PHY 410

HW # 8

[8.1] Fermi-Dirac function

$$f = \frac{1}{e^{\frac{\varepsilon-\mu}{\tau}}+1}$$

$$\Rightarrow -\frac{\partial f}{\partial \varepsilon} = -\frac{\partial}{\partial \varepsilon}\left(\frac{1}{e^{\frac{\varepsilon-\mu}{\tau}}+1}\right) = \frac{1}{\left(e^{\frac{\varepsilon-\mu}{\tau}}+1\right)^2} e^{\frac{\varepsilon-\mu}{\tau}} \times \frac{1}{\tau}$$

At $\varepsilon = \mu$ $\Rightarrow -\frac{\partial f}{\partial \varepsilon}\Big|_{\varepsilon=\mu} = \frac{1}{(1+1)^2} \times e^0 \times \frac{1}{\tau} = \boxed{(4\tau)^{-1}}$

[8.2]

ⓐ $$\mathcal{Z} = \sum_{ASN} \exp\left[(N\mu - \varepsilon_{S(N)})/\tau\right]$$

$N=0$ $\varepsilon_{S(0)}=0$

$N=1$ $\varepsilon_{S(1)}=\varepsilon$

$N=2$ $\varepsilon_{S(2)}=2\varepsilon$

$$\Rightarrow \mathcal{Z} = 1 + \exp\left(\frac{\mu-\varepsilon}{\tau}\right) + \exp\left(\frac{2\mu-2\varepsilon}{\tau}\right)$$

$$= 1 + \lambda e^{-\varepsilon/\tau} + \lambda^2 e^{-2\varepsilon/\tau}$$

$$\langle N\rangle = \lambda\frac{\partial}{\partial\lambda}\ln \mathcal{Z} = \frac{\lambda e^{-\varepsilon/\tau} + 2\lambda^2 e^{-2\varepsilon/\tau}}{1+\lambda e^{-\varepsilon/\tau}+\lambda^2 e^{-2\varepsilon/\tau}}$$

$$= \frac{\exp\left(\frac{\mu-\varepsilon}{\tau}\right) + 2\exp\left(\frac{2(\mu-\varepsilon)}{\tau}\right)}{1+\exp\left(\frac{\mu-\varepsilon}{\tau}\right)+\exp\left(\frac{2(\mu-\varepsilon)}{\tau}\right)}$$

1) In the usual quantum system

orbital 1 ——— orbital 2 ———

$N=0 \rightarrow \varepsilon_0 = 0$ (no particle in both 1 & 2)

$N=1 \rightarrow \varepsilon_1 = \varepsilon$ (1 particle in orbital 1 or orbital 2)

$N=2 \rightarrow \varepsilon_2 = 2\varepsilon$ (1 particle in orbital 1
1 particle in orbital 2)

$$\mathcal{Z} = \sum_{ASN} \exp\left[N\frac{\mu-\varepsilon}{\tau}\right] = 1 + 2\exp\left(\frac{\mu-\varepsilon}{\tau}\right) + \exp\left[\frac{2(\mu-\varepsilon)}{\tau}\right]$$

$$\langle N \rangle = \frac{1\times\exp\left[\frac{\mu-\varepsilon}{\tau}\right] + 1\times\exp\left[\frac{\mu-\varepsilon}{\tau}\right] + 2\times\exp\left[\frac{2(\mu-\varepsilon)}{\tau}\right]}{\mathcal{Z}}$$

$$\langle N \rangle = \frac{2\exp\left(\frac{\mu-\varepsilon}{\tau}\right) + 2\exp\left[\frac{2(\mu-\varepsilon)}{\tau}\right]}{1 + 2\exp\left(\frac{\mu-\varepsilon}{\tau}\right) + \exp\left(\frac{2(\mu-\varepsilon)}{\tau}\right)}$$

$\boxed{8.3}$

(a) chemical potential

$$Z_{int} = \sum_{int} \exp(-\varepsilon_{int}/\tau)$$

We have $\begin{cases} \varepsilon^1_{int} = 0 \\ \varepsilon^2_{int} = \Delta \end{cases} \rightarrow Z_{int} = 1 + e^{-\Delta/\tau}$

$$\mu_{tot} = \tau \left[\log\left(\frac{n}{n_Q}\right) - \log Z_{int} \right]$$

$$\mu_{tot} = \tau \left[\log N - \log V - \frac{3}{2}\log\tau + \frac{3}{2}\log\left(\frac{2\pi\hbar^2}{M}\right) - \log\left(1 + e^{-\Delta/\tau}\right) \right]$$

Here, Using : $n = \frac{N}{V}$

$$n_Q = \left(\frac{M\tau}{2\pi\hbar^2}\right)^{3/2}$$

(b)

$$F_{tot} = F + F_{int} = N\tau\left[\log\left(\frac{n}{n_Q}\right) - 1\right] - N\tau\log Z_{int}$$

$$F_{tot} = N\tau\left[\log N - 1 - \log V - \frac{3}{2}\log\tau + \frac{3}{2}\log\left(\frac{2\pi\hbar^2}{M}\right) - \log\left(1 + e^{-\Delta/\tau}\right)\right]$$

(c)

$$\sigma_{tot} = \sigma + \sigma_{int} = N\left[\log\left(\frac{n_Q}{n}\right) + \frac{5}{2}\right] - \left(\frac{\partial F_{int}}{\partial \tau}\right)_V$$

$$\sigma_{tot} = N\left[-\log N + \log V + \frac{3}{2}\log\tau - \frac{3}{2}\log\left(\frac{2\pi\hbar^2}{M}\right) + 5/2\right] + \frac{\partial}{\partial\tau}\left(N\tau\log Z_{int}\right)$$

We have

$$\frac{\partial}{\partial \tau}\left(N\tau \log Z_{int}\right) = N\tau \frac{\partial}{\partial \tau} \log\left(1+e^{-\Delta/\tau}\right)$$

$$= N\tau \frac{1}{1+e^{-\Delta/\tau}} e^{-\Delta/\tau}\left(\frac{\Delta}{\tau^2}\right)$$

$$= \frac{N\Delta}{\tau} \frac{e^{-\Delta/\tau}}{1+e^{-\Delta/\tau}}$$

$$\Rightarrow \sigma_{tot} = N\left[-\log N + \log V + \frac{3}{2}\log\tau - \frac{3}{2}\log\left(\frac{2\pi\hbar^2}{M}\right) + \frac{5}{2} + \frac{\Delta}{\tau}\frac{1}{e^{\Delta/\tau}+1}\right]$$

(d)

$$P_{tot} = -\left(\frac{\partial F_{tot}}{\partial V}\right)_{\tau,N} \overset{\text{part b}}{=}$$

$$= -\frac{\partial}{\partial V}\left\{N\tau\left(\log N - 1 - \log V - \frac{3}{2}\log\tau + \frac{3}{2}\log\left(\frac{2\pi\hbar^2}{M}\right) - \log\left(1+e^{-\Delta/\tau}\right)\right)\right\}$$

$$= \boxed{\frac{N\tau}{V}}$$

(e)

$$C_V = \tau\left(\frac{\partial\sigma}{\partial\tau}\right)_{V,N} \overset{\text{part d}}{=} N\tau\frac{\partial}{\partial\tau}\left\{-\log N + \log V + \frac{3}{2}\log\tau - \frac{3}{2}\log\left(\frac{2\pi\hbar^2}{M}\right) + \frac{5}{2} + \frac{\Delta}{\tau}\frac{1}{e^{\Delta/\tau}+1}\right\}$$

$$C_V = N\tau\left(\frac{5}{2}\cdot\frac{1}{\tau} - \frac{\Delta}{\tau^2}\cdot\frac{1}{e^{\Delta/\tau}+1} + \frac{\Delta}{\tau}\frac{(-1)}{(e^{\Delta/\tau}+1)^2}\, e^{\Delta/\tau}\left(-\frac{\Delta}{\tau^2}\right)\right)$$ ⑤

$$C_V = N\tau\left(\frac{5}{2\tau} - \frac{\Delta}{\tau^2}\cdot\frac{1}{e^{\Delta/\tau}+1} + \frac{\Delta^2}{\tau^3}\cdot\frac{e^{\Delta/\tau}}{(e^{\Delta/\tau}+1)^2}\right)$$

$$C_V = N\left(\frac{5}{2} - \frac{\Delta}{\tau}\cdot\frac{1}{e^{\Delta/\tau}+1} + \frac{\Delta^2}{\tau^2}\frac{e^{\Delta/\tau}}{(e^{\Delta/\tau}+1)^2}\right)$$

$$C_p = N + C_V = N\left(\frac{5}{2} - \frac{\Delta}{\tau}\frac{1}{e^{\Delta/\tau}+1} + \frac{\Delta^2}{\tau^2}\frac{e^{\Delta/\tau}}{(e^{\Delta/\tau}+1)^2}\right)$$

$\boxed{8.4}$

ⓐ

+) isothermal process : $V_2 = 2V_1$

work done by the gas in the expansion

$$W_{12} = -\int_{V_1}^{V_2} p\,dV = -\int_{V_1}^{V_2} \frac{N\tau}{V}\,dV = -N\tau\log\frac{V_2}{V_1} = -N\tau\log 2$$

Because $T_1 = T_2 \rightarrow U = 0 \Rightarrow W + Q = 0 \Rightarrow$

$$Q_{12} = -W = N\tau\log 2 = Nk_BT\log 2 = RT\log 2$$

$$Q_{12} = 8.31\times 300\times\log 2 = 1728\text{ J}$$

(a) Isentropic expansion

$Q_{23} = 0$

(b) first process is isothermal

$\tau_1 = \tau_2 \rightarrow T_1 = T_2 = 300K$

second process is isentropic

$T_2^{3/2} V_2 = T_3^{3/2} V_3 \Rightarrow T_3 = T_2 \left(\frac{V_2}{V_3} \right)^{2/3}$

$\Rightarrow T_3 = 300\,K \left(\frac{1}{2} \right)^{2/3}$

$T_3 \approx 200\,K$

(c) Irreversible process

$\Delta \sigma = \sigma_2 - \sigma_1 = N \log V_2 - N \log V_1$

$\Delta \sigma = N \log (V_2/V_1) = 6.02 \times 10^{23} \log 2 = 4.17 \times 10^{23}$

$\Delta S = k_B \Delta \sigma = 1.38 \times 10^{-23} J/K \times 4.17 \times 10^{23} = 1.38 \times 4.17\ J/K$

$\boxed{8.5}$ $d\sigma = \frac{dU}{\tau} + \frac{p dV}{\tau} = \frac{1}{\tau} \left(\frac{\partial U}{\partial \tau} \right)_V d\tau + \frac{1}{\tau} \left(\frac{\partial U}{\partial V} \right)_\tau dV + \frac{p dV}{\tau}$

$\left(\frac{\partial \sigma}{\partial \tau} \right)_V = \frac{1}{\tau} \left(\frac{\partial U}{\partial \tau} \right)_V \Rightarrow \left(\frac{\partial U}{\partial \tau} \right)_V = \tau \left(\frac{\partial \sigma}{\partial \tau} \right)_V = C_V$

$$\Rightarrow \quad d\sigma = C_V \frac{d\tau}{\tau} + \frac{1}{\tau}\left(\frac{\partial U}{\partial V}\right)_\tau dV + \frac{p\,dV}{\tau}$$

for ideal gas energy doesnot depend the volume

$$\left(\frac{\partial U}{\partial V}\right)_\tau = 0$$

$$\Rightarrow \quad d\sigma = C_V \frac{d\tau}{\tau} + \frac{P}{\tau} dV = C_V \frac{d\tau}{\tau} + N \frac{dV}{V}$$

$$\sigma = \int C_V \frac{d\tau}{\tau} + \int N \frac{dV}{V}$$

$$\sigma = C_V \log \tau + N \log V + \text{const}$$

$$\boxed{\sigma = C_V \log \tau + N \log V + \sigma_1}$$