PHYS851 Quantum Mechanics I, Fall 2009

FINAL EXAM

NAME:

1. A particle of mass M and wave vector $k > \sqrt{2MV_0}/\hbar$ is scattered by the potential $V(x) = V_1(x) + V_2(x)$, where

$$V_1(x) = \begin{cases} 0; & x < 0 \\ V_0 > 0; & x > 0 \end{cases} \tag{1}$$

$$V_2(x) = g\delta(x) \tag{2}$$

   (a) Compute the reflection and transmission amplitudes, r and t.

   (b) Compute the reflection and transmission probabilities, R and T.

   (c) Show that $|r|^2 + |t|^2 \neq 1$, and explain why probability conservation is not violated.

2. Consider a three-state quantum system, with energy eigenvalues 0, $\hbar\omega_0$, and $4\hbar\omega_0$. The first two normalized energy eigenstates are:

$$|\omega_1\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + |2\rangle\right) \tag{3}$$

$$|\omega_2\rangle = \frac{1}{\sqrt{3}}\left(|1\rangle - |2\rangle + |3\rangle\right) \tag{4}$$

For the following questions, each answer must be given in Dirac notation.

(a) What is the third normalized energy eigenstate?

The initial state of the system is $|\psi(t{=}0)\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle - |2\rangle\right)$.

(b) If the energy of the system were measured at $t = 0$, what are the possible results?

(c) For each possible result, what are the associated probabilities?

(d) For each possible result, give the state of the system at an arbitrary time $t > 0$, assuming no further measurements are made on the system.

(e) Assuming that no measurement was performed at $t = 0$, give the state of the system, $|\psi(t)\rangle$, at an arbitrary time $t > 0$.

In the basis $\{|1\rangle, |2\rangle, |3\rangle\}$, the observable, U, has the matrix representation:

$$U = u_0 \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} \tag{5}$$

(f) What are the possible results of a measurement of U?

(g) For the state, $|\psi(t)\rangle$, computed in part (e), what are the probabilities to obtain each possible result?

3. A particle with mass M, charge q, and spin $s = 1/2$, is confined between two concentric spherical shells. The inner shell has a radius r_0, and the outer shell has radius $r_0 + \epsilon$, where $\epsilon \ll r_0$. The shells act as infinite potential barriers for the particle, so that

$$V(\vec{r}) = V(r) = \begin{cases} \infty; & r < r_0 \\ 0; & r_0 < r < r_0 + \epsilon \\ \infty; & r_0 + \epsilon < r \end{cases} \tag{6}$$

   (a) Write the radial wave equation for the particle in the absence of external electromagnetic fields.

   (b) For the angular momentum term in the radial wave equation, make the approximation $\frac{1}{r^2} \approx \frac{1}{r_0^2}$. Within this approximation, solve the radial wave equation and determine the energy levels of the system.

   (c) For each energy level, give the corresponding degeneracy factor.

   Now assume a uniform magnetic field of strength B_0 is applied. The field is very weak, so that the Zeeman splittings are very small compared the zero-field energy-level spacings.

   (a) What term must be added to the Hamiltonian to account for the effects of the weak magnetic field?

   (b) Assuming that $g = 2$, where g is the magnetic g-factor, what are the energy levels and degeneracies in the presence of the field?

4. A spin-1/2 particle is initially prepared in the state $|\uparrow_z\rangle$. It is passed through a Stern-Gerlach apparatus aligned along the direction $\vec{e}_a(\theta) = \cos(\theta)\vec{e}_z + \sin(\theta)\vec{e_x}$, where θ is an adjustable parameter.

   (a) Write a 2×2 matrix representation of the observable being measured by the device.

   (b) For an arbitrary θ, compute the probabilities for the particle to exit in either the $|\uparrow_a\rangle$ or $|\downarrow_a\rangle$ output channels, where $|\uparrow_a\rangle$ and $|\downarrow_a\rangle$ are the spin-up and spin-down states relative to $\vec{e}_a(\theta)$, respectively.

   (c) The $|\uparrow_a\rangle$ output channel is fed into a second Stern-Gerlach apparatus, aligned along $\vec{e}_z$, and the $|\downarrow_a\rangle$ output channel is fed into a third Stern-Gerlach apparatus, also aligned along $\vec{e}_z$. Four particle detectors (100% efficiency) are arranged so that one is placed at each output channel of both the second and third devices. What are the detection probabilities for each detector?