HW#11

11.1

(i)

The Hydrogen atom consists of one electron and one proton. They both have 1/2 spin. Thus, the spin of Hydrogen atom is 1/2+1/2=1. We can treat it as a boson.

(ii)

Let the average distance to be d, every atom take a average volume of a sphere with radius $r = d/2$, thus

$$N \times \frac{4}{3}\pi r^3 = V \Rightarrow \frac{N}{V} \times \frac{4}{3}\pi r^3 = 1$$

For $\frac{N}{V} = 1.8 \times 10^{14} atoms/cm^3$, we can obtain

$$r = 1.1 \times 10^{-5} cm \Rightarrow d = 2r = 2.2 \times 10^{-5} cm$$

(iii)

We have

$$\frac{N}{V} = 1.8 \times 10^{14} / cm^3 = 1.8 \times 10^{20} / m^3$$

$$M = 1.673 \times 10^{-27} Kg$$

From (7.72), we can obtain

$$\tau_E = \frac{2\pi\hbar^2}{M}\left(\frac{N}{2.612V}\right)^{2/3}$$

$$\Rightarrow T_E = \frac{2\pi\hbar^2}{k_B M}\left(\frac{N}{2.612V}\right)^{2/3} = 5.09 \times 10^{-5} K = 50.9 \mu K$$

The result is very close to the measured value of $50\mu K$.

11.2

We have

$$M = 6.697 \times 10^{-27} Kg$$

$$\frac{N}{V} = \frac{M_{all}/M}{M_{all}/\rho} = \frac{\rho}{M} = 2.165 \times 10^{28} / m^3$$

From (7.72), we can obtain

$$\tau_E = \frac{2\pi\hbar^2}{M}\left(\frac{N}{2.612V}\right)^{2/3}$$

$$\Rightarrow T_E = \frac{2\pi\hbar^2}{k_B M}\left(\frac{N}{2.612V}\right)^{2/3} = 3.1K$$

The result is greater than experimental value of $2.17K$.

11.3

From (7.65), we have

$$D(\varepsilon)=\frac{V}{4\pi^2}(\frac{2M}{\hbar^2})^{3/2}\varepsilon^{1/2}$$

Thus, we can get the energy

$$U=\int_0^\infty \varepsilon D(\varepsilon)f(\varepsilon)d\varepsilon=\int_0^\infty \varepsilon\frac{V}{4\pi^2}(\frac{2M}{\hbar^2})^{3/2}\varepsilon^{1/2}\frac{1}{\mathrm{e}^{(\varepsilon-\mu)/\tau}-1}d\varepsilon$$

$$=\frac{V}{4\pi^2}(\frac{2M}{\hbar^2})^{3/2}\int_0^\infty \varepsilon^{3/2}\frac{1}{\lambda^{-1}\mathrm{e}^{\varepsilon/\tau}-1}d\varepsilon$$

With $x=\varepsilon/\tau,\lambda=1$, we have

$$U=\frac{V}{4\pi^2}(\frac{2M}{\hbar^2})^{3/2}\tau^{5/2}\int_0^\infty\frac{x^{3/2}}{\mathrm{e}^x-1}dx$$

For the integral, we have

$$\int_0^\infty\frac{x^{3/2}}{\mathrm{e}^x-1}dx=\Gamma(5/2)\zeta(5/2)\approx\frac{3}{4}\sqrt{\pi}\times1.341\approx1.783$$

Thus

$$U=1.783\frac{V}{4\pi^2}(\frac{2M}{\hbar^2})^{3/2}\tau^{5/2}$$

For $d\sigma=\left(\frac{dU}{\tau}\right)_V$, we can obtain

$$\sigma=\int\frac{dU}{\tau}+\text{constant}$$

$$=1.783\frac{V}{4\pi^2}(\frac{2M}{\hbar^2})^{3/2}\int_0^\tau\frac{5}{2}\tau'^{3/2}\frac{d\tau'}{\tau'}+\text{constant}$$

$$=1.783\frac{5}{2}\frac{V}{4\pi^2}(\frac{2M}{\hbar^2})^{3/2}\int_0^\tau\tau'^{1/2}d\tau'+\text{constant}$$

$$=1.783\frac{5}{2}\frac{V}{4\pi^2}(\frac{2M}{\hbar^2})^{3/2}\frac{2}{3}\tau'^{3/2}|_0^\tau+\text{constant}$$

$$=1.783\frac{5}{2}\frac{V}{4\pi^2}(\frac{2M}{\hbar^2})^{3/2}\frac{2}{3}\tau'^{3/2}|_0^\tau+\text{constant}$$

$$=1.783\frac{5}{3}\frac{V}{4\pi^2}(\frac{2M}{\hbar^2})^{3/2}\tau^{3/2}+\text{constant}$$

For $\sigma\to0$ as $\tau\to0$, thus

$$\sigma=1.783\frac{5}{3}\frac{V}{4\pi^2}(\frac{2M}{\hbar^2})^{3/2}\tau^{3/2}$$

11.4

For Bose-Einstein distribution function (7.53), we have

$$f(\varepsilon,\tau)=\frac{1}{\exp[(\varepsilon-\mu)/\tau]-1}$$

Thus,

$$f(0)=\frac{1}{\exp(-\mu/\tau)-1}\Rightarrow \exp(-\mu/\tau)=1+\frac{1}{f(0)}$$

$$f(\varepsilon)=\frac{1}{\exp[(\varepsilon-\mu)/\tau]-1}\Rightarrow \exp[(\varepsilon-\mu)/\tau]=1+\frac{1}{f(\varepsilon)}$$

Also, we have

$$\frac{f(0)}{f(\varepsilon)}=2\Rightarrow f(0)=\frac{2}{3}N,\ f(\varepsilon)=\frac{1}{3}N$$

For $N\gg 1$, we can make some approximation

$$\exp(-\mu/\tau)=1+3/2N, \exp[(\varepsilon-\mu)/\tau]=1+3/N$$

$$\Rightarrow \exp(\varepsilon/\tau)=\frac{\exp[(\varepsilon-\mu)/\tau]}{\exp(-\mu/\tau)}=\frac{1+3/N}{1+3/2N}=1+\frac{3/2N}{1+3/2N}\approx 1+3/2N$$

$$\Rightarrow \varepsilon/\tau=\ln(1+3/2N)\approx 3/2N$$

$$\Rightarrow \tau=2N\varepsilon/3$$

11.5

(a)

For reversible heat pump, from (8.4), we have

$$\sigma_l=\sigma_h\Rightarrow Q_l/\tau_l=Q_h/\tau_h$$

$$\Rightarrow Q_l=\frac{\tau_l}{\tau_h}Q_h$$

Thus

$$W=Q_h-Q_l=Q_h-\frac{\tau_l}{\tau_h}Q_h=\frac{\tau_h-\tau_l}{\tau_h}Q_h=\eta_c Q_h$$

$$\Rightarrow \frac{W}{Q_h}=\eta_c=\frac{\tau_h-\tau_l}{\tau_h}$$

For irreversible heat pump, we have

$$\sigma_l<\sigma_h\Rightarrow Q_l/\tau_l<Q_h/\tau_h$$

$$\Rightarrow Q_l<\frac{\tau_l}{\tau_h}Q_h$$

Thus

$$W = Q_h - Q_l > Q_h - \frac{\tau_l}{\tau_h} Q_h = \frac{\tau_h - \tau_l}{\tau_h} Q_h = \eta_c Q_h$$

$$\Rightarrow \frac{W}{Q_h} > \eta_c = \frac{\tau_h - \tau_l}{\tau_h}$$

More work is required for irreversible heat pump.

(b)

We have

$$\frac{W}{Q_h} = \frac{\tau_h - \tau_l}{\tau_h}, \frac{W}{Q_{hh}} = \frac{\tau_{hh} - \tau_l}{\tau_{hh}}$$

Thus

$$\frac{Q_{hh}}{Q_h} = \frac{W / Q_h}{W / Q_{hh}} = \frac{\tau_h - \tau_l}{\tau_h} \times \frac{\tau_{hh}}{\tau_{hh} - \tau_l} = \frac{300 - 270}{300} \times \frac{600}{600 - 270} = \frac{2}{11}$$

(c)