1. T-matrix approach to one-dimensional scattering: In this problem, you will use the Lippman-Schwinger equation

\[ |\psi\rangle = |\psi_0\rangle + GV|\psi\rangle, \tag{1} \]

to solve the one-dimensional problem of tunneling through delta potentials. Take \( \psi_0(z) = e^{ikz} \), and let

\[ V(z) = g\delta(z) + g\delta(z - L). \tag{2} \]

(a) Express Eq. (1) as an integral equation for \( \psi(z) \), and then use the delta-functions to perform the integral. It might be helpful to introduce the dimensionless parameter \( \alpha = \frac{Mg}{\hbar^2k} \). To solve for the two unknown constants, generate two equations by evaluating your solution at \( z = 0 \), and \( z = L \).

Hit with a \( \langle z \rangle \) from the left, and insert \( I = \int dz'|z'|\langle z'| \) after the \( G \) to get the integral equation

\[ \psi(z) = \psi_0(z) + \int dz' G_0(z,z')V(z')\psi(z'). \tag{3} \]

Use \( V(z') = g\delta(z') + g\delta(z' - L) \) to handle the integrals, giving:

\[ \psi(z) = \psi_0(z) + gG_0(z,0)\psi(0) + gG_0(z,L)\psi(L). \tag{4} \]

To find the unknowns, \( \psi(0) \) and \( \psi(L) \), we set first \( z = 0 \), and then \( z = L \), giving

\[ \psi(0) = \psi_0(0) + gG_0(0,0)\psi(0) + gG_0(0,L)\psi(L) \tag{5} \]
\[ \psi(L) = \psi_0(L) + gG_0(L,0)\psi(0) + gG_0(L,L)\psi(L) \tag{6} \]

Solving simultaneously for \( \psi(0) \) and \( \psi(L) \) and taking \( G_0(z, z') \to G_0(|z - z'|) \) gives

\[ \psi(0) = \frac{1 + i\alpha(1 - e^{i2kL})}{1 + 2i\alpha - \alpha^2(1 - e^{i2kL})} \tag{7} \]
\[ \psi(L) = \frac{e^{ikL}}{1 + 2i\alpha - \alpha^2(1 - e^{i2kL})} \tag{8} \]

This gives as the solution:

\[ \psi(z) = e^{ikz} - i\alpha e^{ik(L+|z-L|)} + i\alpha(1 + e^{i2kL}) \left(1 + i\alpha(1 - e^{i2kL})\right) \]
\[ 1 + 2i\alpha - \alpha^2(1 - e^{i2kL}) \] \( \tag{9} \)

(b) Compute the transmission probability \( T = |t|^2 \), with \( t \) defined via

\[ \lim_{z \to \infty} \psi(z) = te^{ikz}. \tag{10} \]

For \( z > L \), this becomes

\[ \psi(z) = te^{ikz} \tag{11} \]
where

\[ t = \frac{1}{1 + 2i\alpha - \alpha^2(1 - e^{i2kL})} \]  

(12)

So that the transmission probability is

\[ T = |t|^2 = \frac{1}{(1 - 2\alpha^2 \sin^2(kL))^2 + 4\alpha^2(1 + \alpha \cos(kL) \sin(kL))^2} \]  

(13)

(c) In the strong-scatterer limit \( \alpha \gg 1 \), at what \( k \)-values is the transmission maximized?

In the limit \( \alpha \gg 1 \), we can keep only the \( \alpha^4 \) term in the denominator, giving

\[ T = \frac{1}{4\alpha^2 \sin^2(kL)} \]  

(14)

which blows up at \( k = n\pi/L \), where \( n \) is any integer.

(d) Consider an infinite square-well of length \( L \). What are the \( k \)-values for each bound-state? How do these compare with the transmission resonances in the strong-scatterer limit?

The bound states correspond to \( k = n\pi/L \), which matches the transmission resonances of the double-delta potential.
2. **The first Born-approximation:** In the first Born-approximation, find the scattering amplitude, \( f(\theta, \phi|k) \), for a Gaussian scattering potential,

\[
V(r) = V_0 e^{-r^2/r_0^2}.
\] (15)

Still within the first Born-approximation, what is the differential cross-section, \( \frac{d\sigma}{d\Omega} \), and total cross-section, \( \sigma_{\text{tot}} \)? First try the integral in spherical coordinates, then when you reach the peak of frustration, try switching to Cartesian coordinates.

In the first-Born approximation, we have

\[
f(\vec{k}', \vec{k}) = -\frac{(2\pi)^2 M}{\hbar^2} \langle \vec{k}' | V | \vec{k} \rangle
\]

\[
\langle \vec{k}' | V | \vec{k} \rangle = \int d^3r \langle \vec{k}' | r \rangle V(r) \langle r | \vec{k} \rangle
\]

\[
= \frac{1}{(2\pi)^3} \int d^3r e^{i(\vec{k}' - \vec{k}) \cdot \vec{r}} V_0 e^{-(r/r_0)^2}
\]

\[
= V_0 \frac{(2\pi)^3}{8\pi^{3/2} r_0^3} \int_0^\infty e^{-k^2 r_0^2} \int_0^{\pi} d\phi \int_0^{2\pi} d\theta \sin \theta e^{i(k'_x - k_x)\phi} e^{i(k'_y - k_y)\theta}
\]

\[
= V_0 \frac{(2\pi)^3}{8\pi^{3/2} r_0^3} \int_0^\infty e^{-k^2 r_0^2} \int_0^{\pi} d\phi \int_0^{2\pi} d\theta \sin \theta e^{i(k'_x - k_x)\phi} e^{i(k'_y - k_y)\theta}
\]

Now \((\vec{k}' - \vec{k})^2 = (\vec{k}' - k_0 \hat{z}) \cdot (\vec{k}' - k_0 \hat{z}) = k'^2 - 2k'_x k_x + k_x^2.

With \( k' = k \) and \( k'_x = k \cos \theta \), this gives

\[
f(\theta|k) = -\frac{\sqrt{\pi}V_0 M r_0^2}{2\hbar^2} r_0 e^{-k^2 r_0^2(1-\cos \theta)}
\]

The differential cross section is then

\[
\frac{d\sigma}{d\Omega} = |f(\theta|k)|^2 = \frac{\pi V_0^2 M^2 r_0^4}{4\hbar^4} r_0^2 e^{-k^2 r_0^2(1-\cos \theta)}
\]

The total cross section is then

\[
\sigma_{\text{tot}} = \int d\Omega \frac{d\sigma}{d\Omega}
\]

\[
= \int_0^{2\pi} d\phi \int_0^{\pi} d(\cos \theta) \frac{\pi V_0^2 M^2 r_0^4}{4\hbar^4} r_0^2 e^{-k^2 r_0^2(1-\cos \theta)}
\]

\[
= \frac{\pi^2 V_0^2 M^2 r_0^4}{2\hbar^4} r_0^2 \int_0^1 du e^{-k^2 r_0^2(1-u)}
\]

\[
= \frac{\pi^2 V_0^2 M^2 r_0^4}{2\hbar^4} r_0^2 \left( 1 - e^{-2k^2 r_0^2} \right)
\]
3. The Huang-Fer mi pseudopotential: First, try to compute the T-matrix in three dimensions for a three-dimensional delta-function scatter, \( V(\vec{r}) = g\delta^3(\vec{r}) \). What happens?

A workable zero-range potential in three-dimensions is called the Huang-Fer mi pseudo-potential, \( V_{HF} \), defined via

\[
\langle \vec{r}|V_{HF}|\psi \rangle = g\delta^3(\vec{r})\psi_{\text{reg}}(\vec{r}),
\]

where

\[
\psi_{\text{reg}}(\vec{r}) = \frac{d}{dr}\psi(\vec{r}).
\]

This potential is also referred to as a “regularized delta-function”.

(a) By expanding \( \psi(\vec{r}) \) in powers of \( r \), starting with \( r^{-1} \), show that the effect of the regularization operator, \( \frac{d}{dr} \) is to remove the \( 1/r \) term in the expansion. Thus \( \psi_{\text{reg}}(\vec{r}) \), is always non-singular at \( r = 0 \).

\[
\psi(\vec{r}) = c_{-1}(\theta, \phi) \frac{1}{r} + c_0(\theta, \phi) + c_1(\theta, \phi)r + \ldots,
\]

Then we have

\[
\frac{d}{dr}r\psi(\vec{r}) = \frac{d}{dr}\left[c_{-1}(\theta, \phi) + c_0(\theta, \phi)r + c_1(\theta, \phi)r^2 + \ldots\right] = c_0(\theta, \phi) + 2c_1(\theta, \phi)r + \ldots
\]

so we see that the singular term has been removed. Thus \( \psi_{\text{reg}}(\vec{r}) \) is non-singular at \( r = 0 \). In fact, we can use the sifting property of the delta function to give

\[
\langle \vec{r}|V_{HF}|\psi \rangle = \delta^3(\vec{r})\psi_{\text{reg}}(0)
\]

(b) Compute the T-matrix for \( V_{HF} \), using the regularization property to solve the singularity problem encountered with the simple delta-function.

We start from the Born-series expansion

\[
T = V + VG_0V + VG_0VG_0V + \ldots
\]

\[
\langle \vec{r}|T|\psi \rangle = g\delta^3(\vec{r})\psi_{\text{reg}}(0) + g^2\delta^3(\vec{r})\frac{d}{dr}r\int d^3r'G_0(\vec{r}, \vec{r}')\delta^3(\vec{r}')\psi_{\text{reg}}(0)
\]

\[
+ g^3\delta^3(\vec{r})\frac{d}{dr}r\int d^3r'd^3r''G_0(\vec{r}, \vec{r}')\delta^3(\vec{r}')\frac{d}{dr'}r'G_0(\vec{r}', \vec{r}'')\delta^3(\vec{r}'')\psi_{\text{reg}}(0) + \ldots
\]

\[
= g\delta^3(\vec{r})\psi_{\text{reg}}(0) \left[1 + g\frac{d}{dr}rG_0(\vec{r}, 0) + g^2\frac{d}{dr}r\int d^3r'G_0(\vec{r}, \vec{r}')\delta^3(\vec{r}')\frac{d}{dr'}r'G_0(\vec{r}', 0) + \ldots\right]
\]

\[
= g\delta^3(\vec{r})\psi_{\text{reg}}(0) \left[1 + gG_{0,\text{reg}}(0, 0) + g^2\frac{d}{dr}rG_0(\vec{r}, 0)G_{0,\text{reg}}(0, 0) + \ldots\right]
\]

\[
= g\delta^3(\vec{r})\psi_{\text{reg}}(0) \left[1 + gG_{0,\text{reg}}(0, 0) + g^2G^2_{0,\text{reg}}(0, 0) + \ldots\right]
\]

\[
= \frac{g\delta^3(\vec{r})}{1 - gG_{0,\text{reg}}(0, 0)}\psi_{\text{reg}}(0)
\]

Expanding \( G(\vec{r}, 0) \) in powers of \( r \) gives

\[
G_0(\vec{r}, 0) = -\frac{M}{2\pi\hbar^2} \left[\frac{1}{r} + ik - \frac{k^2}{2}r + \ldots\right]
\]
so that
\[ G_{0,\text{reg}}(0,0) = -ika \]  \hfill (24)

where
\[ a = \frac{Mg}{2\pi \hbar^2} \]  \hfill (25)

so that finally, we have
\[ T = \frac{V_{HF}}{1 + ika} \]  \hfill (26)

where
\[ g' = \frac{g}{1 + ika} \]  \hfill (27)

is called the ‘re-normalized coupling constant’.

(c) Use your answer to part (b) to compute the differential cross-section, \( \frac{d\sigma}{d\Omega} \), as well as the total cross-section, \( \sigma_{\text{tot}} \), for the Huang-Fermi pseudo-potential.

The Fourier transform of \( T \) is then
\[ T(\vec{k}', \vec{k}) = \frac{1}{(2\pi)^3} \int d^3r' d^3r' e^{-i\vec{k}' \cdot \vec{r}' - i\vec{k} \cdot \vec{r}} T(\vec{r}', \vec{r}) e^{i\vec{k} \cdot \vec{r}} \]

from \( f(\vec{k}', \vec{k}) = -\frac{(2\pi)^2 M}{\hbar^2} T(\vec{k}', \vec{k}) \), we find
\[ f(\vec{k}', \vec{k}) = \frac{(2\pi)^2 M}{\hbar^2} \frac{g}{(2\pi)^3(1 + ika)} \]
\[ = \frac{a}{1 + ika} \]

The differential cross-section is then
\[ \frac{d\sigma}{d\Omega} = \frac{a^2}{1 + (ka)^2} \]

As this doesn’t depend on \( \theta \) or \( \phi \), we have simply
\[ \sigma_{\text{tot}} = \frac{4\pi a^2}{1 + (ka)^2} \]