HW#4

4.1

(a)

For a two states system, we have

$$Z = \exp(-0/\tau) + \exp(-\varepsilon/\tau) = 1 + \exp(-\varepsilon/\tau)$$

Thus, the free energy

$$F = -\tau \log Z = -\tau \log[1 + \exp(-\varepsilon/\tau)]$$

(b)

From (3.49)

$$\left(\frac{\partial F}{\partial \tau}\right)_v = -\sigma$$

$$\Rightarrow \sigma = -\left(\frac{\partial F}{\partial \tau}\right)_v$$

$$= \log[1 + \exp(-\varepsilon/\tau)] + \tau \times \frac{1}{1 + \exp(-\varepsilon/\tau)} \times \exp(-\varepsilon/\tau) \times (\varepsilon/\tau^2)$$

$$= \log[1 + \exp(-\varepsilon/\tau)] + \frac{\varepsilon}{\tau}\frac{\exp(-\varepsilon/\tau)}{1 + \exp(-\varepsilon/\tau)}$$

$$= \log[1 + \exp(-\varepsilon/\tau)] + \frac{\varepsilon}{\tau}\frac{1}{\exp(\varepsilon/\tau) + 1}$$

From (3.35)

$$F = U - \tau\sigma$$

$$\Rightarrow U = F + \tau\sigma = \varepsilon\frac{\exp(-\varepsilon/\tau)}{1 + \exp(-\varepsilon/\tau)} = \frac{\varepsilon}{\exp(\varepsilon/\tau) + 1}$$

4.2

(a)

For one magnet, the partition function is

$$Z = \exp(mB/\tau) + \exp(-mB/\tau) = 2\cosh(mB/\tau)$$

$$< m >= \frac{m\exp(mB/\tau) - m\exp(-mB/\tau)}{Z} = m\tanh(mB/\tau)$$

Thus

$$M = n < m >= nm\tanh(mB/\tau)$$

$$\chi = \frac{dM}{dB} = nm \times \frac{m}{\tau} \times \sec^2(mB/\tau) = \frac{nm^2}{\tau}\sec^2(mB/\tau)$$

(b)

$$F = -n\tau \log Z = -n\tau \log[2\cosh(mB/\tau)]$$

For

$$x = M / nm = \tanh(mB/\tau)$$

$$x^2 = \tanh^2(mB/\tau) = \frac{\sinh^2(mB/\tau)}{\cosh^2(mB/\tau)} = 1 - \frac{1}{\cosh^2(mB/\tau)}$$

$$\Rightarrow 1 - x^2 = \cosh^{-2}(mB/\tau)$$

$$F = -n\tau \log[2\cosh(mB/\tau)] = -n\tau\{\log 2 - \frac{1}{2}\log[\cosh^{-2}(mB/\tau)]\} = -n\tau[\log 2 - \frac{1}{2}\log(1-x^2)]$$

(c)

From (a), we have

$$\chi = \frac{dM}{dB} = \frac{nm^2}{\tau}\sec^2(mB/\tau)$$

In the limit of $mB \ll \tau$, we have

$$\sec(mB/\tau) \approx 1$$

$$\Rightarrow \chi \approx \frac{nm^2}{\tau}$$

4.3

(a)

The partition function is

$$Z = \sum_{s=0}^{\infty} \exp(-\varepsilon_s/\tau) = \sum_{s=0}^{\infty} \exp(-s\hbar\omega/\tau) = \frac{1}{1-\exp(-\hbar\omega/\tau)}$$

The free energy is

$$F = -\tau \log Z = \tau \log[1-\exp(-\hbar\omega/\tau)]$$

At high temperature $\tau \gg \hbar\omega$, we have

$$\exp(-\hbar\omega/\tau) \approx 1 - \hbar\omega/\tau$$

$$\Rightarrow F = \tau \log[1-\exp(-\hbar\omega/\tau)] \approx \tau \log(\hbar\omega/\tau)$$

(b)

From (3.49)

$$\left(\frac{\partial F}{\partial \tau}\right)_v = -\sigma \Rightarrow \sigma = -\left(\frac{\partial F}{\partial \tau}\right)_v$$

$$= -\tau \times \frac{1}{1-\exp(-\hbar\omega/\tau)} \times [-\exp(-\hbar\omega/\tau)] \times (\hbar\omega/\tau^2) - \log[1-\exp(-\hbar\omega/\tau)]$$

$$= \frac{\hbar\omega}{\tau}\frac{\exp(-\hbar\omega/\tau)}{1-\exp(-\hbar\omega/\tau)} - \log[1-\exp(-\hbar\omega/\tau)]$$

$$= \frac{\hbar\omega/\tau}{\exp(\hbar\omega/\tau)-1} - \log[1-\exp(-\hbar\omega/\tau)]$$

3.5

There are 2^3=8 microstates.

Microstate	Energy(mB)
↑ ↑ ↑	3
↑ ↑ ↓	1
↑ ↓ ↑	1
↑ ↓ ↓	-1
↓ ↑ ↑	1
↓ ↑ ↓	-1
↓ ↓ ↑	-1
↓ ↓ ↓	-3

$$Z(3,\tau) = \exp(3mB/\tau) + 3\exp(mB/\tau) + 3\exp(-mB/\tau) + \exp(-3mB/\tau)$$

$$[Z(1,\tau)]^3 = [2\cosh(mB/\tau)]^3 = [\exp(mB/\tau) + \exp(-mB/\tau)]^3$$
$$= \exp(3mB/\tau) + 3\exp(2mB/\tau)\exp(-mB/\tau) + 3\exp(mB/\tau)\exp(-2mB/\tau) + \exp(-3mB/\tau)$$
$$= \exp(3mB/\tau) + 3\exp(mB/\tau) + 3\exp(-mB/\tau) + \exp(-3mB/\tau)$$

Thus

$$Z(3,\tau) = [Z(1,\tau)]^3$$

We can get this result in another way. All the microstates come from

(↑ + ↓) (↑ + ↓) (↑ + ↓)

The partition function includes all the microstates, thus

$$Z(3,\tau)$$
$$= [\exp(mB/\tau) + \exp(-mB/\tau)][\exp(mB/\tau) + \exp(-mB/\tau)][\exp(mB/\tau) + \exp(-mB/\tau)]$$
$$= [Z(1,\tau)]^3$$

Also,

$$Z(n,\tau)$$
$$= [\exp(mB/\tau) + \exp(-mB/\tau)][\exp(mB/\tau) + \exp(-mB/\tau)]...[\exp(mB/\tau) + \exp(-mB/\tau)]$$
$$= [Z(1,\tau)]^n$$