HW#10

10.1

(a)

From (7.5),

$$\varepsilon_F = \frac{\hbar^2}{2m}(\frac{\pi n_F}{L})^2 \Rightarrow n_F = \frac{L}{\pi\hbar}(2m\varepsilon_F)^{1/2}$$

For one-dimension, we have

$$N = 2\times n_F = 2n_F = \frac{2L}{\pi\hbar}(2m\varepsilon_F)^{1/2}$$

Thus

$$D_1(\varepsilon) = \frac{dN}{d\varepsilon} = \frac{2L}{\pi\hbar}\times\frac{1}{2}(2m\varepsilon_F)^{-1/2}\times 2m = \frac{L}{\pi\hbar}(2m/\varepsilon_F)^{1/2}$$

(b)

From (7.5),

$$\varepsilon_F = \frac{\hbar^2}{2m}(\frac{\pi n_F}{L})^2 \Rightarrow n_F = \frac{L}{\pi\hbar}(2m\varepsilon_F)^{1/2}$$

For two-dimension, we have

$$N = 2\times\frac{1}{4}\times\pi {n_F}^2 = \frac{1}{2}\pi {n_F}^2 = \frac{1}{2}\pi\times\frac{L^2}{\pi^2\hbar^2}2m\varepsilon_F = \frac{mL^2}{\pi\hbar^2}\varepsilon_F$$

Thus

$$D_2(\varepsilon) = \frac{dN}{d\varepsilon} = \frac{mL^2}{\pi\hbar^2} = \frac{Am}{\pi\hbar^2}$$

10.2

(a)

For particle in a cube of volume $V = L^3$, we have $L = \frac{n}{2}\lambda,$ the momentum is

$$p = h/\lambda = n\hbar\pi/L$$

For relativistic ideal gas, we have

$$\varepsilon_n \cong pc = n\hbar\pi c/L$$

From (7.6), we have

$$N = \frac{\pi {n_F}^3}{3}, n_F = (3N/\pi)^{1/3}$$

Thus

$$\varepsilon_F \cong p_F c = n_F\hbar\pi c/L = \hbar\pi c(3N/\pi)^{1/3}/L = \hbar\pi c(3N/V\pi)^{1/3} = \hbar\pi c(3n/\pi)^{1/3}$$

(b)

The total energy of ground state of the gas is

$$U_0 = \sum_{n \le n_F} \varepsilon_n = 2 \times \frac{1}{8} \times 4\pi \int_0^{n_F} dn n^2 \varepsilon_n = \pi \int_0^{n_F} dn n^2 \times \frac{n\hbar\pi c}{L} = \frac{\hbar\pi^2 c}{L} \int_0^{n_F} dn n^3$$

$$= \frac{\hbar\pi^2 c}{L} \frac{{n_F}^4}{4} = \frac{3}{4} \frac{\pi {n_F}^3}{3} \frac{n_F \hbar\pi c}{L} = \frac{3}{4} N\varepsilon_{n_F}$$

10.3

(a)

For a Fermi electron gas in the ground state, the pressure is

$$p = -\left(\frac{\partial U}{\partial V}\right)_\sigma = -\left(\frac{\partial U_0}{\partial V}\right)_N$$

From (7.10) and (7.7),

$$U_0 = \frac{3}{5} N\varepsilon_F , \varepsilon_F = \frac{\hbar^2}{2m} (3\pi^2 n)^{2/3}$$

Thus

$$U_0 = \frac{3}{10} \frac{N\hbar^2}{m} (3\pi^2 n)^{2/3} = \frac{3}{10} \frac{N\hbar^2}{m} (3\pi^2 \frac{N}{V})^{2/3}$$

$$\Rightarrow p = -\left(\frac{\partial U_0}{\partial V}\right)_N = -\frac{3}{10} \frac{N\hbar^2}{m} (3\pi^2 N)^{2/3} \times -\frac{2}{3} \times V^{-5/3} = \frac{(3\pi^2)^{2/3}}{5} \frac{\hbar^2}{m} (\frac{N}{V})^{5/3}$$

(b)

For $\tau \ll \varepsilon_F$, from (7.37)

$$C_{el} = \frac{1}{2} \pi^2 N\tau / \tau_F = \frac{1}{2} \pi^2 N\tau / \varepsilon_F$$

Also, we have

$$C = \tau \left(\frac{\partial \sigma}{\partial \tau}\right)_V$$

$$\Rightarrow \left(\frac{\partial \sigma}{\partial \tau}\right)_V = \frac{C}{\tau} = \frac{1}{2} \pi^2 N / \varepsilon_F = \text{constant}$$

Let $\sigma = A\tau + B$

For $\left(\frac{\partial \sigma}{\partial \tau}\right)_V = \frac{1}{2} \pi^2 N / \varepsilon_F , A = \frac{1}{2} \pi^2 N / \varepsilon_F$

For $\sigma \to 0$ as $\tau \to 0$, $B = 0$

Thus

$$\sigma = \frac{1}{2} \pi^2 N\tau / \varepsilon_F$$

10.4

(a)

$$m(^3He) = 3\times 1.66\times 10^{-27}kg = 4.98\times 10^{-27}kg$$

$$\rho(^3He) = 0.081gcm^{-3} = 81kgm^{-3}$$

$$n(^3He) = \frac{N}{V} = \frac{M/m}{M/\rho} = \frac{\rho}{m} = 1.627\times 10^{28}m^{-3}$$

From (7.40), we have

$$\varepsilon_F = (\hbar^2/2m)(3\pi^2 n)^{2/3} = 6.86\times 10^{-23}J = 4.29\times 10^{-4}eV$$

From (7.41), we can get

$$\frac{1}{2}mv_F{}^2 = \varepsilon_F \Rightarrow v_F = \sqrt{2\varepsilon_F/m} = 166ms^{-1} = 1.66\times 10^4 cms^{-1}$$

Also,

$$T_F = \varepsilon_F / k_B = 4.97K$$

(b)

For $T \ll T_F$, from (7.38), we can get

$$C_{el} = \frac{1}{2}\pi^2 Nk_B T/T_F = 0.993K^{-1}Nk_B T$$

10.5

From question 10.1, we have

$$D_2(\varepsilon) = \frac{mL^2}{\pi\hbar^2} = \frac{Am}{\pi\hbar^2}$$

From (10.25), we have

$$N = \int_0^\infty d\varepsilon D(\varepsilon) f(\varepsilon,\tau,\mu) = \frac{Am}{\pi\hbar^2}\int_0^\infty \frac{d\varepsilon}{\exp[(\varepsilon-\mu)/\tau]+1}$$

Let $t = \exp[(\varepsilon-\mu)/\tau]$, we have

$$dt = \exp[(\varepsilon-\mu)/\tau]\times d\varepsilon/\tau$$

$$\Rightarrow d\varepsilon = \frac{\tau}{\exp[(\varepsilon-\mu)/\tau]}dt = \frac{\tau}{t}dt$$

$$N = \frac{Am}{\pi\hbar^2}\int_{e^{-\mu/\tau}}^\infty \frac{\frac{\tau}{t}dt}{t+1} = \frac{Am\tau}{\pi\hbar^2}\int_{e^{-\mu/\tau}}^\infty \frac{dt}{t(t+1)} = \frac{Am\tau}{\pi\hbar^2}\int_{e^{-\mu/\tau}}^\infty (\frac{1}{t}-\frac{1}{t+1})dt = \frac{Am\tau}{\pi\hbar^2}[\ln(t)-\ln(t+1)]|_{e^{-\mu/\tau}}^\infty$$

$$= \frac{Am\tau}{\pi\hbar^2}\ln[t/(t+1)]|_{e^{-\mu/\tau}}^\infty = \frac{Am\tau}{\pi\hbar^2}\ln(1+e^{\mu/\tau})$$

Thus

$$\mu = \tau \ln[\exp(\frac{N}{A}\frac{\pi\hbar^2}{m\tau}) - 1]$$

From (7.5),

$$\varepsilon_F = \frac{\hbar^2}{2m}(\frac{\pi n_F}{L})^2$$

For two-dimension, we have

$$N = 2 \times \frac{1}{4} \times \pi n_F{}^2 = \frac{1}{2}\pi n_F{}^2 = \frac{1}{2}\pi \times \frac{L^2}{\pi^2\hbar^2} 2m\varepsilon_F = \frac{mA}{\pi\hbar^2}\varepsilon_F$$

Thus

$$\varepsilon_F = \frac{N}{A}\frac{\pi\hbar^2}{m} \Rightarrow \mu = \tau \ln[\exp(\varepsilon_F / \tau) - 1]$$

For $\tau \ll \varepsilon_F$, we have

$$\mu = \tau \ln[\exp(\varepsilon_F / \tau) - 1] \cong \tau \times \varepsilon_F / \tau = \varepsilon_F$$

For $\tau \gg \varepsilon_F$, we have

$$\mu = \tau \ln[\exp(\varepsilon_F / \tau) - 1] \cong \tau \ln(1 + \varepsilon_F / \tau - 1) = \tau \ln(\varepsilon_F / \tau)$$