PHY 410 – Spring 2010
Exam #2
(1 Hour)

PLEASE WAIT UNTIL YOU ARE TOLD TO BEGIN THE EXAM

While waiting, carefully fill in the information requested below

Your Name:…………………………………………………………….

Your Student Number:…………………………………………………

There are 4 problems. Please answer them all showing your work clearly (for partial credit).

USEFUL CONSTANTS AND INTEGRALS

$$\textit{Thermal wavelength } \lambda_{th} = \sqrt{\frac{2\pi\hbar^2}{M\tau}}$$

$$\textit{Quantum concentration } n_Q = \left(\frac{M\tau}{2\pi\hbar^2}\right)^{3/2} = \frac{1}{\lambda_{th}^3}$$

Boltzmann constant $k_B = 1.38066 x 10^{-23} JK^{-1}$

Planck's constant $\hbar = 1.05459 x 10^{-34} Js$

Energy of a photon in mode $\vec{n}$ inside a cubic box of volume $L^3 : \hbar\omega_{\vec{n}} = \hbar n\pi c / L$

$$\int_0^\infty \frac{x^3}{e^x - 1} dx = \frac{\pi^4}{15}$$

Problem 1 (10 points)

A solid surface has 2 binding sites to each of which one Ar atom can be bound with energy ε. Ar atoms bound to different sites do not interact with each other. The solid surface is in contact with a bath of Ar atoms at temperature τ and chemical potential μ. Let $\lambda = e^{\mu/\tau}$ be the absolute activity of the Ar atoms.

(1) What is the probability of finding 2 Ar atoms bound to the surface?

(2) What is the probability that the surface has no bound Ar atoms?

(3) What is the probability that the surface has 1 bound Ar atom?

(4) What is the average number of Ar atoms bound to the surface?

$$\tilde{Z} = \sum_{N=0}^{2} \lambda^N \sum_{s(N)} e^{-\varepsilon_{s(N)}/\tau}$$

$$= 1 + 2\lambda e^{-\varepsilon/\tau} + \lambda^2 e^{-2\varepsilon/\tau}$$

(1) $P(N=2) = \dfrac{\lambda^2 e^{-2\varepsilon/\tau}}{\tilde{Z}}$

(2) $P(N=0) = \dfrac{1}{\tilde{Z}}$

(3) $P(N=1) = \dfrac{2\lambda e^{-\varepsilon/\tau}}{\tilde{Z}}$

(4) $< N >= \sum_{N=0}^{2} NP(N) = \dfrac{2\lambda e^{-\varepsilon/\tau} + 2\lambda^2 e^{-2\varepsilon/\tau}}{\tilde{Z}} = \dfrac{2\lambda e^{-\varepsilon/\tau} + 2\lambda^2 e^{-2\varepsilon/\tau}}{1 + 2\lambda e^{-\varepsilon/\tau} + \lambda^2 e^{-2\varepsilon/\tau}}$

Problem 2 (15 points)
The energy density of black body radiation confined in a box of volume V at temperature τ is given by

$$u = \frac{U}{V} = \frac{\pi^2}{15c^3\hbar^3}\tau^4 .$$

(i) What is its heat capacity per unit volume?(3 points)
(ii) Starting from the expression relating heat capacity and entropy σ (use thermodynamic identity) calculate σ . (6 points)
(iii) If the black body radiation undergoes an adiabatic expansion by a factor of 8 then by what factor does its temperature change? (3 points)
(iv) If the temperature of the black body increases by a factor of 2 then by what factor its entropy/volume changes? (3 points)

$$(i)\ U = V\frac{\pi^2}{15c^3\hbar^3}\tau^4$$

$$C_V = \left(\frac{\partial U}{\partial \tau}\right)_V = V\frac{4\pi^2}{15c^3\hbar^3}\tau^3$$

$$(ii)\ C_V = \left(\frac{\partial U}{\partial \tau}\right)_V = \tau\left(\frac{\partial \sigma}{\partial \tau}\right)_V$$

$$\int_0^\sigma d\sigma = \int \frac{C_V(\tau')}{\tau'}d\tau'$$

$$\sigma = V\frac{4\pi^2}{45c^3\hbar^3}\tau^3;\ Used\ \sigma(\tau=0)=0$$

$$(iii)\ Adiabatic\ \mathrm{Pr}ocess\ \sigma = const$$

$$V\tau^3 = const$$

$$If\ V\ increases\ by\ 2^3\ then\ \tau\ shoud\ decrease\ by\ 2$$

$$(iv)\ If\ \tau\ increases\ by\ 2\ then$$

$$\sigma / V\ increases\ by\ 2^3 = 8$$

Problem 3 (10 points)

(i) A classical ideal gas of *N* atoms of mass *M* is confined inside a cubic box of volume $V=L^3$ at temperature τ. What is the chemical potential of the gas?(4 points)

(ii) Each atom has a charge *Q*. A potential *V* is applied to the top plate of the box and the bottom plate is kept at zero potential. Derive an expression for the density of the gas as a function of the distance from the bottom plate, $n(z)$, where $z=0(L)$ for the bottom (top) plate.

<u>Treat the charged gas as ideal and assume that the electric field inside is uniform.</u> (6 points)

$$(i)\ \mu = \tau \ln\left(\frac{n}{n_Q}\right);\ n = N/V \ or < N > /V,\ n_Q = \frac{1}{\lambda_{th}^3}$$

(ii) In the presence of external potential energy (due to charge and potentials

$$\mu_{tot} = \mu_{\text{int}} + \mu_{ext} = \tau \ln\left(\frac{n(z)}{n_Q}\right) + QEz = \tau \ln\left(\frac{n(z)}{n_Q}\right) + \frac{QVz}{L}$$

Since in diffusive equilibrium $\mu_{tot} = const = C$

$$\tau \ln\left(\frac{n(z)}{n_Q}\right) + \frac{QVz}{L} = C$$

$$n(z) = n_Q e^{\frac{c - QVz/L}{\tau}} = n(0) e^{-\frac{QVz}{L\tau}}$$

Problem 4 (15 points)

(i) Write down the Fermi-Dirac and Bose-Einstein distribution functions for the occupation of a single orbital of energy ε in terms of temperature τ and chemical potential μ (Don't derive it). Plot these as a function of ε.
(2 points)

(ii) What is the distribution function in the classical regime? When is the classical regime applicable? (3 points)

(iii) Using the above classical limit of the distribution function find μ in terms of the average number of particles <*N*>, volume *V*, and the quantum concentration n_Q for a 3-dimensional ideal classical gas. Use this equation to find the Helmholtz's free energy.
(8 points)

(iv) What is the value of μ when the concentration $\frac{<N>}{V} = n_Q$?(2 points)

(i)

$$f_{FD}(\varepsilon) = <N(\varepsilon)> = \frac{1}{e^{(\varepsilon-\mu)/\tau}+1}$$

$$f_{BE}(\varepsilon) = <N(\varepsilon)> = \frac{1}{e^{(\varepsilon-\mu)/\tau}-1}$$

(ii)

$$f_{classical}(\varepsilon) = e^{-(\varepsilon-\mu)/\tau};\ when\ e^{(\varepsilon-\mu)/\tau} >> 1;\ (n << n_Q)$$

(iii)

$$<N> = \sum_{\varepsilon} <N(\varepsilon)> = e^{\mu/\tau}\sum_{\varepsilon} e^{-\varepsilon/\tau} = e^{\mu/\tau} Z_1 = e^{\mu/\tau}\frac{V}{\lambda_{th}^3}$$

$$\mu = \tau \ln\left(\frac{<N>\lambda_{th}^3}{V}\right) \equiv \tau \ln\left(\frac{N\lambda_{th}^3}{V}\right) = \tau \ln\left(\frac{n}{n_Q}\right)$$

Since chemical potentil μ *and Helmholtz Free energy* $F(N,\tau,V)$ *are related by* $\mu = \left(\frac{\partial F}{\partial N}\right)_{\tau,V}$

$$F(N,\tau,V) = \int_0^N \mu(N',\tau,V)dN' = N\tau\left[\ln\frac{n}{n_Q} - 1\right]$$

(iv) *When* $n = n_Q,\ \mu = \tau \ln(1) = 0$