Topics Covered: Algebraic approach to the quantized harmonic oscillator, coherent states.

Some Key Concepts: Oscillator length, creation and annihilation operators, the phonon number operator.

1. Start from the harmonic oscillator Hamiltonian $H = \frac{1}{2M}P^2 + \frac{1}{2}M\omega^2X^2$. Make the change of variables $X \rightarrow \lambda \bar{X}$, $P \rightarrow \hbar \lambda \bar{P}$, and $H \rightarrow \hbar^2\lambda^2 \bar{H}$. Find the value of $\lambda$ for which $\bar{H} = \frac{1}{2}(\bar{X}^2 + \bar{P}^2)$.

2. Write down the harmonic oscillator Hamiltonian in terms of $\omega$, $A$, and $A^\dagger$, and then write the commutation relation between $A$ and $A^\dagger$. Use these to derive the equation of motion for the expectation value $a(t) = \langle \psi(t)|A|\psi(t)\rangle$. Solve this equation for the general case $a(0) = a_0$. Prove that $a^*(t) := \langle A^\dagger \rangle = [a(t)]^*$.

3. Starting from $\langle x|X|n-1 \rangle = x\phi_{n-1}(x)$, express $X$ in terms of $A$ and $A^\dagger$, to derive a recursion relation of the form:

$$\phi_n(x) = f_n(x)\phi_{n-1}(x) + g_n(x)\phi_{n-2}.$$ (1)

Starting from $\phi_0(x) = [\sqrt{\pi} \lambda]^{-1/2} e^{-\frac{1}{2}(x/\lambda)^2}$, use your recursion relation to compute $\phi_2(x)$, $\phi_3(x)$, and $\phi_4(x)$.

4. Make the definition $\phi_n(p) = i^n (p|n)$. Start from $\langle p|P|n-1 \rangle = p(p|n-1)$ and derive a recursion relation for $\phi_n(p)$ by making an analogy to your result from the last problem.

5. Consider a particle in the potential

$$V(x) = \begin{cases} \frac{1}{2}M\omega^2 x^2; & x > 0 \\ \infty; & x < 0 \end{cases}.$$ (2)

What boundary condition must the eigenstates satisfy at $x = 0$? To find the eigenstates and eigenvalues, consider that the wave-function must also satisfy the harmonic oscillator wave equation for $x > 0$, as well be normalizable ($\lim_{x \to \infty} \psi(x) = 0$). Can you think of any states that you already know of that satisfy all three conditions?

6. Consider the potential $V(x) = a + bX + cX^2$. Let $H = \frac{1}{2M}P^2 + V(X)$, so that the energy eigenstates and eigenvalues are defined via $H|E_n\rangle = E_n|E_n\rangle$. Make a change of variables to complete the square and map the problem onto the harmonic oscillator problem and then determine the allowed energies, $\{E_n\}$ and corresponding eigenfunctions $\psi_n(x) := \langle x|E_n\rangle$. 


7. Consider the potential

\[ V(x) = \begin{cases} 
0; & 0 < x < W < L \\
V_0 > 0; & W < x < L \\
\infty; & \text{otherwise}
\end{cases} \quad (3) \]

There are important boundary conditions at \( x = 0, x = W, \) and \( x = L, \) what are they? Assume that \( E < V_0, \) and make an ansatz for each of the two regions, which automatically satisfies the boundary conditions at \( x = 0 \) and \( x = L. \)

Show that the two boundary conditions at \( x = W \) can only be satisfied for certain values of \( E, \) and give a transcendental equation whose solutions yield the allowed energies.

8. A three level system is described by the Hamiltonian \( H = \hbar \Omega(t) (|1\rangle\langle 3| - |3\rangle\langle 1|) + \hbar \Delta(t)|2\rangle\langle 2|. \)

Determine the eigenvalues and eigenvectors of \( H. \)

At time \( t = 0, \) the system is prepared in state \( |1\rangle, \) with \( \Delta(0) = 0 \) and \( \Omega(0) = 0. \) Then \( \Omega \) is suddenly increased to a value of \( \Omega_0, \) and held for a duration of \( T. \) What is the state of the system at time \( t = T? \)

At time \( T, \) the operator \( J = j_0 (i|1\rangle\langle 2| - i|2\rangle\langle 1| + 3|3\rangle\langle 3|) \) is measured. What are the possible outcomes of the measurement and the associated probabilities?

For each possible outcome, what is the state immediately after the measurement?