HW#3

3.1

(a)

From these formulas

$$\left(\frac{\partial \sigma}{\partial U}\right)_N = \frac{1}{\tau}, \sigma = \log g(N,U), g(U) = CU^{3N/2}$$

We can get

$$\left(\frac{\partial \sigma}{\partial U}\right)_N = \frac{1}{g(N,U)}\left(\frac{\partial g}{\partial U}\right)_N = \frac{1}{CU^{3N/2}} \times C \times \frac{3}{2} NU^{3/2N-1} = \frac{3}{2}\frac{N}{U} = \frac{1}{\tau}$$

$$\Rightarrow U = \frac{3}{2} N\tau$$

(b)

From part (a), we have

$$\left(\frac{\partial \sigma}{\partial U}\right)_N = \frac{3}{2}\frac{N}{U}$$

Thus

$$\left(\frac{\partial^2 \sigma}{\partial U^2}\right)_N = \frac{3}{2} N \left(\frac{\partial \frac{1}{U}}{\partial U}\right)_N = -\frac{3}{2}\frac{N}{U^2}$$

It is negative.

3.2

(a)

We have

$$Cv \times m \times \tau_1 + Cv \times m \times \tau_2 = Cv \times 2m \times \tau_F$$

$$\Rightarrow \tau_F = \frac{\tau_1 + \tau_2}{2}$$

$$\Rightarrow T_F = \frac{T_1 + T_2}{2}$$

(b)

$$\Delta S_1 = \int \frac{dU}{T} = \int_{T_1}^{T_F} \frac{mC_v dT}{T} = mC_v \ln T \Big|_{T_1}^{T_F} = mC_v \ln(\frac{T_F}{T_1}) = mC_v \ln(\frac{\tau_F}{\tau_1})$$

$$\Delta S_2 = \int \frac{dU}{T} = \int_{T_2}^{T_F} \frac{mC_v dT}{T} = mC_v \ln T \Big|_{T_2}^{T_F} = mC_v \ln(\frac{T_F}{T_2}) = mC_v \ln(\frac{\tau_F}{\tau_2})$$

$$\Delta S = \Delta S_1 + \Delta S_2$$

$$\Rightarrow k_B \ln g = mC_v \ln(\frac{\tau_F}{\tau_1}) + mC_v \ln(\frac{\tau_F}{\tau_2}) = mC_v \ln\left(\frac{\tau_F{}^2}{\tau_1\tau_2}\right)$$

$$\Rightarrow g = \left(\frac{\tau_F{}^2}{\tau_1\tau_2}\right)^{\frac{mC_v}{k_B}}$$

3.3

We have

$$\sigma(s) \simeq \log g(N,0) - 2s^2 / N$$

For spin excess 2s, the energy

$$U = -2mBs$$

$$\Rightarrow s = -\frac{U}{2mB}$$

Thus

$$\sigma(U) \simeq \sigma_0 - U^2 / 2m^2B^2N, \sigma_0 = \log g(N,0)$$

$$\left(\frac{\partial\sigma}{\partial U}\right)_N = -\frac{U}{m^2B^2N} = \frac{1}{\tau}$$

$$\Rightarrow U = \langle U \rangle = -\frac{m^2B^2N}{\tau} = -MB$$

$$\Rightarrow M = \frac{m^2BN}{\tau}$$

$$\Rightarrow \frac{M}{Nm} = \frac{mB}{\tau}$$

3.4

(a)

From (1.55)

$$g(N,n) = \frac{(N+n-1)!}{n!(N-1)!}$$

We get get

$$\sigma = \log g(N,n) = \log\frac{(N+n-1)!}{n!(N-1)!} = \log(N+n-1)! - \log n! - \log(N-1)!$$

$$= \log(N+n)! - \log n! - \log N! + \log(\frac{N}{N+n})$$

$$\approx \log(N+n)! - \log n! - \log N!$$

$$= (N+n)\log(N+n) - (N+n) - (n\log n - n) - (N\log N - N)$$

$$= (N+n)\log(N+n) - n\log n - N\log N$$

$$= (N+n)\log(N+n) - (N+n)\log N - (n\log n - n\log N)$$

$$= (N+n)\log(1+n/N) - n\log(n/N)$$

(b)

We have $U = n\hbar\omega$, thus

$$\left(\frac{\partial\sigma}{\partial U}\right)_N = \frac{1}{\hbar\omega}\left(\frac{\partial\sigma}{\partial n}\right)_N$$

$$= \frac{1}{\hbar\omega}[(N+n)\times\frac{1}{N+n} + \log(N+n) - n\times\frac{1}{n} - \log n]$$

$$= \frac{1}{\hbar\omega}\log\frac{N+n}{n} = \frac{1}{\tau}$$

$$\Rightarrow \frac{N}{n} + 1 = \exp(\hbar\omega/\tau)$$

$$=> n = \frac{N}{\exp(\hbar\omega/\tau) - 1} = \frac{U}{\hbar\omega}$$

$$=> U = \frac{N\hbar\omega}{\exp(\hbar\omega/\tau) - 1}$$

3.5

(a)

From (2.17)

$$g_1(N_1,\hat{s}_1+\delta)g_2(N_2,\hat{s}_2-\delta) = (g_1g_2)_{\max}\exp(-\frac{2\delta^2}{N_1} - \frac{2\delta^2}{N_2})$$

Let $N_1 = N_2 = 10^{22}, \delta = 10^{11}$, we have

$$g_1g_2 = (g_1g_2)_{\max}\exp(-\frac{2\delta^2}{N_1} - \frac{2\delta^2}{N_2})$$

$$\Rightarrow g_1g_2/(g_1g_2)_{\max} = \exp(-\frac{2\delta^2}{N_1} - \frac{2\delta^2}{N_2}) = \exp(-4) \approx 0.0183$$

(b)

When s is a huge number, we can use (1.35)

$$g(N,s) \simeq g(N,0)\exp(-2s^2/N), g(N,0) \simeq (2/\pi N)^{1/2}2^N$$

From (2.14), for maximum value, we need

$$\frac{\hat{s}_1}{N_1} = \frac{\hat{s}_2}{N_2} = \frac{s}{N}$$

$$\Rightarrow \hat{s}_1 = \hat{s}_2 = \frac{s}{2}$$

Thus

$$(g_1 g_2)_{\max} = g(N_1, \hat{s}_1)^2 \simeq (2/\pi N_1) 2^{2N_1} \exp(-4\hat{s}_1^{\,2}/N_1)$$

Also, we know that

$$\sum_{s_1} g_1(N_1, s_1) g_2(N_2, s - s_1) = g(2N_1, 2\hat{s}_1) = (1/\pi N_1)^{1/2} 2^{2N_1} \exp(-4\hat{s}_1^{\,2}/N_1)$$

We can get the factor

$$\frac{\sum_{s_1} g_1(N_1, s_1) g_2(N_2, s - s_1)}{(g_1 g_2)_{\max}} = (\pi N_1)^{1/2}/2 = 8.86 \times 10^{10}$$

(c)

The real entropy is

$$\sigma_{real} = \log g(2N_1, 2\hat{s}_1) = -\frac{1}{2}\log(\pi N_1) + 2N_1 \log 2 - 4\hat{s}_1^{\,2}/N_1 \approx 1.3862 \times 10^{22}$$

If we ignore the factor, we have

$$\sigma_{ignore} \approx \log(g_1 g_2)_{\max} = \log 2 - \log(\pi N_1) + 2N_1 \log 2 - 4\hat{s}_1^{\,2}/N_1 \approx 1.3862 \times 10^{22}$$

It is almost the same. We can get the error by

$$\sigma_{real} - \sigma_{ignore} = \log \frac{g(2N_1, 2\hat{s}_1)}{(g_1 g_2)_{\max}} = \log(8.86 \times 10^{10}) \approx 25.2$$

Very small difference compared to $\sigma_{real}, \sigma_{ignore}$.