PHY 410
HW # 6

6.1

$$\langle s_n \rangle = \frac{1}{\exp\left(\frac{\hbar\omega_n}{\tau}\right)-1}$$

$$N = \sum_n \langle s_n \rangle = \sum_n \frac{1}{\exp\left(\frac{\hbar\omega_n}{\tau}\right)-1}$$

$$N = \frac{1}{8}\int_0^\infty 4\pi n^2\,dn \frac{1}{\exp\left(\frac{\hbar\omega_n}{\tau}\right)-1} \times 2$$

$$N = \pi \int_0^\infty n^2 dn \frac{1}{\exp\left(\frac{\hbar\pi c}{L\tau} n\right)-1}$$

$$x = \frac{\pi\hbar c}{L\tau} \rightarrow dx = \frac{\pi\hbar c}{L\tau}\,dn$$

$$\rightarrow N = \frac{\pi L\tau}{\pi\hbar c}\int_0^\infty dx \left(\frac{L\tau}{\pi\hbar c}\right)^2 x^2 \frac{1}{e^x-1}$$

$$N = \frac{1}{\pi^2} L^3 \left(\frac{\tau}{\hbar c}\right)^3 \int_0^\infty \frac{x^2}{e^x-1}\,dx$$

$$L^3 = V \;\&\; \int_0^\infty \frac{x^2}{e^x-1}\,dx = 2.404$$

$$\rightarrow \boxed{N = 2.404\,\pi^{-2} V \left(\frac{\tau}{\hbar c}\right)^3}$$

6.2

(a) $p = -\left(\frac{\partial U}{\partial V}\right)_\sigma$

$$U = \sum_j s_j \hbar\omega_j \Rightarrow p = -\sum_j s_j \hbar \left(\frac{d\omega_j}{dV}\right)_\sigma \quad (1)$$

(b) $\omega_j = \frac{j\pi c}{L} = \frac{j\pi c}{V^{1/3}}$

$$\frac{d\omega_j}{dV} = -\frac{1}{3}\frac{j\pi c}{V^{4/3}} = \boxed{-\frac{\omega_j}{3V}}$$

(c) plug in (1):

$$p = -\sum_j s_j \hbar \left(-\frac{\omega_j}{3V}\right) = \frac{\sum s_j \hbar\omega_j}{3V} = \boxed{\frac{U}{3V}}$$

(d)

* pressure of thermal radiation

$$P_{rad} = \frac{U}{3V} = \frac{1}{3}\left(\frac{\pi^2}{15\hbar^3 c^3}\tau^4\right) = \frac{1}{3}\left(\frac{\pi^2 k_B^4 T^4}{15\hbar^3 c^3}\right)$$

* pressure of kinetic energy

$$P_{kin} = N. k_B T$$

$$P_{rad} = P_{kin} \Leftrightarrow \frac{\pi^2 k_B^4 T^4}{45\hbar^3 c^3} = N k_B T \Rightarrow T^3 = \frac{45 N\hbar^3 c^3}{\pi^2 k_B^3}$$

$$\Rightarrow T = \frac{\hbar c}{k_B}\sqrt[3]{\frac{45N}{\pi^2}} = \frac{1.054\times10^{-34}\times 3\times10^{8}}{1.381\times10^{-23}}\sqrt[3]{\frac{45}{\pi^2}.6.(}$$

$$= \boxed{32.07\times10^{6}\,(K)}$$

6.3

• Energy of photons

$$U = \frac{\pi^2 V}{15\hbar^3 c^3} \tau^4 \rightarrow C_v = \frac{\partial U}{\partial \tau} = \frac{4\pi^2 V}{15\hbar^3 c^3} \tau^3 = \frac{4\pi^2 V}{15\hbar^3 c^3} k_B^3 T^3$$

• Heat capacity contributed by phonon

$$C_v' = \frac{12\pi^4 N}{5} \left(\frac{\tau'}{k_B \theta}\right)^3 = \frac{12\pi^4 N}{5} \left(\frac{T'}{\theta}\right)^3$$

$$C_v = C_v' \Leftrightarrow \frac{4\pi^2 V}{15\hbar^3 c^3} k_B^3 T^3 = \frac{12\pi^4 N}{5} \left(\frac{T'}{\theta}\right)^3$$

$$\Rightarrow T = \frac{T'}{k_B \theta} \hbar c \left(9\pi^2 \frac{N}{V}\right)^{1/3}$$

$$T = \frac{T'}{\theta} \cdot \frac{\hbar c}{k_B} \left(9\pi^2 \frac{N}{V}\right)^{1/3}$$

$$T = \frac{1}{100} \times \frac{1.054 \times 10^{-34} \times 3 \times 10^8}{1.381 \times 10^{-23}} \left(9\pi^2 \times 10^{28}\right)^{1/3}$$

$$\boxed{T = 22 \times 10^4 \, K}$$

6.4

Total chemical potential

$$\mu = \tau \log\left(\frac{n}{n_Q}\right) + Mgh$$

$$\Rightarrow \mu - Mgh = \tau \log\left(\frac{n}{n_Q}\right)$$

$$\Rightarrow n = n_Q \exp\left(\frac{\mu - Mgh}{\tau}\right)$$

$$\Rightarrow n = n_Q \exp\left(\frac{\mu}{\tau}\right) \cdot \exp\left(-\frac{Mgh}{\tau}\right)$$

$$\boxed{n = n(0) \exp\left(-\frac{Mgh}{\tau}\right)}, \quad n(0) = n_Q \exp\left(\frac{\mu}{\tau}\right)$$

$$p = n\tau$$

$$\Rightarrow \boxed{p(h) = p(0) \exp\left(-\frac{Mgh}{\tau}\right)} \qquad p(0) = n(0)\tau = n_Q \tau \exp($$