PHY 410

Homework 9

problem 1

(a) 1 dimension : each orbital filled with one electron up to energy

$$\varepsilon_F = \frac{\hbar^2}{2m}\left(\frac{\pi n_F}{L}\right)^2$$

For the system to hold N electrons, the orbitals must be filled up to n_F such as:

$$N = 2 \times n_F \quad \text{(each electron has 2 possible spin orientation)}$$

$$\rightarrow n_F = \frac{N}{2} \rightarrow \varepsilon_F = \frac{\hbar^2}{2m}\left(\frac{\pi N}{2L}\right)^2 \rightarrow N = \frac{2L}{\pi}\left(\frac{2m\varepsilon_F}{\hbar^2}\right)^{1/2}$$

$$\rightarrow N(\varepsilon) = \frac{2L}{\pi\hbar}(2m\varepsilon)^{1/2}$$

$$D_1(\varepsilon) = \frac{dN}{d\varepsilon} = \frac{L}{\pi\hbar}(2m\varepsilon)^{-1/2} \cdot 2m$$

$$D_1(\varepsilon) = \frac{L}{\pi}\left(\frac{2m}{\hbar^2\varepsilon}\right)^{1/2}$$

(b) 2-dimension

$$\varepsilon_F = \frac{\hbar^2}{2m}\left(\frac{\pi n_F}{L}\right)^2$$

$$N(\varepsilon_F) = 2 \times \frac{1}{4}\pi n_F^2 = \frac{1}{2}\pi n_F^2 \rightarrow n_F = \left(\frac{2N}{\pi}\right)^{1/2}$$

$$\rightarrow \varepsilon_F = \frac{\hbar^2\pi^2}{2mL^2} \cdot \frac{2N}{\pi} = \frac{\hbar^2\pi}{mA} \cdot N(\varepsilon_F)$$

$$\rightarrow N(\varepsilon) = \frac{mA}{\pi\hbar^2}\varepsilon \rightarrow D_2(\varepsilon) = \frac{dN}{d\varepsilon} = \frac{mA}{\pi\hbar^2}$$

<u>Problem 2</u>

$$\varepsilon_F = pc = \left(\frac{\pi\hbar}{L} n_F\right) c = \frac{\pi\hbar c}{L} \cdot n_F \qquad \left(n_F = \sqrt{n_x^2 + n_y^2 + n_z^2}\right)$$

$$N = 2 \times \frac{1}{8} \times \frac{4\pi}{3} n_F^3 = \frac{\pi}{3} n_F^3 \Rightarrow n_F = \left(\frac{3N}{\pi}\right)^{1/3}$$

$$\Rightarrow \varepsilon_F = \frac{\pi\hbar c}{L}\left(\frac{3N}{\pi}\right)^{1/3} = \pi\hbar c \left(\frac{3N}{\pi L^3}\right)^{1/3} = \pi\hbar c \left(\frac{3}{\pi} \cdot \frac{N}{V}\right)^{1/3}$$

$$\boxed{\varepsilon_F = \pi\hbar c \left(\frac{3n}{\pi}\right)^{1/3}}$$

$$U_0 = 2 \sum_{n \le n_F} \varepsilon_n = 2 \times \frac{1}{8} \times 4\pi \int_0^{n_F} dn\, n^2 \varepsilon_n$$

$$U_0 = \pi \int_0^{n_F} dn\, n^2 \frac{\pi\hbar c}{L} n = \frac{\pi^2 \hbar c}{L} \int_0^{n_F} dn\, n^3$$

$$U_0 = \frac{\pi^2 \hbar c}{4L} n_F^4 = \frac{\pi^2 \hbar c}{4L} \left(\frac{3N}{\pi}\right)^{4/3}$$

$$U_0 = \frac{3}{4} N \frac{\pi\hbar c}{L} \left(\frac{3N}{\pi}\right)^{1/3} = \frac{3}{4} N \pi\hbar c \left(\frac{3N}{\pi L^3}\right)^{1/3}$$

$$U_0 = \frac{3}{4} N \pi\hbar c \left(\frac{3}{\pi} n\right)^{1/3} = \frac{3}{4} N \varepsilon_F$$

problem 3

(a)

$$U_0 = \frac{3}{5} N \varepsilon_F = \frac{3}{5} N \frac{\hbar^2}{2m} \left(\frac{3\pi N}{V} \right)^{2/3}$$

$$U_0 = \frac{3 N \hbar^2}{10 m} \left(\frac{3\pi^2 N}{V} \right)^{2/3}$$

$$P = -\left(\frac{\partial U}{\partial V} \right)_\sigma = -\frac{3N\hbar^2}{10m} \cdot \frac{2}{3} \left(\frac{3\pi^2 N}{V} \right)^{-1/3} \left(-\frac{3\pi^2 N}{V^2} \right)$$

$$P = \frac{(3\pi^2)^{2/3}}{5} \frac{\hbar^2}{m} \left(\frac{N}{V} \right)^{5/3}$$

(b)

$$C_V = \tau \left(\frac{\partial \sigma}{\partial \tau} \right)_V = \frac{1}{2} \pi^2 N k_B T / T_F$$

$$\Rightarrow \quad \frac{\partial \sigma}{\partial \tau} = \frac{\pi^2 N}{2 T_F}$$

$$\Rightarrow \quad \sigma = \frac{\pi^2 N}{2 T_F} \tau = \frac{\pi^2 N k_B}{2} \frac{\tau}{\varepsilon_F}$$

problem 4

(a)

$$n = \frac{N}{V} = \frac{\frac{0.081}{0.5\times10^{-22}}}{1} = 162\times10^{20}\ \text{atom}/\text{cm}^3 = 162\times10^{26}\ \frac{\text{atom}}{\text{m}^3}$$

$$\mathcal{E}_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3} = \frac{(1.054\times10^{-34})^2}{2\times9.1\times10^{-31}}(3\pi^2\times1.62\times10^{26})^{2/3}$$

$$\mathcal{E}_F = 3.74\times10^{-19}\,J = 2.34\ eV$$

$$\frac{1}{2}m v_F^2 = \mathcal{E}_F \Rightarrow v_F = \sqrt{\frac{2\mathcal{E}_F}{m}} = \sqrt{\frac{2\times3.74\times10^{-19}}{9.1\times10^{-31}}} = 9.1\times10^5\,m/s$$

$$T_F = \mathcal{E}_F/k_B = \frac{3.74\times10^{-19}}{1.38\times10^{-23}} = 2.7\times10^4\ K$$

(b)

$$C_{el} = \frac{1}{3}\pi^2\,\frac{3N}{2\mathcal{E}_F}\,\tau = \frac{1}{2}\pi^2\,\frac{Nk_B\tau}{\mathcal{E}_F} = \frac{1}{2}\pi^2\,\frac{Nk_BT}{\mathcal{E}_F}$$

$$C_{el} = \frac{1\times(3.14)^2}{2.\ 2.34}\,Nk_BT = 2.11\,Nk_BT$$

problems

2-Dimension : $\mathcal{D}_2(\varepsilon) = \dfrac{Am}{\pi\hbar^2}$

$$N = \int_0^\infty d\varepsilon\, \mathcal{D}_2(\varepsilon)\, f(\varepsilon,\tau,\mu)$$

$$N = \frac{Am}{\pi\hbar^2}\int_0^\infty \frac{d\varepsilon}{e^{\frac{\varepsilon-\mu}{\tau}}+1}$$

$$x = e^{\frac{\varepsilon-\mu}{\tau}} \rightarrow dx = e^{\frac{\varepsilon-\mu}{\tau}}\cdot\frac{1}{\tau}\,d\varepsilon = \frac{x}{\tau}\,d\varepsilon$$

$$N = \frac{Am}{\pi\hbar^2}\int_{e^{-\mu/\tau}}^\infty \frac{\tau\,dx}{x}\cdot\frac{1}{x+1}$$

$$N = \frac{Am}{\pi\hbar^2}\tau\int_{e^{-\mu/\tau}}^\infty \frac{dx}{x(x+1)}$$

$$N = \frac{Am\tau}{\pi\hbar^2}\int_{e^{-\mu/\tau}}^\infty dx\left(\frac{1}{x}-\frac{1}{x+1}\right)$$

$$N = \frac{Am\tau}{\pi\hbar^2}\left(\ln x - \ln(x+1)\right)\Big|_{e^{-\mu/\tau}}^\infty$$

$$N = \frac{Am\tau}{\pi\hbar^2}\ln\frac{e^{-\mu/\tau}+1}{e^{-\mu/\tau}} = \frac{Am\tau}{\pi\hbar^2}\ln\left(1+e^{\mu/\tau}\right)$$

$$\Rightarrow\ 1+e^{\mu/\tau} = e^{N\pi\hbar^2/Am\tau} \rightarrow \mu = \tau\ln\left[e^{N\pi\hbar^2/Am\tau}-1\right]$$

$$\mu = \tau\ln\left[e^{\varepsilon_F/\tau}-1\right]$$

+ $\underline{\tau \ll \varepsilon_F}$

$$\mu = \tau \ln\left[e^{\varepsilon_F/\tau} - 1\right] \simeq \tau \ln\left(e^{\varepsilon_F/\tau}\right) = \varepsilon_F$$

+ $\underline{\tau \gg \varepsilon_F}$

$$\mu = \tau \ln\left(1 + \frac{\varepsilon_F}{\tau} - 1\right) = \tau \ln\left(\frac{\varepsilon_F}{\tau}\right)$$