g_1 g_2}{(g_1 g_2)_{max}} = \exp\left(-\frac{2\delta^2}{N_1} - \frac{2\delta^2}{N_2}\right) \Rightarrow \frac{g_1 g_2}{(g_1 g_2)_{max}} = \exp(-4) = \boxed{0.018}$$

(b)

$$\frac{2\delta^2}{N_1} = \frac{2\delta^2}{N_2} = \frac{2 \cdot 10^{22}}{10^{22}} = 2$$

(b) $s = 10^{20}$

$$\frac{\hat{s}_1}{N_1} = \frac{\hat{s}_2}{N_2} \Rightarrow \hat{s}_1 = \hat{s}_2 \quad (\text{cause' } N_1 = N_2 = 10^{22}) \quad \Big| \Rightarrow \hat{s}_1 = \hat{s}_2 = \frac{10^{20}}{2}$$

$$\hat{s}_1 + \hat{s}_2 = s = 10^{20}$$

(*) $$\sum_{s_1} g_1(N_1, s_1)\, g_2(N_2, s - s_1) = \sum_{s_1} g_1(0)\, g_2(0) \exp\left(-\frac{2s_1^2}{N_1} - \frac{2(s-s_1)^2}{N_2}\right)$$

$$\simeq \int_{-\infty}^{\infty} ds_1\, g_1(0) g_2(0)\, e^{-\frac{2s_1^2}{N_1} - \frac{2(s-s_1)^2}{N_2}}$$

Note that $N_1 = N_2 \Rightarrow \frac{2s_1^2}{N_1} + \frac{2(s-s_1)^2}{N_2} = \frac{2s_1^2 + 2(s-s_1)^2}{N_1}$

$$= \frac{2}{N_1}\left[s_1^2 + s^2 - 2ss_1 + s_1^2\right]$$

$$= \frac{2}{N_1}\left[2s_1^2 - 2ss_1 + s^2\right]$$

$$= \frac{4}{N_1}\left[s_1^2 - ss_1 + \frac{s^2}{2}\right] = \frac{4}{N_1}\left[\left(s_1 - \frac{s}{2}\right)^2 + \frac{s^2}{4}\right]$$

$$= \frac{4}{N_1}\left[s_1 - s/2\right]^2 + \frac{s^2}{N_1}$$

$$\Rightarrow \sum_{s_1} g_1(N_1, s_1)\, g_2(N_2, s - s_1) \approx g_1(0) g_2(0)\, e^{-\frac{s^2}{N_1}} \int_{-\infty}^{\infty} ds_1\, e^{-\frac{4}{N_1}(s_1 - s/2)}$$

$$\simeq g_1(0) g_2(0)\, e^{-s^2/N_1} \cdot \sqrt{\frac{N_1 \pi}{4}}$$

Other hand, $$(g_1 g_2)_{max} \approx g_1(0) g_2(0)\, e^{-2s^2/N} \simeq g_1(0) g_2(0)\, e^{-2s^2/2N_1}$$

$$\simeq g_1(0) g_2(0)\, e^{-s^2/N_1}$$

$\Rightarrow$ The multiplied factor is $\sqrt{\frac{N_1 \pi}{4}} = \sqrt{\frac{10^{22} \cdot \pi}{4}} = \boxed{\frac{10^{11}}{2}\sqrt{\pi}}$