HW#5

5.1

For

$$\lambda_{th} = \sqrt{\frac{2\pi\hbar^2}{mK_BT}}, n_Q = \lambda_{th}^{-3}$$

We can get

(i)

$$m = 28\times1.66\times10^{-27}kg = 4.65\times10^{-26}kg$$

$$\lambda_{th} = \sqrt{\frac{2\pi\hbar^2}{mK_BT}} \approx 1.90\times10^{-11}m$$

$$n_Q = \lambda_{th}^{-3} = 1.45\times10^{32}m^{-3}$$

$$n = 10^{19} / cc = 10^{25}m^{-3}$$

$$n / n_Q = 6.91\times10^{-8} \ll 1$$

It can be treated as classical system.

(ii)

$$m = 9.1\times10^{-31}kg$$

$$\lambda_{th} = \sqrt{\frac{2\pi\hbar^2}{mK_BT}} \approx 4.30\times10^{-9}m$$

$$n_Q = \lambda_{th}^{-3} = 1.25\times10^{25}m^{-3}$$

$$n = 10^{22} / cc = 10^{28}m^{-3}$$

$$n / n_Q = 797.5 \gg 1$$

It cannot be treated as classical system.

(iii)

$$m = 4\times1.66\times10^{-27}kg = 6.64\times10^{-27}kg$$

$$\lambda_{th} = \sqrt{\frac{2\pi\hbar^2}{mK_BT}} = 8.73\times10^{-10}m$$

$$n_Q = \lambda_{th}^{-3} = 1.50\times10^{27}m^{-3}$$

$$n = 10^{22} / cc = 10^{28}m^{-3}$$

$$n / n_Q = 6.65 \gg 1$$

It cannot be treated as classical system.

5.2

From (3.34a)

$$\tau d\sigma = dU + pdV$$

We can get

(i)

$$\tau d\sigma = dU + pdV = pdV = \frac{N\tau}{V} dV$$

$$\Delta\sigma = \int_V^{V/2} \frac{N}{V} dV = N \ln V \Big|_V^{V/2} = -N \ln 2$$

(ii)

$$\tau d\sigma = dU + pdV = dU = \frac{3}{2} N d\tau$$

$$\Delta\sigma = \int_\tau^{2\tau} \frac{3}{2} \frac{N}{\tau} d\tau = \frac{3}{2} N \ln \tau \Big|_\tau^{2\tau} = \frac{3}{2} N \ln 2$$

Also, we can use the formula for ideal gas (3.76)

$$\sigma = N[\ln(n_Q / n) + \frac{5}{2}]$$

$$n_Q = (m\tau / 2\pi\hbar^2)^{3/2}$$

(i)

$$V \to V/2 \Rightarrow n \to 2n$$

$$\Delta\sigma = \sigma_2 - \sigma_1 = -N \ln 2$$

(ii)

$$\tau \to 2\tau \Rightarrow n_Q \to 2^{3/2} n_Q$$

$$\Delta\sigma = \sigma_2 - \sigma_1 = N \ln 2^{3/2} = \frac{3}{2} N \ln 2$$

5.3

(a)

The partition function is

$$Z_R(\tau) = \sum_{s=0}^{\infty} \exp(-\varepsilon_s / \tau) = \sum_{j=0}^{\infty} g(j) \exp(-\varepsilon_j / \tau) = \sum_{j=0}^{\infty} (2j+1) \exp[-j(j+1)\varepsilon_0 / \tau]$$

(b)

Let $f(j) = (2j+1)\exp[-j(j+1)\varepsilon_0 / \tau]$, we can get the approximated value by

$$Z = \sum_{j=0}^{\infty} f(j) \approx \int_{-1/2}^{\infty} f(j) dj$$

We need to choose the lower integration limit -1/2 to get more accurate result.

Let $t = j(j+1)\varepsilon_0 / \tau, dt = dj(2j+1)\varepsilon_0 / \tau$, thus

$$\int f(j)dj = \int (2j+1)\exp[-j(j+1)\varepsilon_0/\tau]dj$$
$$= \frac{\tau}{\varepsilon_0}\int \exp(-t)dt$$
$$= -\frac{\tau}{\varepsilon_0}\exp(-t)$$
$$= -\frac{\tau}{\varepsilon_0}\exp[-j(j+1)\varepsilon_0/\tau]$$

Thus,

$$Z = \sum_{j=0}^{\infty} f(j) \approx \int_{-1/2}^{\infty} f(j)dj = -\frac{\tau}{\varepsilon_0}\exp[-j(j+1)\varepsilon_0/\tau]\Big|_{-1/2}^{\infty}$$
$$= \frac{\tau}{\varepsilon_0}\exp(\frac{\varepsilon_0}{4\tau})$$

For $\tau \gg \varepsilon_0, \frac{\varepsilon_0}{4\tau} \ll 1$, we have

$$Z = \frac{\tau}{\varepsilon_0}\exp(\frac{\varepsilon_0}{4\tau}) \approx \frac{\tau}{\varepsilon_0}(1+\frac{\varepsilon_0}{4\tau}) = \frac{\tau}{\varepsilon_0} + \frac{1}{4}$$

(c)

For $\tau \ll \varepsilon_0, \frac{\varepsilon_0}{4\tau} \gg 1$, we cannot use the approximation. By truncating the sum after the second term, we can get

$$Z = \sum_{j=0}^{\infty}(2j+1)\exp[-j(j+1)\varepsilon_0/\tau]$$
$$\approx 1 + 3\exp(-2\varepsilon_0/\tau)$$

(d)

For

$$U(\tau) = \frac{\tau^2}{Z}\left(\frac{\partial Z}{\partial \tau}\right)$$
$$C(\tau) = \frac{\partial U}{\partial \tau}$$

We can get

For $\tau \gg \varepsilon_0$,

$$U(\tau) = \frac{\tau^2}{Z}\left(\frac{\partial Z}{\partial \tau}\right) \approx \frac{\tau^2}{\frac{\tau}{\varepsilon_0}+\frac{1}{4}} \times \frac{1}{\varepsilon_0} = \frac{\tau}{1+\frac{\varepsilon_0}{4\tau}} \approx \tau(1-\frac{\varepsilon_0}{4\tau}) = \tau - \frac{\varepsilon_0}{4}$$
$$C(\tau) = \frac{\partial U}{\partial \tau} \approx 1$$

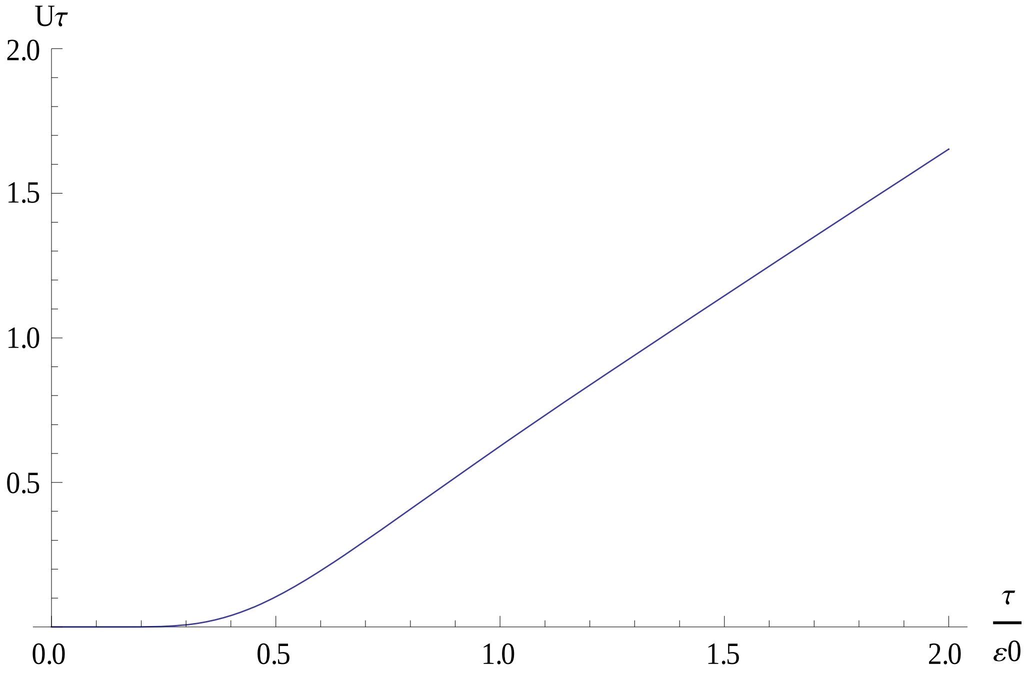
For $\tau \ll \varepsilon_0$,

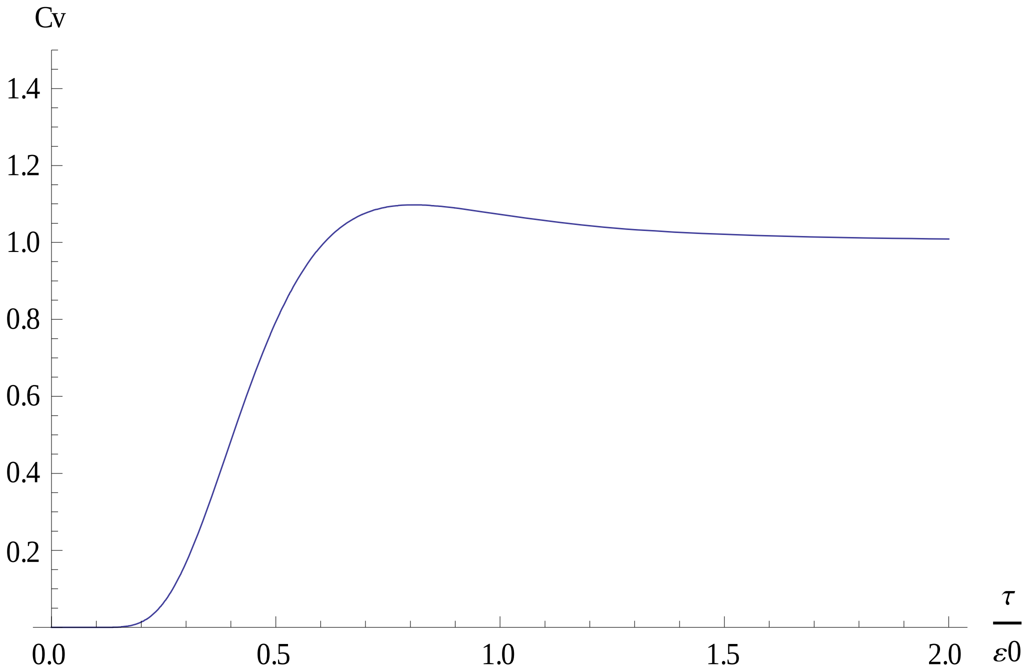
$$U(\tau) = \frac{\tau^2}{Z}\left(\frac{\partial Z}{\partial \tau}\right) \approx \frac{\tau^2}{1+3\exp(-2\varepsilon_0/\tau)} \times 3\exp(-2\varepsilon_0/\tau) \times 2\frac{\varepsilon_0}{\tau^2} \approx 6\varepsilon_0 \exp(-2\varepsilon_0/\tau)$$

$$C(\tau) = \frac{\partial U}{\partial \tau} \approx 6\varepsilon_0 \exp(-2\varepsilon_0/\tau) \times 2\frac{\varepsilon_0}{\tau^2} = 12\frac{\varepsilon_0{}^2}{\tau^2}\exp(-2\varepsilon_0/\tau)$$

(e)

The figure is drawing by Mathmatica.





5.4

From (3.59), we have

$$\varepsilon_n = \frac{\hbar^2}{2M}(\frac{\pi}{L})^2(n_x{}^2 + n_y{}^2 + n_z{}^2)$$

For ground orbital, we have $n_x = n_y = n_z = 1$, thus

$$\varepsilon_{ground} = \frac{3}{2}\frac{\hbar^2}{M}(\frac{\pi}{L})^2$$

Thus,

$$\tau = \frac{3}{2}\frac{\hbar^2}{M}(\frac{\pi}{L})^2$$
$$\Rightarrow L = (\frac{3}{2}\frac{\hbar^2\pi^2}{M\tau})^{1/2}$$
$$\Rightarrow n_0 = 1/L^3 = (\frac{4}{3\pi})^{3/2}(\frac{M\tau}{2\pi\hbar^2})^{3/2} = (\frac{4}{3\pi})^{3/2} n_Q \approx 0.2765 n_Q$$

5.5

For one-dimension, the orbital energies are

$$\varepsilon_n = \frac{\hbar^2}{2M}(\frac{\pi}{L})^2 n^2 = \varepsilon_1 n^2$$

The partition function of one particle is

$$Z_1 = \sum_{n=1}^{\infty}\exp(-\varepsilon_n/\tau) = \sum_{n=1}^{\infty}\exp(-\varepsilon_1 n^2/\tau)$$
$$\approx \int_0^{\infty}\exp(\varepsilon_1 n^2/\tau)dn = (\frac{\pi\tau}{4\varepsilon_1})^{1/2} = (\frac{M\tau}{2\pi\hbar^2})^{1/2} L = n_{Q1}L$$
$$\Rightarrow n_{Q1} = (\frac{M\tau}{2\pi\hbar^2})^{1/2}$$

For N particles, we have

$$Z_N = \frac{Z_1{}^N}{N!}$$
$$\Rightarrow F = -\tau\ln Z_N = -\tau[N\ln Z_1 - \ln(N!)]$$
$$\approx -\tau[N\ln(n_{Q1}L) - N\ln N - N]$$
$$= -N\tau[\ln(n_{Q1}L) - \ln(nL) - 1]$$
$$= -N\tau[\ln(n_{Q1}/n) + 1]$$

Thus,

$$\sigma = -\left(\frac{\partial F}{\partial \tau}\right)_N = N[\ln(n_{Q1}/n)+1] + N\tau \frac{\partial \ln n_{Q1}}{\partial \tau}$$

$$= N[\ln(n_{Q1}/n)+1] + N\tau \frac{1}{2}\frac{\partial \ln \tau}{\partial \tau}$$

$$= N[\ln(n_{Q1}/n)+1] + \frac{1}{2}N$$

$$= N[\ln(n_{Q1}/n) + \frac{3}{2}]$$