1. **The continuity equation:** The probability that a particle of mass $m$ lies on the interval $[a,b]$ at time $t$ is

$$P(t|a,b) = \int_a^b dx \, |\psi(x,t)|^2 \quad (1)$$

Differentiate (1) and use the definition of the probability current, $j = -\frac{i\hbar}{2m} (\psi^* \frac{d}{dx} \psi - \psi \frac{d}{dx} \psi^*)$, to show that

$$\frac{d}{dt} P(t|a,b) = j(a,t) - j(b,t). \quad (2)$$

Next, take the limit as $b-a \to 0$ of both (1) and (2), and combine the results to derive the continuity equation:

$$\frac{d}{dx} j(x,t) = -\frac{d}{dt} \rho(x,t).$$

2. **Bound-states of a delta-well:** The inverted delta-potential is given by

$$V(x) = -g \delta(x), \quad (3)$$

where $g > 0$. For a particle of mass $m$, this potential supports a single bound-state for $E = E_b < 0$.

(a) Based on dimensional analysis, estimate the energy, $E_b$, using the only available parameters, $h$, $m$, and $g$.

(b) Assume a solution of the form:

$$\psi_b(x) = c e^{-|x|/\lambda}, \quad (4)$$

and use the delta-function boundary conditions at $x = 0$ to determine $\lambda$, as well as the energy, $E_b$. You can then use normalization to determine $c$. What is $\langle X^2 \rangle$ for this bound-state?

3. **Inverted delta scattering:** Consider a particle of mass $m$, subject to the inverted delta-potential, $V(x) = -g \delta(x)$, with $g > 0$. Only this time, consider an incoming particle with energy $E > 0$. What are the transmission and reflection probabilities, $T$, and $R$?
4. **Combination of delta and step:** Consider a particle of mass $m$, whose potential energy is

$$V(x) = V_0 u(x) + g\delta(x),$$

where $u(x)$ is the unit step function and $V_0 > 0$.

(a) What are the two boundary conditions at $x = 0$ that $\psi(x)$ must satisfy?

(b) For an incident wave of the form $e^{ikx}$, use the ‘plug and chug’ approach to find the reflection and transmission amplitudes, $r$ and $t$.

(c) Compute the reflection probability, $R$, and the transmission probability, $T$. What is the relationship between $T$ and $|t|^2$?

(d) Lastly, compute the transfer matrix for this potential at the discontinuity point, $x = 0$.

(e) Compare your answer to the matrices

$$M_{\delta,\text{step}} = M_{\text{step}}(K,k)M_{\delta}(ka),$$

and

$$M_{\text{step},\delta} = M_{\delta}(Ka)M_{\text{step}}(K,k),$$

where $K = \sqrt{k^2 - \frac{2mV_0}{\hbar^2}}$ and $a = \frac{\hbar^2}{(Mg)}$. Comment on your result.

5. **Delta function Fabry Perot Resonator:** Consider transmission of particles of mass $m$ through two delta-function barriers, described by the potential

$$V(x) = g\delta(x) + g\delta(x - L),$$

where $g > 0$ and $L > 0$.

(a) First, compute the allowed $k$-values for an infinite square well of length $L$, where $k = \sqrt{2mE/\hbar}$.

(b) Next, use the transfer-matrix approach to compute the full transfer matrix of the resonator.

(c) Use the full transfer-matrix to compute the transmission probability, $T$, in terms of the dimensionless parameters $\theta = 2kL$ and $\Delta = 1/ka$, where $a = \hbar^2/(Mg)$.

(d) Make plots of $T$ versus $\theta$ for $\Delta = 1$, $\Delta = 2$, and $\Delta = 4$. Compare the location of the transmission resonances on each plot to the locations of the allowed $k$-values from part (a).

6. Consider a particle of mass $m$ incident on a square potential barrier of height $V_0 > 0$, and width $W$. Consider the case where the incident energy, $E$, is smaller than $V_0$.

(a) Compute the probability to tunnel through the barrier, $T$, as function of the incident wave-vector, $k$.

(b) Write out the full form of the wavefunction of the particle in the tunneling region.

(c) Take limit as $W \to 0$ and $V_0 \to \infty$, while holding $V_0W$ constant, and show that your answer agrees with the result for a delta-function potential, $V(x) = g\delta(x)$, with $g = V_0W$. 