HW#6

6.1

We have

$$N=\sum_n <s_n>=\sum_n \frac{1}{\exp(\hbar\omega_n/\tau)-1}=\sum_n \frac{1}{\exp(x)-1}$$

$$\omega_n = n\pi c/L, x=\hbar\omega_n/\tau, n=\frac{\tau L}{\hbar\pi c}x$$

Similar to (4.17), a factor 2 will applied for 2 independent polarizations.

$$N=\sum_n \frac{1}{\exp(x)-1}\approx \pi\int_0^\infty \frac{1}{\exp(x)-1}n^2dn=\pi(\frac{\tau L}{\hbar\pi c})^3\int_0^\infty \frac{x^2}{\exp(x)-1}dx=\pi^{-2}V(\frac{\tau}{\hbar c})^3\int_0^\infty \frac{x^2}{\exp(x)-1}dx$$

We can get the integral value

$$\int_0^\infty \frac{x^2}{\exp(x)-1}dx=\int_0^\infty \lim_{n\to\infty}[\frac{1-\exp(-nx)}{1-\exp(-x)}-1]x^2dx=\sum_{n=1}^\infty\int_0^\infty \exp(-nx)x^2dx$$

$$=\sum_{n=1}^\infty -\frac{\exp(-nx)(2+2nx+n^2x^2)}{n^3}|_0^\infty=\sum_{n=1}^\infty\frac{2}{n^3}=2\zeta(3)\approx 2.40411$$

Thus,

$$N\approx 2.404\pi^{-2}V(\frac{\tau}{\hbar c})^3$$

6.2

We have

$$C_L=\left(\frac{\partial U}{\partial\tau}\right)_L$$

$$U=\sum_n <s_n>\hbar\omega_n=\sum_n\frac{\hbar\omega_n}{\exp(\hbar\omega_n/\tau)-1}=\tau\sum_n\frac{x}{\exp(x)-1}$$

$$\omega_n=n\pi v/L, x=\hbar\omega_n/\tau, n=\frac{\tau L}{\hbar\pi v}x$$

Thus,

$$U=\tau\sum_n\frac{x}{\exp(x)-1}\approx\tau\int_0^\infty\frac{x}{\exp(x)-1}dn=\frac{\tau^2L}{\hbar\pi v}\int_0^\infty\frac{x}{\exp(x)-1}dx$$

Similar to problem 1, we can get

$$\int_0^\infty\frac{x}{\exp(x)-1}dx=\int_0^\infty\lim_{n\to\infty}[\frac{1-\exp(-nx)}{1-\exp(-x)}-1]xdx=\sum_{n=1}^\infty\int_0^\infty\exp(-nx)xdx$$

$$=\sum_{n=1}^\infty-\frac{\exp(-nx)(1+nx)}{n^2}|_0^\infty=\sum_{n=1}^\infty\frac{1}{n^2}=\zeta(2)=\frac{\pi^2}{6}$$

Thus,

$$U = \frac{\pi\tau^2 L}{6\hbar v}$$

$$\Rightarrow C_L = \left(\frac{\partial U}{\partial \tau}\right)_L = \frac{\pi\tau L}{3\hbar v}$$

6.3

For capacity of matter, we have

$$C_v = \frac{3}{2} N k_B$$

For radiation, from (4.20)

$$\frac{U}{V} = \frac{\pi^2}{15\hbar^3 c^3}\tau^4$$

$$\Rightarrow U = \frac{\pi^2 V}{15\hbar^3 c^3} k_B{}^4 T^4$$

$$\Rightarrow C_{rad} = \left(\frac{\partial U}{\partial T}\right)_V = \frac{4\pi^2 V}{15\hbar^3 c^3} k_B{}^4 T^3$$

Thus, the ratio of heat capacity of matter to that of radiation is

$$\frac{C_v}{C_{rad}} = \frac{3}{2} N k_B \times \frac{15\hbar^3 c^3}{4\pi^2 V k_B{}^4 T^3} = \frac{45 n\hbar^3 c^3}{8\pi^2 k_B{}^3 T^3} \approx 2.8\times 10^{-10}$$

This ratio is $\sim 10^{-9}$.

6.4

The partition function is

$$Z = \sum_n s_n \exp(-\hbar\omega_n / \tau)$$

s_n is the number of photons in the mode n.

For

$$F = U - \tau\sigma, F = -\tau \ln Z$$

We can get

$$p = \tau\left(\frac{\partial \sigma}{\partial V}\right)_U = -\left(\frac{\partial (U-\tau\sigma)}{\partial V}\right)_{U,\tau} = -\left(\frac{\partial F}{\partial V}\right)_\tau = \tau\left(\frac{\partial \ln Z}{\partial V}\right)_\tau = \frac{\tau}{Z}\left(\frac{\partial Z}{\partial V}\right)_\tau$$

$$= \frac{\tau}{Z}\sum_n s_n \exp(-\hbar\omega_n / \tau) \times -\hbar / \tau \times \frac{d\omega_n}{dV}$$

$$= -\frac{1}{Z}\sum_n s_n \exp(-\hbar\omega_n / \tau) \times \hbar \times \frac{d\omega_n}{dV}$$

From (3.15)

$$\omega_n = n\pi c / L$$

$$\Rightarrow \ln \omega_n = -\ln L + \text{constant} = -\frac{1}{3}\ln V + \text{constant}$$

$$\Rightarrow \frac{d \ln \omega_n}{dV} = \frac{1}{\omega_n}\frac{d\omega_n}{dV} = -\frac{1}{3V}$$

$$\Rightarrow \frac{d\omega_n}{dV} = -\frac{\omega_n}{3V}$$

Thus,

$$p = -\frac{1}{Z}\sum_n s_n \exp(-\hbar\omega_n / \tau) \times \hbar \times \frac{d\omega_n}{dV}$$

$$= -\frac{1}{Z}\sum_n s_n \exp(-\hbar\omega_n / \tau) \times \hbar \times -\frac{\omega_n}{3V}$$

$$= \frac{1}{3V}\frac{1}{Z}\sum_n s_n \exp(-\hbar\omega_n / \tau) \times \hbar\omega_n$$

$$= \frac{U}{3V}$$