PHYS852 Quantum Mechanics II, Spring 2010
HOMEWORK ASSIGNMENT 10

Topics covered: Green's function, Lippman-Schwinger Eq., T-matrix, Born Series.

1. **T-matrix approach to one-dimensional scattering:** In this problem, you will use the Lippman-Schwinger equation

$$|\psi\rangle = |\psi_0\rangle + GV|\psi\rangle, \tag{1}$$

to solve the one-dimensional problem of tunneling through delta potentials. Take $\psi_0(z) = e^{ikz}$, and let

$$V(z) = g\delta(z) + g\delta(z-L). \tag{2}$$

(a) Express Eq. (1) as an integral equation for $\psi(z)$, and then use the delta-functions to perform the integral. It might be helpful to introduce the dimensionless parameter $\alpha = \frac{Mg}{\hbar^2 k}$. To solve for the two unknown constants, generate two equations by evaluating your solution at $z = 0$, and $z = L$.

Hit with a $\langle z|$ from the left, and insert $I = \int dz'|z'\rangle\langle z'|$ after the G to get the integral equation

$$\psi(z) = \psi_0(z) + \int dz'\, G_0(z,z')V(z')\psi(z'). \tag{3}$$

Use $V(z') = g\delta(z') + g\delta(z'-L)$ to handle the integrals, giving:

$$\psi(z) = \psi_0(z) + gG_0(z,0)\psi(0) + gG_0(z,L)\psi(L). \tag{4}$$

To find the unknowns, $\psi(0)$ and $\psi(L)$, we set first $z = 0$, and then $z = L$, giving

$$\psi(0) = \psi_0(0) + gG_0(0,0)\psi(0) + gG_0(0,L)\psi(L) \tag{5}$$

$$\psi(L) = \psi_0(L) + gG_0(L,0)\psi(0) + gG_0(L,L)\psi(L) \tag{6}$$

Solving simultaneously for $\psi(0)$ and $\psi(L)$ and taking $G_0(z,z') \to G_0(|z-z'|)$ gives

$$\psi(0) = \frac{1 + i\alpha(1-e^{i2kL})}{1 + 2i\alpha - \alpha^2(1-e^{i2kL})} \tag{7}$$

$$\psi(L) = \frac{e^{ikL}}{1 + 2i\alpha - \alpha^2(1-e^{i2kL})} \tag{8}$$

This gives as the solution:

$$\psi(z) = e^{ikz} - i\alpha\frac{e^{ik(L+|z-L|} + e^{ik|z|}\left(1 + i\alpha(1-e^{i2kL})\right)}{1 + 2i\alpha - \alpha^2(1-e^{i2kL})}. \tag{9}$$

(b) Compute the transmission probability $T = |t|^2$, with t defined via

$$\lim_{z\to\infty}\psi(z) = te^{ikz}. \tag{10}$$

For $z > L$, this becomes

$$\psi(z) = te^{ikz} \tag{11}$$

where

$$t = \frac{1}{1 + 2i\alpha - \alpha^2(1 - e^{i2kL})} \tag{12}$$

So that the transmission probability is

$$T = |t|^2 = \frac{1}{(1 - 2\alpha^2 \sin^2(kL))^2 + 4\alpha^2(1 + \alpha\cos(kL)\sin(kL))^2} \tag{13}$$

(c) In the strong-scatterer limit $\alpha \gg 1$, at what k-values is the transmission maximized?
In the limit $\alpha \gg 1$, we can keep only the α^4 term in the denominator, giving

$$T = \frac{1}{4\alpha^2 \sin^2(kL)} \tag{14}$$

which blows up at $k = n\pi/L$, where n is any integer.

(d) Consider an infinite square-well of length L. What are the k-values for each bound-state? How do these compare with the transmission resonances in the strong-scatterer limit?
The bound states correspond to $k = n\pi/L$, which matches the transmission resonances of the double-delta potential.

2. **The first Born-approximation:** In the first Born-approximation, find the scattering amplitude, $f(\theta, \phi|k)$, for a Gaussian scattering potential,

$$V(r) = V_0 e^{(-r/r_0)^2}. \tag{15}$$

Still within the first Born-approximation, what is the differential cross-section, $\frac{d\sigma}{d\Omega}$, and total cross-section, σ_{tot}? First try the integral in spherical coordinates, then when you reach the peak of frustration, try switching to Cartesian coordinates.

In the first-Born approximation, we have

$$f(\vec{k}', \vec{k}) = -\frac{(2\pi)^2 M}{\hbar^2} \langle \vec{k}' | V | \vec{k} \rangle$$

$$\begin{aligned}
\langle \vec{k}' | V | \vec{k} \rangle &= \int d^3r \, \langle \vec{k}' | \vec{r} \rangle V(\vec{r}) \langle \vec{r} | \vec{k} \rangle \\
&= \frac{1}{(2\pi)^3} \int d^3r e^{i(\vec{k}-\vec{k}')\cdot\vec{r}} V_0 e^{-(r/r_0)^2} \\
&= \frac{V_0}{(2\pi)^3} \int_{-\infty}^{\infty} dx e^{ik'_x x - \left(\frac{x}{r_0}\right)^2} \int_{-\infty}^{\infty} dy e^{ik'_y y - \left(\frac{y}{r_0}\right)^2} \int_{-\infty}^{\infty} dz e^{i(k'_z - k)z - \left(\frac{z}{r_0}\right)^2} \\
&= \frac{V_0}{8\pi^{3/2}} r_0^3 e^{-\frac{1}{4}((\vec{k}'-\vec{k})^2 r_0^2)}
\end{aligned}$$

Now $(\vec{k}' - \vec{k})^2 = (\vec{k}' - k\vec{e}_z) \cdot (\vec{k}' - k\vec{e}_z) = k'^2 - 2k'_z k + k^2$.
With $k' = k$ and $k'_z = k\cos\theta$, this gives

$$f(\theta|k) = -\frac{\sqrt{\pi} V_0 M r_0^2}{2\hbar^2} r_0 e^{-\frac{k^2}{2}(1-\cos\theta)}$$

The differential cross section is then

$$\frac{d\sigma}{d\Omega} = |f(\theta|k)|^2 = \frac{\pi V_0^2 M^2 r_0^4}{4\hbar^4} r_0^2 e^{-k^2 r_0^2 (1-\cos\theta)}$$

The total cross section is then

$$\begin{aligned}
\sigma_{tot} &= \int d\Omega \frac{d\sigma}{d\Omega} \\
&= \int_0^{2\pi} d\phi \int_0^{\infty} d(\cos\theta) \, \frac{\pi V_0^2 M^2 r_0^4}{4\hbar^4} r_0^2 e^{-k^2 r_0^2 (1-\cos\theta)} \\
&= \frac{\pi^2 V_0^2 M^2 r_0^4}{2\hbar^4} r_0^2 \int_{-1}^{1} du \, e^{-k^2 r_0^2 (1-u)} \\
&= \frac{\pi^2 V_0^2 M^2 r_0^4}{2\hbar^4} r_0^2 \frac{1 - e^{-2k^2 r_0^2}}{k^2 r_0^2}
\end{aligned}$$

3. **The Huang-Fermi pseudopotential:** First, try to compute the T-matrix in three dimensions for a three-dimensional delta-function scatter, $V(\vec{r}) = g\delta^3(\vec{r})$. What happens?

   A workable zero-range potential in three-dimensions is called the Huang-Fermi pseudo-potential, V_{HF}, defined via

$$\langle\vec{r}|V_{HF}|\psi\rangle = g\delta^3(\vec{r})\psi_{reg}(\vec{r}), \tag{16}$$

   where

$$\psi_{reg}(\vec{r}) = \frac{d}{dr} r\,\psi(\vec{r}). \tag{17}$$

   This potential is also referred to as a "regularized delta-function".

   (a) By expanding $\psi(\vec{r})$ in powers of r, starting with r^{-1}, show that the effect of the regularization operator, $\frac{d}{dr}r$ is to remove the $1/r$ term in the expansion. Thus $\psi_{reg}(\vec{r})$, is always non-singular at $r = 0$.

$$\psi(\vec{r}) = c_{-1}(\theta,\phi)\frac{1}{r} + c_0(\theta,\phi) + c_1(\theta,\phi)r + \ldots, \tag{18}$$

   Then we have

$$\begin{aligned} \frac{d}{dr} r\psi(\vec{r}) &= \frac{d}{dr}\left[c_{-1}(\theta,\phi) + c_0(\theta,\phi)r + c_1(\theta,\phi)r^2 + \ldots\right] \\ &= c_0(\theta,\phi) + 2c_1(\theta,\phi)r + \ldots \end{aligned} \tag{19}$$

   so we see that the singular term has been removed. Thus $\psi_{reg}(\vec{r})$ is non-singular at $r = 0$. In fact, we can use the sifting property of the delta function to give

$$\langle\vec{r}|V_{HF}|\psi\rangle = \delta^3(\vec{r})\psi_{reg}(0) \tag{20}$$

   (b) Compute the T-matrix for V_{HF}, using the regularization property to solve the singularity problem encountered with the simple delta-function.
   We start from the Born-series expansion

$$T = V + VG_0V + VG_0VG_0V + \ldots \tag{21}$$

$$\begin{aligned} \langle\vec{r}|T|\psi\rangle &= g\delta^3(\vec{r})\psi_{reg}(0) + g^2\delta^3(\vec{r})\frac{d}{dr}r\int d^3r'\,G_0(\vec{r},\vec{r}')\delta^3(\vec{r}')\psi_{reg}(0) \\ &+ g^3\delta^3(\vec{r})\frac{d}{dr}r\int d^3r'd^3r''\,G_0(\vec{r},\vec{r}')\delta^3(\vec{r}')\frac{d}{dr'}r'G_0(\vec{r}',\vec{r}'')\delta_3(\vec{r}'')\psi_{reg}(0) + \ldots \\ &= g\delta^3(\vec{r})\psi_{reg}(0)\left[1 + g\frac{d}{dr}rG_0(\vec{r},0) + g^2\frac{d}{dr}r\int d^3r'G_0(\vec{r},\vec{r}')\delta^3(\vec{r}')\frac{d}{dr'}r'G_0(\vec{r}',0) + \ldots\right] \\ &= g\delta^3(\vec{r})\psi_{reg}(0)\left[1 + gG_{0,reg}(0,0) + g^2\frac{d}{dr}rG_0(\vec{r},0)G_{0,reg}(0,0) + \ldots\right] \\ &= g\delta^3(\vec{r})\psi_{reg}(0)\left[1 + gG_{0,reg}(0,0) + g^2G_{0,reg}^2(0,0) + \ldots\right] \\ &= \frac{g\delta^3(\vec{r})}{1 - gG_{0,reg}(0,0)}\psi_{reg}(0) \end{aligned} \tag{22}$$

   Expanding $G(\vec{r},0)$ in powers of r gives

$$G_0(\vec{r},0) = -\frac{M}{2\pi\hbar^2}\left[\frac{1}{r} + ik - \frac{k^2}{2}r + \ldots\right] \tag{23}$$

so that

$$G_{0,reg}(0,0) = -ika \tag{24}$$

where

$$a = \frac{Mg}{2\pi\hbar^2} \tag{25}$$

so that finally, we have

$$T = \frac{V_{HF}}{1+ika} \tag{26}$$

where

$$g' = \frac{g}{1+ika} \tag{27}$$

is called the 're-normalized coupling constant'.

(c) Use your answer to part (b) to compute the differential cross-section, $\frac{d\sigma}{d\Omega}$, as well as the total cross-section, σ_{tot}, for the Huang-Fermi pseudo-potential.
The Fourier transform of T is then

$$\begin{aligned} T(\vec{k}', \vec{k}) &= \frac{1}{(2\pi)^3} \int d^3r' d^3r\, e^{-i\vec{k}'\cdot\vec{r}'} T(\vec{r}', \vec{r}) e^{i\vec{k}\cdot\vec{r}} \\ &= \frac{g}{(2\pi)^3(1+ika)} \end{aligned}$$

from $f(\vec{k}', \vec{k}) = -\frac{(2\pi)^2 M}{\hbar^2} T(\vec{k}', \vec{k})$, we find

$$\begin{aligned} f(\vec{k}', \vec{k}) &= -\frac{(2\pi)^2 M}{\hbar^2} \frac{g}{(2\pi)^3(1+ika)} \\ &= -\frac{a}{1+ika} \end{aligned}$$

The differential cross-section is then

$$\frac{d\sigma}{d\Omega} = \frac{a^2}{1+(ka)^2}$$

As this doesn't depend on θ or ϕ, we have simply

$$\sigma_{tot} = \frac{4\pi a^2}{1+(ka)^2}$$