PHY 410 – Spring 2010 Exam #1

PLEASE WAIT UNTIL YOU ARE TOLD TO BEGIN THE EXAM

While waiting, carefully fill in the information requested below

Your Name:……………………………………………………

Your Student Number:………………………………………

There are 4 problems. Please answer them all. Total time is 1 Hour

USEFUL CONSTANTS AND EQUATIONS

Stirling's formula: ln N! ~ N lnN – N when N >> 1

$$Thermal\ \ wavelength\ \lambda_{th} = \sqrt{\frac{2\pi\hbar^2}{M\tau}}$$

$$Quantum\ \ concentration\ n_Q = \left(\frac{M\tau}{2\pi\hbar^2}\right)^{3/2} = \frac{1}{\lambda_{th}^3}$$

$$Planck's\ const.\ \hbar = 1.05459x10^{-34}\ Js$$

$$Boltzmann\ const.\ k_B = 1.38066x10^{-23}\ J/K$$

$$amu = 1.66057x10^{-27}\ kg$$

Problem 1 (10 points)

Consider a system containing 6 spins. Each spin can point either up or down. Magnetic moment associated with each spin is m. There is no external magnetic field. (4 points)

(i) What is the total number of microstates for this system?

$$2^6 = 64$$

(ii) If all the microstates are equally accessible then what is the probability of finding the system in any one of these microstates?

$$\frac{1}{64}$$

(iii) What is the probability of finding the system in the macrostate $N_\uparrow = 4$?

$$g(N, N_\uparrow)\frac{1}{2^N} = g(6,4)\frac{1}{2^6} = \frac{6!}{4!2!}\frac{1}{64} = \frac{15}{64}$$

Now apply an external magnetic field *B*. Energy of each spin is either –*mB* (when ↑) or +*mB* (when ↓). The system is in equilibrium with a reservoir at temperature τ. (6 points)

(i) What is the probability of finding the system with total energy $U = 0$?

$$U = 0 \textit{ implies } N_\uparrow = N_\downarrow = 3$$

$$g(6,3) = \frac{6!}{3!3!} = 20$$

$$\Pr ob = g(6,3)\frac{e^{-0/\tau}}{Z} = \frac{20}{Z}$$

$$Z = (Z_1)^6 = \left(e^{mB/\tau} + e^{-mB/\tau}\right)^6 = \left(e^{6mB/\tau} + 6e^{4mB/\tau} + 15e^{2mB/\tau} + 20 + 15e^{-2mB/\tau} + 6e^{-4mB/\tau} + e^{-6mB/\tau}\right)$$

(ii) What is the probability of finding the system with total energy $U = 6mB$?

$$\Pr ob = g(6,0)\frac{e^{-6mB/\tau}}{Z}$$, where *Z* is the same as in (i) above

Problem 2 (15 points)

Thermal properties of an impurity atom adsorbed on the surface of a solid can be described by a simple model consisting of finite number of microstates with different energies. A simple such model consists of 4 microstates, one with energy 0 and the other three, each with energy $\Delta > 0$. The impurity atom and the solid are in equilibrium at temperature τ.

(i) What is the partition function for this impurity atom? (3 points)

$$Z = 1 + 3e^{-\Delta/\tau}$$

(ii) What is its average energy $U(\tau)$? Plot qualitatively U as a function of τ/Δ. (6 points)

$$U = \tau^2 \frac{\partial}{\partial \tau}(\ln Z) = \tau^2 \frac{\frac{\partial Z}{\partial \tau}}{Z} = \frac{3\Delta e^{-\Delta/\tau}}{1 + 3e^{-\Delta/\tau}} = \frac{3\Delta}{e^{\Delta/\tau} + 3}$$

(iii) What is the heat capacity?(3 points)

$$C = \frac{\partial U}{\partial \tau} = 3\left(\frac{\Delta}{\tau}\right)^2 \frac{e^{\Delta/\tau}}{\left(e^{\Delta/\tau} + 3\right)^2} = 3\left(\frac{\Delta}{\tau}\right)^2 \frac{e^{-\Delta/\tau}}{\left(1 + 3e^{-\Delta/\tau}\right)^2}$$

(iv) Find the temperature τ_0 when the average energy $U(\tau_0)$ is equal to $\Delta/2$.(3 points)

$$\frac{3\Delta e^{-\Delta/\tau}}{1 + 3e^{-\Delta/\tau}} = \frac{\Delta}{2}; e^{-\Delta/\tau} = 1/3; \tau = \frac{\Delta}{\ln 3} = 0.91\Delta$$

Problem 3 (13 points)

An ideal gas of N indistinguishable He atoms is contained inside a cubic box of volume V at $T = 100K$ and density $n = 10^{24} / m^3$.

(i) What is the thermal wave length $\lambda_{th,He}$ associated with these atoms? Can we treat this system classically or not, why? (5 points)

$$\lambda_{th} = \sqrt{\frac{2\pi\hbar^2}{M\tau}} = \sqrt{\frac{2\pi(1.05)^2 x10^{-68}}{4x1.66x10^{-27} x1.38x10^{-23} x10^2}} = 0.87x10^{-10} m$$

$$n_Q = \frac{1}{\lambda_{th}^3} = 1.52x10^{30} / m^3. \ Since\ n << n_Q,\ the\ system\ can\ be\ treated\ classically$$

(ii)What is the partition function of this gas Z_{He} if the partition function of 1 He atom is $Z_{1,He} = \left(\frac{V}{\lambda_{th,Her}^3}\right)$?(3 points) (Express your answer in terms of N,V, and $\lambda_{th,He}$.

Drop the He suffix for simplicity

$$Z_N = \frac{(Z_1)^N}{N!}$$

(iii) What is the Helmholtz free energy/atom and the average energy/ atom? (5 points) (Show your work)

$$F/N = -\tau \ln Z_N / N = -\tau[N \ln Z_1 - lmN!]/N \approx -\tau\left[\ln\left(\frac{Z_1}{N}\right) + 1\right] = -\tau\left[\ln\left(\frac{V}{N\lambda_{th}^3} + 1\right)\right]$$

$$U/N = \tau^2\left[\frac{\partial}{\partial\tau}\ln Z_N\right]/N = \tau^2 \frac{\partial}{\partial\tau}\left[\ln \tau^{3/2} + Cons\tan t\, independent\ of\ \tau\right] = \frac{3}{2}\tau$$

Problem 4 (12 points)

Consider a system consisting of 3 quantum harmonic oscillators, all with the same frequency ω. Microstates of each oscillator are characterized by an integer s = 0,1,2,… with energy $\varepsilon_s = s\hbar\omega$.

(i) If the total energy of this system $U = 2\hbar\omega$, how many microstates are associated with this energy? Enumerate these microstates using a suitable notation (4points)
Use notation $(N; s_1, s_2, s_3)$

$$(3;0,0,2);(3;0,2,0);(3;2,0,0);(3;1,1,0);(3;1,0,1);(3;0,1,1)\ \textit{Total}\ \ 6\ \textit{microstates}$$

(ii) Bring a second system consisting of 4 magnets in an external field B, each magnet can point either parallel or antiparallel to B, with total magnetic moment exactly equal to 0. Don't let these systems exchange energy. What is the total number of microstates associated with the combined oscillator-magnet system?(3 points)

Total magnetic moment is exactly equal to zero $N_{\uparrow} = N_{\downarrow} = 2$

Multiplicity of magnetic system is $\frac{4!}{2!2!} = 6$

Multiplicity of the non-exchanging magnet-QHO system is 6x6=36

(iii) Now consider a different physical situation where a system of N_1 QHOs with average energy/atom, $U_1/N_1 = 4\hbar\omega$ is brought in thermal contact with a system of N_2 noninteracting magnets in the presence of an external field B, with average energy/magnet $U_2/N_2 = 0$. In which direction will the energy flow, from oscillators to magnets or vice-versa? (Hint: compare the temperatures of the two systems) Give reasons for your answer. (5 points)

$$QHO: \frac{U_1}{N_1} = \frac{\hbar\omega}{e^{\hbar\omega/\tau_1} - 1} = 4\hbar\omega \quad ; \quad MAGNET: \frac{U_2}{N_2} = -mB\tanh\frac{mB}{\tau_2} = 0$$

This gives $\tau_2 \equiv \tau_{magnet} = \infty$ *BUT* $\tau_1 \equiv \tau_{QHO}$ *is finite*

Magnets are hotter than the QHOs. Therefore the energy will flow from the magnets to the QHP's