PHY 410

Final Examination, Spring 2009

May 4, 2009 (5:45-7:45 p.m.)

PLEASE WAIT UNTIL YOU ARE TOLD TO BEGIN THE EXAM.

While waiting, carefully fill in the information requested below

Your Name:____

Your Student Number:____

DO NOT TURN THIS PAGE UNTIL THE EXAM STARTS

USEFUL CONSTANTS AND INTEGRALS

Avogadro's Number $N_A = 6.022 \times 10^{23}$

Boltzmann's constant $k = 1.381 \times 10^{-23}$ J/K

$= 8.617 \times 10^{-5}$ eV/K

Planck constant $\hbar = 1.054 \times 10^{-34}$ J.s

Electron charge (magnitude) $e = 1.602 \times 10^{-19}$ C

Electron mass $m = 9.109 \times 10^{-31}$ kg

Speed of light $= 2.998 \times 10^{8}$ m/s

1 atm = 1.013 bar

1 bar $= 10^{5}$ N/m^2

1 eV $= 1.602 \times 10^{-19}$ J

There are 7 problems. To receive full credit for each answer, *you must work neatly, show your work and simplify your answer to the extent possible.*

Problem 1 (5 points)

A power plant produces 1 GW of electricity at an efficiency of 40%.

(i) What is the rate at which the power plant expels waste heat to the environment?

$$\eta = \frac{W}{Q_h}; W = \eta Q_h = \eta(W + Q_l) \Rightarrow W = \frac{\eta}{1-\eta} Q_l; Q_l = \frac{1-\eta}{\eta} W$$

$$Given\ \frac{dW}{dt} = 1GW\ \ we\ have\ \frac{dQ_l}{dt} = \frac{1-\eta}{\eta}\frac{dW}{dt} = \frac{1-0.4}{0.4} 1GW = 1.5\,GW$$

(ii) What is the rate at which it uses up energy at the hot end?

$$\frac{dQ_h}{dt} = \frac{1}{\eta}\frac{dW}{dt} = \frac{1}{0.4} 1GW = 2.5\,GW$$

$$Alternatively\ \frac{dQ_h}{dt} = \frac{dQ_l}{dt} + \frac{dW}{dt} = 1.5\,GW + 1GW = 2.5GW$$

Problem 2 (7.5 points)

Consider a fuel cell that uses methane (gas) as fuel. The reaction is

$$CH_4 + 2O_2 \rightarrow 2H_2O + CO_2$$

The difference in Gibb's free energy (final – initial) is -800 kJ per mole of CH_4.

(i) Assuming ideal performance, how much electrical work can be produced by the cell for each mole of CH_4.

$$800\ kJ$$

(ii)The two steps of this reaction are

At –ve electrode $CH_4 + 2H_2O \rightarrow CO_2 + 8H^+ + 8e^-$ and At +ve electrode $2O_2 + 8H^+ + 8e^- \rightarrow 4H_2O$

What is the maximum voltage generated by the cell?

$$(8e)N_A V_0 = 800\,kJ$$

$$eV_0 = \frac{800x10^3 J}{8N_A} = \frac{800x10^3 J}{8x6x10^{23}} \frac{1eV}{1.6x10^{-19} J} = 1.04\,eV$$

$$Therefore \max voltage\ V_0 = 1.04\,V$$

Problem 3 (10 points)

An ideal gas of N bosons of mass M is in a cubical box of volume V. The bosons are nonrelativistic. The density of orbitals (or states) is given by

$$D_3(\varepsilon) = \frac{V}{4\pi^2}\left(\frac{2M}{\hbar^2}\right)^{3/2} \varepsilon^{1/2} \ ; \varepsilon \geq 0.$$

(i)Write down an expression for the number of particles in terms of $D_3(\varepsilon)$ and the Bose-Einstein distribution function $f_{BE}(\varepsilon)$.

$$N = \int_0^\infty D_3(\varepsilon)\frac{1}{e^{(\varepsilon-\mu)/\tau}-1}d\varepsilon$$

(ii) Derive an expression for the BE condensation temperature τ_E in terms M and the density N/V .(Use $\int_0^\infty \frac{x^{1/2}}{e^x-1}dx = 1.306\sqrt{\pi}$)

$$N = \int_0^\infty D_3(\varepsilon)\frac{1}{e^{\varepsilon/\tau_E}-1}d\varepsilon = \frac{V}{4\pi^2}\left(\frac{2M}{\hbar^2}\right)^{3/2}\int_0^\infty \frac{\varepsilon^{1/2}}{e^{\varepsilon/\tau_E}-1}d\varepsilon$$

$$Put\, x = \frac{\varepsilon}{\tau_E} : N = \frac{V}{4\pi^2}\left(\frac{2M}{\hbar^2}\right)^{3/2}\tau_E^{3/2}\int_0^\infty \frac{x^{1/2}}{e^x-1}dx$$

$$\tau_E = \frac{\hbar^2}{2M}\left(\frac{4\pi^{3/2}}{1.306}\frac{N}{V}\right)^{2/3}$$

(iii) How does τ_E change when the mass of the boson doubles?

It reduces by a factor of 2

Problem 4 (7.5 points)

The measured constant volume heat capacity of a metal at low temperature ($\tau << \tau_F = \varepsilon_F$ *and* $\tau << k_B \theta_{Debye}$) is given by

$C_V = A\tau + B\tau^3$, where *A* and *B* are constants.

(i)What are the physical origins of the above two different contributions to C_V?

The linear terms comes from the Fermi Dirac statistics of electrons in a metal.
The cubic terms comes from the phonons (lattice vibrations) of the atoms

(ii) What is the temperature dependence of entropy of the metal?

$$C_V = \frac{dQ}{d\tau} = \tau \frac{d\sigma}{d\tau}$$

$$\int d\sigma = \int \frac{C_V(\tau')}{\tau'} d\tau'$$

$$\sigma(\tau) - \sigma(0) = \int_0^\tau \frac{C_V(\tau')}{\tau'} d\tau' = A\int_0^\tau d\tau' + B\int_0^\tau (\tau')^2 d\tau' = A\tau + \frac{B\tau^3}{3}$$

Usually

$$\sigma(0) = 0$$

Problem 5 (10 points)

Consider a defect in a solid which has two energy levels with energy 0 and $\varepsilon > 0$. The degeneracy of the ground level is 2 and that of the excited level is 1. The solid is at a temperature τ.

(i) What is the partition function for this defect?

(ii) What is its average energy $U(\tau)$? Plot U as a function of τ.

(iii) What is $U(\tau)$ at $\tau = \infty$?

(iv) Find the temperature τ_0 when the average energy $U(\tau_0)$ is half the average energy at $\tau = \infty$, $U(\infty)$.

$$z = 2 + e^{-\varepsilon/\tau}$$

$$U(\tau) = \frac{\varepsilon\, e^{-\varepsilon/\tau}}{2 + e^{-\varepsilon/\tau}}$$

$$U(\infty) = \frac{\varepsilon}{3}$$

$$U(\tau_0) = \frac{U(\infty)}{2} = \frac{\varepsilon}{6} = \frac{\varepsilon\, e^{-\varepsilon/\tau_0}}{2 + e^{-\varepsilon/\tau_0}}$$

$$\text{Solve for } \frac{\varepsilon}{\tau_0} \text{ and then } \tau_0 = -\frac{\varepsilon}{\ln \frac{2}{5}}$$

Problem 6 (15 points)

The energy per unit volume of a black body radiation and the radiation pressure p are given by

$$u = \frac{U}{V} = A\tau^4;\ where\ A\ is\ a\,const\,ant, \quad p = \frac{u}{3}$$

A gas of photon is used as the working medium of a heat engine. The gas undergoes an isothermal expansion at temperature τ_h from volume V_1 to $V_2 = 2V_1$.

(i) What is the work done by the photon gas, W_{12}? If instead of a photon gas we use an ideal gas of *N* Ar atoms, what is the work done in this case?

$$PHOTONS:\ W_{12} = \int_1^2 p\,dV = \frac{1}{3}A\tau^4 \int_{V_1}^{V_2} dV = \frac{A\tau_h^4}{3}(V_2 - V_1) = \frac{A\tau_h^4 V_1}{3}$$

$$ARGON\ ATOMS:\ W_{12} = \int_1^2 p\,dV = N\tau_h \int_{V_1}^{V_2} \frac{dV}{V} = N\tau_h \ln\left(\frac{V_2}{V_1}\right) = N\tau_h \ln(2)$$

(ii) Now the photon gas undergoes an adiabatic expansion from (τ_h, V_2) to (τ_l, V_3) ($where\ \tau_l = \tau_h / 2, V_3 = 2V_2$), and the work done by the photon gas is W_{23}. What is W_{23}/W_{12}?

$$Start\ \ from\ the\,1st\ law\ of\ thermo\,dynamics\ (Energy\ onservation)$$

$$dU = dQ + dW\ \ where\ dW\ is\ the\ work\ done\ ON\ the\ gas$$

$$Work\ done\ BY\ the\ gas = -dW = dQ - dU$$

$$For\ an\ adiabatic\ process\ dQ = 0$$

$$Work\ done\ BY\ the\ gas = -dW = -dU$$

$$W_{23} = \int_2^3 -dU = U(2) - U(3) = AV_2\tau_h^4 - AV_3\tau_l^4$$

$$Use\ result\ \ for\ W_{12}\ from\ (i)\ above$$

$$W_{23}/W_{12} = \frac{AV_2\tau_h^4 - AV_3\tau_l^4}{\left(A\tau_h^4 V_1\right)/3} = \frac{3V_2}{V_1}\left(1 - \left(\tau_l/\tau_h\right)^4 \frac{V_3}{V_2}\right)$$

$$NOW\ \ Plug\ in\ the\ different\ ratios.$$

Problem 7 (15 points)

Consider an ideal 2-dimensional gas of non-relativisitic electrons (spin ½) of density *n=N/A* at <u>zero temperature</u>. The density of states for this system is given by $D_2(\varepsilon) = \frac{A}{2\pi}\left(\frac{2m}{\hbar^2}\right); \varepsilon \geq 0.$

(i) Derive an equation relating the Fermi energy ε_F as a function of n, mass m, and other fundamental constant(s).

$$N = \int_0^\infty D_2(\varepsilon)\frac{1}{e^{(\varepsilon-\mu)/\tau}+1}d\varepsilon = \int_0^{\varepsilon_F} D_2(\varepsilon)d\varepsilon = \frac{A}{2\pi}\left(\frac{2m}{\hbar^2}\right)\varepsilon_F$$

$$\varepsilon_F = \frac{\hbar^2}{2m}\left(\frac{2\pi N}{A}\right)$$

(ii) Calculate the average energy (energy/particle) in terms of the Fermi energy ε_F .

$$U = \int_0^{\varepsilon_F} \varepsilon D_2(\varepsilon)d\varepsilon = \frac{A}{2\pi}\left(\frac{2m}{\hbar^2}\right)\frac{\varepsilon_F^2}{2}$$

$$\frac{U}{N} = \frac{\varepsilon_F}{2}$$

(iii) What is the Fermi energy ε_F (in electron volts) if $n = 10^{18}\, m^{-2}$

$$\varepsilon_F = 2\pi\, x 10^{18}\, \tfrac{1}{m^2}\, x \frac{\left(1.054x10^{-34}\, J.s\right)^2}{2x9.1x10^{-31}\, kg}\, x \frac{1eV}{1.6x10^{-19}\, J}$$

$$= 0.24\, eV$$