PHY 410
HW #4

4.1

(a)

$$F = -\tau \log Z \quad ; \quad Z = \sum_s \exp\left(-\frac{\varepsilon_s}{\tau}\right)$$

$$\Rightarrow Z = 1 + e^{-\frac{\varepsilon}{\tau}}$$

$$\Rightarrow \boxed{F = -\tau \log\left[1 + e^{-\frac{\varepsilon}{\tau}}\right]}$$

(b) entropy

$$\sigma = -\frac{\partial F}{\partial \tau} = +\left[\log\left[1 + e^{-\varepsilon/\tau}\right] + \frac{\tau}{1 + e^{-\varepsilon/\tau}} e^{-\varepsilon/\tau} \frac{\varepsilon}{\tau^2}\right]$$

$$\sigma = +\left(\log\left[1 + e^{-\varepsilon/\tau}\right] + \frac{e^{-\varepsilon/\tau}}{1 + e^{-\varepsilon/\tau}} \cdot \frac{\varepsilon}{\tau}\right)$$

$$\boxed{\sigma = +\log\left[1 + e^{-\varepsilon/\tau}\right] + \frac{\varepsilon}{\tau} \frac{1}{e^{\varepsilon/\tau} + 1}}$$

$$U = F + \tau\sigma$$

$$= -\tau \log\left(1 + e^{-\varepsilon/\tau}\right) + \tau \log\left(1 + e^{-\varepsilon/\tau}\right) + \varepsilon \cdot \frac{1}{e^{\varepsilon/\tau} + 1}$$

$$\Rightarrow \boxed{U = +\frac{\varepsilon}{e^{\varepsilon/\tau} + 1}}$$

4.2

(a) when put a magnet into a magnetic field B, the the system has 2 state

+ $\varepsilon_\uparrow = -mB$. (if $\vec{m} \uparrow\uparrow \vec{B}$)

+ $\varepsilon_\downarrow = mB$ (if $\vec{m} \uparrow\downarrow \vec{B}$)

$$\rightarrow Z = e^{-\frac{mB}{\tau}} + e^{\frac{mB}{\tau}}$$

The number of magnets/volume having $\vec{m} \uparrow\uparrow \vec{B}$:

$$N_\uparrow = n . P(\varepsilon_\uparrow) = n . \frac{e^{mB/\tau}}{Z} \quad ; n = \frac{N}{V}$$

The number of magnets having $\vec{m} \uparrow\downarrow \vec{B}$

$$N_\downarrow = n P(\varepsilon_\downarrow) = \frac{n . e^{-mB/\tau}}{Z}$$

Spin access: $N_\uparrow - N_\downarrow = 2S$

The magnetization: $M = 2Sm = mn \frac{e^{mB/\tau} - e^{-mB/\tau}}{Z}$
(Magnetic moment/Volume)

$$\boxed{M = mn . \frac{e^{mB/\tau} - e^{-mB/\tau}}{e^{mB/\tau} + e^{-mB/\tau}}} = mn . \frac{\sinh\left(\frac{mB}{\tau}\right)}{\cosh\left(\frac{mB}{\tau}\right)}$$

$$= \boxed{mn . \tanh\left(\frac{mB}{\tau}\right)}$$

(+) susceptibility $\chi = \frac{dM}{dB}$

$$\chi = \frac{dM}{dB} = mn \cdot \frac{d}{dB}\left(\tanh\left(\frac{mB}{\tau}\right)\right)$$

$$= mn \cdot \frac{1}{\cosh^2\left(\frac{mB}{\tau}\right)} \cdot \frac{m}{\tau} = \boxed{\frac{m^2 n}{\tau} \cdot \frac{1}{\cosh^2\left(\frac{mB}{\tau}\right)}}$$

(b)

$$Z = e^{-\frac{mB}{\tau}} + e^{\frac{mB}{\tau}} = 2\cosh\frac{mB}{\tau}$$

$$F = -\tau \log Z = -\tau \log\left[e^{-\frac{mB}{\tau}} + e^{\frac{mB}{\tau}}\right]$$

$$\boxed{F = -\tau \log\left(2\cosh\frac{mB}{\tau}\right)}$$

+ We have

$$1 - \tanh^2\left(\frac{mB}{\tau}\right) = \frac{1}{\cosh^2\left(\frac{mB}{\tau}\right)} \Rightarrow \cancel{\cosh^2\left(\frac{mB}{\tau}\right) = \frac{1}{1-\left(\frac{M}{mn}\right)^2}}$$

Put $x = \tanh\frac{mB}{\tau}$

$$\Rightarrow \cosh\frac{mB}{\tau} = \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \boxed{F = -\tau \log\frac{2}{\sqrt{1-x^2}} = -\frac{\tau}{2}\log\frac{4}{1-x^2}}$$

(e)

$$\chi = \frac{m^2 n}{\tau} \frac{1}{\cosh^2\left(\frac{mB}{\tau}\right)}$$

$$mB \ll \tau \Rightarrow \frac{mB}{\tau} \ll 1 \qquad e^{\frac{mB}{\tau}} \approx 1 + \frac{mB}{\tau}$$

$$e^{-\frac{mB}{\tau}} \approx 1 - \frac{mB}{\tau}$$

$$\Rightarrow \cosh\frac{mB}{\tau} = \frac{1}{2}\left(e^{\frac{mB}{\tau}} + e^{-\frac{mB}{\tau}}\right) \approx$$

$$\Rightarrow \boxed{\chi \approx \frac{m^2 n}{\tau}}$$

$\boxed{4.3}$

(a)

$$Z = \sum_s \exp(-\varepsilon_s/\tau) = \sum_{s=0}^{\infty} \exp\left(-\frac{s\hbar\omega}{\tau}\right)$$

$$= \sum_{s=0}^{\infty}\left[\exp\left(-\frac{\hbar\omega}{\tau}\right)\right]^s = \frac{1}{1-\exp\left(-\frac{\hbar\omega}{\tau}\right)}.$$

$$\left(\text{cause : } \sum_n x^n = \frac{1}{1-x}, \quad |x| \ll 1\right)$$

$$F = -\tau \log Z = -\tau \log \frac{1}{1-e^{-\frac{\hbar\omega}{\tau}}}$$

$$\Rightarrow \boxed{F = \tau \log\left[1-\exp\left(-\frac{\hbar\omega}{\tau}\right)\right]}$$

(b)

$$\sigma = -\frac{\partial F}{\partial \tau} = -\log\left[1-\exp\left(-\frac{\hbar\omega}{\tau}\right)\right] - \tau \cdot \frac{1}{1-e^{-\frac{\hbar\omega}{\tau}}} \cdot \left(-e^{-\frac{\hbar\omega}{\tau}}\right) \cdot$$

$$\boxed{\sigma = \frac{\hbar\omega/\tau}{\exp\left(\frac{\hbar\omega}{\tau}\right)-1} - \log\left[1-\exp\left(\frac{\hbar\omega}{\tau}\right)\right]}$$

$\boxed{4.4}$

(a)

Microstates	Energy
$\uparrow_1 \uparrow_2 \uparrow_3$	$-3mB$
$\uparrow_1 \uparrow_2 \downarrow_3$	$-mB$
$\uparrow_1 \downarrow_2 \uparrow_3$	$-mB$
$\downarrow_1 \uparrow_2 \uparrow_3$	$-mB$
$\uparrow_1 \downarrow_2 \downarrow_3$	mB
$\downarrow_1 \downarrow_2 \uparrow_3$	mB
$\downarrow_1 \uparrow_2 \downarrow_3$	mB
$\downarrow_1 \downarrow_2 \downarrow_3$	$3mB$

(b) $$Z = \sum_s \exp(-\varepsilon_s/\tau) = e^{\frac{3mB}{\tau}} + e^{-\frac{3mB}{\tau}} + 3e^{\frac{mB}{\tau}} + 3e^{-\frac{mB}{\tau}}$$

$$\boxed{Z(3,\tau) = 2\cosh\left(\frac{3mB}{\tau}\right) + 6\cosh\frac{mB}{\tau}}$$

$$Z(1,\tau) = 2\cosh\left(\frac{mB}{\tau}\right) = e^{\frac{mB}{\tau}} + e^{-\frac{mB}{\tau}}$$

$$\left[Z(1,\tau)\right]^3 = \left(e^{\frac{mB}{\tau}} + e^{-\frac{mB}{\tau}}\right)^3$$

$$= e^{\frac{3mB}{\tau}} + e^{-\frac{3mB}{\tau}} + 3e^{\frac{2mB}{\tau}} \cdot e^{-\frac{mB}{\tau}} + 3e^{-\frac{2mB}{\tau}} e^{\frac{mB}{\tau}}$$

$$= e^{\frac{3mB}{\tau}} + e^{-\frac{3mB}{\tau}} + 3e^{\frac{mB}{\tau}} + 3e^{-\frac{mB}{\tau}}$$

$$\boxed{\left[Z(1,\tau)\right]^3 = 2\cosh\frac{3mB}{\tau} + 2\cosh\frac{mB}{\tau} = Z(3,\tau)}$$