1. **The Parity Operator:** [20 pts] Determine the matrix element \( \langle x | \Pi | x' \rangle \) and use it to simplify the identity \( \Pi = \int dx \, dx' |x\rangle \langle x| \Pi |x'\rangle \langle x'| \), then use this identity to compute \( \Pi^2 \), \( \Pi^3 \), and \( \Pi^n \).

From these results find an expression for \( S(u) = \exp[\Pi u] \) in the form \( f(u) + g(u) \Pi \).

What is \( \langle x | S(0) | \psi \rangle \)? Express your answer in terms of \( \psi_{\text{even}}(x) = \frac{1}{2}(\psi(x) + \psi(-x)) \) and \( \psi_{\text{odd}}(x) = \frac{1}{2}(\psi(x) - \psi(-x)) \).

Compute \( \langle x | S(0) | \psi \rangle \), \( \lim_{u \to \infty} \langle x | S(u) | \psi \rangle \), and \( \lim_{u \to -\infty} \langle x | S(u) | \psi \rangle \).

2. [15 pts] The coherent state \( |\alpha\rangle \) is defined by \( |\alpha\rangle = \frac{\exp[|\alpha|^2]}{\cosh u} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \), where the states \( \{ |n\rangle \} \) are the harmonic oscillator energy eigenstates.

First, show that for \( \alpha = 0 \), the coherent state \( |\alpha=0\rangle \) is exactly equal to the harmonic oscillator ground-state, \( |0\rangle \).

Then show that any other coherent state can be created by acting on the ground-state, \( |0\rangle \), with the ‘displacement operator’ \( D(\alpha) \), i.e. show that \( |\alpha\rangle = D(\alpha)|0\rangle \), where

\[
D(\alpha) := e^{\alpha A^\dagger - \alpha^* A} \tag{1}
\]

You may need the Zassenhaus formula \( e^{B+C} = e^B e^C e^{-[B,C]/2} \), which is valid only when \( [B, [B, C]] = [C, [B, C]] = 0 \).

What is \( D(\alpha_2)|\alpha_1\rangle \)?

3. [15 pts] Consider a system described by the Hamiltonian \( H = \hbar \kappa (A + A^\dagger) \). Use your results from the previous problem to determine \( |\psi(t)\rangle \) for a system initially in the ground-state, \( |\psi(0)\rangle = |0\rangle \).