1. **Heisenberg Picture** Consider a single-particle system described by the Hamiltonian

\[ H = i\hbar\chi (A - A^\dagger), \]

where

\[ A = \frac{1}{\sqrt{2}} \left( \frac{1}{\lambda} X + i\frac{\lambda}{\hbar} P \right) \]

so that \([A, A^\dagger] = 1\).

(a) Derive the Heisenberg equations of motion for \(A_H(t)\) and \(A^\dagger_H(t)\).

(b) Solve these equations and give the solutions in terms of \(A_S\) and \(A^\dagger_S\).

(c) From these solutions, express \(X_H(t)\) and \(P_H(t)\) in terms of \(X_S\) and \(P_S\).

(d) At \(t = 0\) the wavefunction of the particle is known to be \(\psi(x, 0) = \mathcal{N}e^{-((x/\sigma)\sin(\kappa_0 x))^3}\), where \(\mathcal{N}\) is a normalization constant, and \(\sigma\) and \(\kappa_0\) are arbitrary constants. What is the wavefunction at any later time \(t\)? Hint: recall that \(T(d) = e^{-\frac{\pi d}{\hbar}P}\).
2. Consider a pair of identical spin-1/2 particles in a uniform magnetic field. Neglect the motion of the particles, and consider only their spin degrees of freedom. The Hamiltonian is then

\[ H = -\gamma \left( \vec{S}_1 \cdot \vec{B} + \vec{S}_2 \cdot \vec{B} \right), \]

where \( \vec{S}_1 \) and \( \vec{S}_2 \) are the spin operators of particles 1 and 2, respectively, and \( \gamma \) is a constant. Take the z-axis to lie along the magnetic field, so that \( \vec{B} = B_0 \vec{e}_z \).

(a) Using the concept of a tensor product space, construct a suitable basis for this system where each basis vector is a simultaneous eigenstate of \( S_{1z} \) and \( S_{2z} \).

(b) What are the energy eigenvalues of the system?

(c) What is the degeneracy of each energy level?

(d) What are the energy eigenstates?
3. Let $|1\rangle$, $|2\rangle$, and $|3\rangle$ be a set of three orthonormal state vectors. Consider a system described by the Hamiltonian

$$H = \hbar \Omega (|1\rangle\langle 2| + |2\rangle\langle 1|) + \hbar \omega_3 |3\rangle\langle 3|,$$

where $\omega_3 \neq \Omega$.

Let $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |3\rangle)$.

(a) What are the eigenvalues and normalized eigenvectors of $H$?
(b) What is $|\psi(t)\rangle$?
(c) What is the probability that the system is in state $|3\rangle$ at time $t$?
(d) What is the probability that the system is in state $|2\rangle$ at time $t$?
4. (a) Make a sketch of the energy level diagram for a hydrogen atom, with energy on the vertical axis and angular momentum on the horizontal. (i.e. draw a separate level for each state with well defined \( n \) and \( \ell \) eigenvalues, even though some of them may be degenerate.) Show all levels up to \( n = 4 \).

(b) Indicate the total degeneracy of each \( n \) level, as well as the individual degeneracies of each \( n, \ell \) state.

(c) Write a formula for the wavelength of the photon emitted when the atom decays from level \( n = 3, \ell = 2 \) to level \( n = 1, \ell = 0 \).

(d) Note that the magnetic dipole moment of the hydrogen atom is \( \vec{\mu} = \frac{e}{2m} \vec{L} \). What is the paramagnetic term (i.e. the term linear in \( B \)) that must be added to the Hydrogen Hamiltonian when a uniform magnetic field is applied?

(e) Indicate on your diagram what happen to each \( n, \ell \) level in a uniform magnetic field (neglect diamagnetic effects). Clearly indicate which of the new levels are degenerate, and give the complete degeneracy of each energy eigenstate of the atom+field system.
5. Consider two annihilation operators $A$ and $B$ which satisfy $[A, A^\dagger] = [B, B^\dagger] = 1$ and $[A, B] = [A, B^\dagger] = 0$. Show that the operators

$$J_x = \frac{\hbar}{2}(A^\dagger B + B^\dagger A)$$

$$J_y = i\frac{\hbar}{2}(A^\dagger B - B^\dagger A)$$

$$J_z = \hbar\frac{1}{2}(B^\dagger B - A^\dagger A)$$

satisfy the angular momentum commutation relations.

What are the energy eigenvalues and corresponding degeneracies of the following Hamiltonian

$$H = \hbar\Omega(2A^\dagger AB^\dagger B + A^\dagger A + B^\dagger B)?$$

Hint: write out the terms in $J_x^2 + J_y^2$. 