PHY 410

HW #5

5.1

$$\lambda_{th} = \sqrt{\frac{2\pi\hbar^2}{m k_B T}}$$

(a) N_2, $T = 300K$

$$\lambda_{N_2} = \sqrt{\frac{2\pi \times (1.054\times10^{-34} J.S)^2}{4.65\times10^{-26} kg\ 1.38\times10^{-23} m^2 kg\, s^{-2} T^{-1} \times 300K}} = 1.904\times10^{-11}\ m)$$

The inter particle space

$$d_{N_2} = \left(\frac{V}{N}\right)^{1/3} = \left(\frac{1}{n_{N_2}}\right)^{1/3} = \left(\frac{1}{10^{19}/cc}\right)^{1/3} = 4.642\times10^{-7}\ cm$$
$$= 4.642\times10^{-9}\ m$$

$\lambda_{N_2} \ll d_{N_2} \rightarrow$ can be treated as classical

(b) Electron, $T = 300K$

$$\lambda_e = \sqrt{\frac{2\pi\times(1.054\times10^{-34})^2}{9.1\times10^{-31}\times1.38\times10^{-23}\times300}} = 4.303\times10^{-9}\ m$$

$$d_e = \left(\frac{V}{N}\right)^{1/3} = (10^{-22})^{1/3} = 4.642\times10^{-8}\ cm = 4.642\times10^{-10}\ m$$

$\lambda_e \sim 10 d_e \rightarrow$ Can not be treated as classical

(c) He, $T = 1K$

$$\lambda_{He} = \sqrt{\frac{2\pi\times(1.054\times10^{-34})^2}{6.692\times10^{-27}\times1.38\times10^{-23}\times1}} = 8.691\times10^{-10}\ m$$

$$d_{He} = \left(\frac{V}{N}\right)^{1/3} = (10^{-22})^{1/3} = 4.642\times10^{-10}\ m$$

$\lambda_{He} \sim d_{He} \rightarrow$ cannot be treated as classical

5.2 Sackur-Tetrode equation

$$\sigma = N\left[\log\left(\frac{n_Q}{n}\right) + \frac{5}{2}\right] \equiv N\left[\log\left(\frac{v}{\lambda_{th}^3}\right) + \frac{5}{2}\right] \quad , v = \frac{1}{n}, \lambda_{th}^3 = \frac{1}{n_Q}$$

$$\Rightarrow \frac{\Delta\sigma}{N} = \frac{\sigma_A - \sigma_B}{N} = \log\left(\frac{n_Q^A}{n^A}\right) - \log\left(\frac{n_Q^B}{n^B}\right)$$

$$= \log\left(\frac{n_Q^A}{n_Q^B} \cdot \frac{n^B}{n^A}\right)$$

(a) isothermal process $\quad \tau_A = \tau_B \rightarrow n_Q^A = n_Q^B$

$$V_A = \frac{1}{2}V_B \rightarrow n^B = \frac{1}{2}n^A$$

$$\Rightarrow \frac{\Delta\sigma}{N} = \log\frac{1}{2} = \boxed{-\log 2}$$

(b) isochoric process $\quad V_A = V_B \Rightarrow n^A = n^B$

$$\tau_A = 2\tau_B \Rightarrow \frac{n_Q^A}{n_Q^B} = \left(\frac{\tau_A}{\tau_B}\right)^{3/2} = (2)^{3/2}$$

$$\frac{\Delta\sigma}{N} = \log(2)^{3/2} = \boxed{+\frac{3}{2}\log 2}$$

5.3

(a)

$$Z_R = \sum_{j=0}^{\infty} g(j)\, e^{-\frac{\varepsilon(j)}{\tau}} = \boxed{\sum_{j=0}^{\infty} (2j+1)\, e^{-\frac{\varepsilon_0}{\tau} j(j+1)}}$$

(b)

$$= \int_0^\infty dj\, (2j+1)\, e^{-\frac{\varepsilon_0}{\tau} j(j+1)}$$

or $\rightarrow$ $\boxed{\text{Define } \frac{\varepsilon_0}{\tau} j(j+1) = x \quad \frac{\varepsilon_0}{\tau}(2j+1)\,dj = dx \quad \frac{\tau}{\varepsilon_0}\int_0^\infty dx\, e^{-x} = \frac{\tau}{\varepsilon_0}}$

If $\varepsilon_0 \ll \tau$

$$= \int_0^\infty dj\, (2j+1)\, e^{-\frac{\varepsilon_0}{\tau}\left[\left(j+\frac{1}{2}\right)^2 - \frac{1}{4}\right]}$$

$n = j + \frac{1}{2}$

$$\Rightarrow Z_R = \int_{1/2}^{\infty} dn \; 2n \; e^{-\frac{\varepsilon_0}{\tau}\left[n^2 - \frac{1}{4}\right]}$$

$$= 2 e^{\frac{\varepsilon_0}{4\tau}} \int_{1/2}^{\infty} dn \; n \; e^{-\frac{\varepsilon_0}{\tau} n^2}$$

$$= e^{\frac{\varepsilon_0}{4\tau}} \int_{1/2}^{\infty} d(n^2) \; e^{-\frac{\varepsilon_0}{\tau} n^2}$$

$$= e^{\frac{\varepsilon_0}{4\tau}} \cdot e^{-\frac{\varepsilon_0}{\tau} n^2} \left(-\frac{\tau}{\varepsilon_0}\right) \Big|_{1/2}^{\infty}$$

$$= e^{\frac{\varepsilon_0}{4\tau}} \left(-\frac{\tau}{\varepsilon_0}\right)\left(- e^{-\frac{\varepsilon_0}{4\tau}}\right)$$

$$= \boxed{\frac{\tau}{\varepsilon_0}}$$

(c) If $\varepsilon_0 \gg \tau$

$$Z_R \approx g(0)\, e^{-\frac{\varepsilon(0)}{\tau}} + g(1)\, e^{-\frac{\varepsilon(1)}{\tau}}$$

$$\boxed{Z_R = 1 + 3 e^{-\frac{2\varepsilon_0}{\tau}}}$$

(d)

$\underline{\varepsilon_0 \ll \tau}$ $\qquad Z_R = \frac{\tau}{\varepsilon_0}$

$$U = \langle \varepsilon \rangle = \frac{\sum_j (2j+1)\, \varepsilon_j\, e^{-\varepsilon_j/\tau}}{Z}$$

use

$$\boxed{\begin{aligned} U = \langle \varepsilon \rangle &= \tau^2 \frac{\partial}{\partial \tau} \ln Z_R \\ &= \tau^2 \frac{\partial}{\partial \tau} \ln(\tau/\varepsilon_0) = \tau^2 \cdot \frac{1}{\tau} \\ &= \tau \end{aligned}}$$

$$U = \frac{\varepsilon_0}{\tau} \sum_j (2j+1)\, j(j+1)\varepsilon_0\, e^{-\frac{\varepsilon_0}{\tau} j(j+1)}$$

$$U = 2\frac{\varepsilon_0^2}{\tau}\int_0^\infty dj\,(j+\tfrac{1}{2})\left[(j+\tfrac{1}{2})^2-\tfrac{1}{4}\right] e^{-\frac{\varepsilon_0}{\tau}\left[(j+\frac{1}{2})^2-\frac{1}{4}\right]}$$

$$n = j+\frac{1}{2}$$

$$U = 2\frac{\varepsilon_0^2}{\tau}\int_{1/2}^\infty dn\; n\left(n^2-\frac{1}{4}\right) e^{-\frac{\varepsilon_0}{\tau}n^2}\, e^{\frac{\varepsilon_0}{4\tau}}$$

$$x = n^2$$

$$U = \frac{\varepsilon_0^2}{\tau}\, e^{\frac{\varepsilon_0}{4\tau}}\int_{1/4}^\infty dx\,\left(x-\frac{1}{4}\right) e^{-\frac{\varepsilon_0}{\tau}x}$$

$$U = \frac{\varepsilon_0^2}{\tau}\, e^{\frac{\varepsilon_0}{4\tau}}\left[\int_{1/4}^\infty x\, e^{-\frac{\varepsilon_0}{\tau}x}dx - \frac{1}{4}\int_{1/4}^\infty e^{-\frac{\varepsilon_0}{\tau}x}dx\right]$$

$$(+)\quad \int_{1/4}^\infty e^{-\frac{\varepsilon_0}{\tau}x}dx = -\frac{\tau}{\varepsilon_0}\, e^{-\frac{\varepsilon_0}{\tau}x}\Big|_{1/4}^\infty = \frac{\tau}{\varepsilon_0}\, e^{-\frac{\varepsilon_0}{4\tau}}$$

$$(\dagger)\quad \int_{1/4}^\infty x\, e^{-\frac{\varepsilon_0}{\tau}x}dx = -\frac{\tau}{\varepsilon_0}\int_{1/4}^\infty x\, d\left(e^{-\frac{\varepsilon_0}{\tau}x}\right)$$

$$= -\frac{\tau}{\varepsilon_0}\left[x\, e^{-\frac{\varepsilon_0}{\tau}}\Big|_{1/4}^\infty - \int_{1/4}^\infty e^{-\varepsilon_0/\tau\, x}dx\right]$$

$$= -\frac{\tau}{\varepsilon_0}\left[-\frac{1}{4}\, e^{-\frac{\varepsilon_0}{4\tau}} + \frac{\tau}{\varepsilon_0}\, e^{-\varepsilon_0/\tau\, x}\Big|_{1/4}^\infty\right]$$

$$= -\frac{\tau}{\varepsilon_0}\left[-\frac{1}{4}\, e^{-\frac{\varepsilon_0}{4\tau}} - \frac{\tau}{\varepsilon_0}\, e^{-\frac{\varepsilon_0}{4\tau}}\right]$$

$$= \frac{\tau}{4\varepsilon_0}\, e^{-\varepsilon_0/4\tau} + \frac{\tau^2}{\varepsilon_0^2}\, e^{-\frac{\varepsilon_0}{4\tau}}$$

$$\Rightarrow \quad U = \frac{\varepsilon_0^2}{\tau} e^{\frac{\varepsilon_0}{4\tau}} \left[\frac{\tau}{4\varepsilon_0} e^{-\frac{\varepsilon_0}{4\tau}} + \frac{\tau^2}{\varepsilon_0^2} e^{-\frac{\varepsilon_0}{4\tau}} - \frac{\tau}{4\varepsilon_0} e^{-\frac{\varepsilon_0}{4\tau}} \right]$$

$$\boxed{U = \tau}$$

$$\boxed{C_V = \frac{\partial U}{\partial \tau} = 1}$$

$\underline{\varepsilon_0 \gg \tau}$

$$Z_R = 1 + 3 e^{-\frac{2\varepsilon_0}{\tau}}$$

; or

$$\tau^2 \frac{\partial}{\partial \tau} \ln \left[1 + 3 e^{-2\varepsilon_0/\tau} \right] = \tau^2 \frac{3 \cdot (-2\varepsilon_0) e^{-2\varepsilon_0/\tau} \cdot (-1/\tau^2)}{1 + 3 e^{-2\varepsilon_0/\tau}} = \frac{6\varepsilon_0 e^{-2\varepsilon_0/\tau}}{1 + 3 e^{-2\varepsilon_0/\tau}} = \frac{6\varepsilon_0}{3 + e^{2\varepsilon_0/\tau}}$$

$$U = \langle \varepsilon \rangle = \frac{1}{Z} \sum_{J=0}^{\infty} (2j+1)\, \varepsilon(j)\, e^{-\frac{\varepsilon(j)}{\tau}}$$

$$= \frac{1}{Z} \left[\varepsilon(0)\, e^{-\varepsilon(0)/\tau} + 3\, \varepsilon(1)\, e^{-\frac{\varepsilon(1)}{\tau}} \right]$$

$$= \frac{1}{Z} \left[3 \times 2\varepsilon_0 \times e^{-\frac{2\varepsilon_0}{\tau}} \right] = \frac{6\varepsilon_0 e^{-\frac{2\varepsilon_0}{\tau}}}{1 + 3 e^{-\frac{2\varepsilon_0}{\tau}}}$$

$$\boxed{U = \frac{6\varepsilon_0}{3 + e^{\frac{2\varepsilon_0}{\tau}}}}$$

$$C_V = \frac{\partial U}{\partial \tau} = -\frac{6\varepsilon_0}{(3 + e^{2\varepsilon_0/\tau})^2} e^{\frac{2\varepsilon_0}{\tau}} \left(-\frac{2\varepsilon_0}{\tau^2} \right)$$

$$\boxed{C_V = \frac{12\varepsilon_0^2}{\tau^2} \times \frac{e^{2\varepsilon_0/\tau}}{(3 + e^{2\varepsilon_0/\tau})^2}}$$

e

(e)

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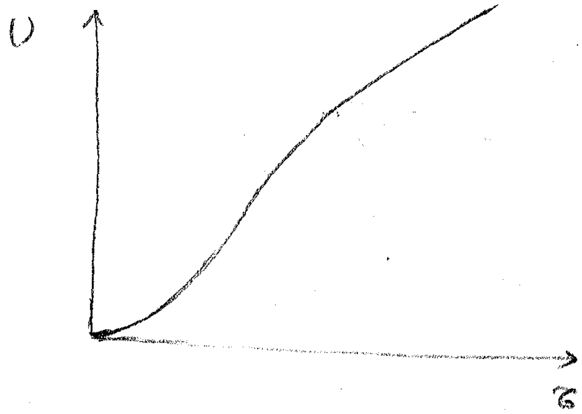

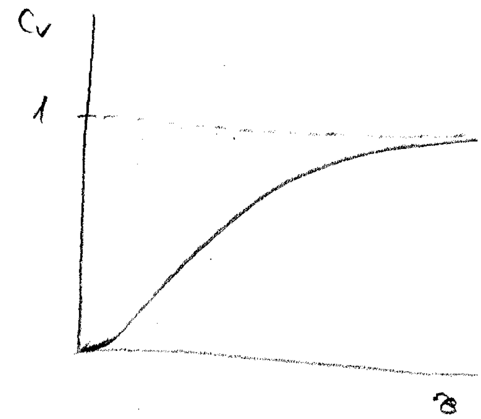

5.4

$$\varepsilon_n = \frac{\hbar^2\pi^2}{2ML^2}\left(n_x^2+n_y^2+n_z^2\right)$$

At the ground orbital $n_x = n_y = n_z = 1$

$$\varepsilon_{ground} = \boxed{\frac{3}{2}\frac{\hbar^2\pi^2}{ML^2}}$$

Inorder $\varepsilon_{ground} = \tau$

$$\Rightarrow \quad \frac{3}{2}\frac{\hbar^2\pi^2}{ML^2} = \tau \Rightarrow L^2 = \frac{3}{2}\frac{\hbar^2\pi^2}{M\tau}$$

$$n_0 = \frac{1}{L^3} = \left(\frac{2M\tau}{3\hbar^2\pi^2}\right)^{3/2} = \left(\frac{4}{3\pi}\right)^{3/2}\left(\frac{M\tau}{2\pi\hbar^2}\right)^{3/2}$$

$$n_0 = \left(\frac{4}{3\pi}\right)^{3/2} n_Q = \boxed{0.3\, n_Q}$$

5.5

1 particles in 1 dimension: $\varepsilon_n = \frac{\hbar^2}{2M}\left(\frac{\pi}{L}\right)^2 n^2$

$$Z_1 = \sum_n e^{-\varepsilon_n/\tau} = \int_0^\infty e^{-\frac{\hbar^2\pi^2}{2ML^2\tau}n^2}\, dn = \frac{1}{2}\sqrt{\frac{2ML^2\tau}{\pi\hbar^2}}$$

N particles: $$Z_N = \frac{1}{N!}(Z_1)^N$$

$$F = -\tau \log Z_N = -\tau \log Z_1^N + \tau \log N!$$

$$F = -\tau N \log Z_1 + \tau(N\log N - N)$$

$$F = -\tau N \log Z_1 + \tau N \log N - \tau N$$

$$F = -\frac{\tau N}{2}\log\left(\frac{ML^2\tau}{2\pi\hbar^2}\right) + \tau N \log N - \tau N$$

$$\sigma = -\frac{\partial F}{\partial \tau} = \frac{N}{2}\log\left(\frac{ML^2\tau}{2\pi\hbar^2}\right) + \frac{\tau N}{2}\cdot\frac{2\pi\hbar^2}{ML^2\tau}\cdot\frac{ML^2}{2\pi\hbar^2} - N\log N + N$$

$$\sigma = \frac{N}{2}\log\left(\frac{ML^2\tau}{2\pi\hbar^2}\right) + \frac{N}{2} - N\log N + N$$

$$\sigma = N\log\frac{\left(\frac{ML^2\tau}{2\pi\hbar^2}\right)^{1/2}}{N} + \frac{3N}{2}$$

$$\sigma = N\log\frac{\left(\frac{M\tau}{2\pi\hbar^2}\right)^{1/2}}{\frac{N}{L}} + \frac{3N}{2}$$

$$n = \frac{N}{L}$$

$$n_Q = \sqrt{\frac{M\tau}{2\pi\hbar^2}}$$

$$\Rightarrow \boxed{\sigma = N\left[\log\left(\frac{n_Q}{n}\right) + \frac{3}{2}\right]}$$