PHYS852 Quantum Mechanics II, Spring 2010
HOMEWORK ASSIGNMENT 11

Topics covered: Scattering amplitude, differential cross-section, scattering probabilities.

1. Using only the definition, $G_0 = (E - H_0 + i\epsilon)^{-1}$, show that the free-space Green's function is the solution to
$$\left[E + \frac{\hbar^2}{2M}\nabla^2_{\vec{r}}\right] G_0(\vec{r},\vec{r}\,') = \delta^3(\vec{r}-\vec{r}\,'). \tag{1}$$
The purpose of this problem is just to establish the equivalence between our operator-based approach, and the standard Green's function formalism encountered, e.g., in classical EM.

2. If we define the operator F via $f(\vec{k}',\vec{k}) = \langle\vec{k}'|F|\vec{k}\rangle$, then it follows that $F = -\frac{(2\pi)^2 M}{\hbar^2}T$, where T is the T-matrix operator. In principle, one would like to deduce the form of the potential V from scattering data.

   First, derive an expression for the operator V in terms of the operators G_0 and T only.

   In preparation for problem 11.4, use this expression for V to prove that the full Green's function, $G = (E - H_0 - V + i\epsilon)^{-1}$ is related to the background Green's function, G_0 via the simple relation:
$$G = G_0 + G_0 T G_0. \tag{2}$$
(Hint: don't forget that order matters in operator inversion $(AB)^{-1} = B^{-1}A^{-1}$.)

3. Consider a system described by H_0 that has no bound states, but has a continuum of states for $E > 0$. This means that
$$G_0(E) = \int_0^\infty dE' \frac{|E'^{(0)}\rangle\langle E'^{(0)}|}{E - E' + i\epsilon}, \tag{3}$$
where we have assumed that the bare states $|E^{(0)}\rangle$ are non-degenerate. Incorporating any degeneracy is accomplished by adding additional quantum numbers and summing/integrating over them.

   Now consider a different system, described by $H = H_0 + V$, that in addition to a continuum of states for $E > 0$, may have a set of negative energy bound states, $\{E_n\}$, . In this case, it follows from the definition $G = (E - H + i\epsilon)^{-1}$, that
$$G = \sum_n \frac{|E_n\rangle\langle E_n|}{E - E_n + i\epsilon} + \int_0^\infty dE' \frac{|E'\rangle\langle E'|}{E - E' + i\epsilon}. \tag{4}$$
Show that for $E < 0$, as $\epsilon \to 0$, G remains finite unless E matches the energy of one of the bound states. Thus the negative energy singularities of a system's Green's function correspond to the energies of the bound states of the potential V. Show that the bound-state wavefunction is given by the formula
$$\psi_n(\vec{r}) = \sqrt{\langle\vec{r}|\lim_{E\to E_n}(E - E_n)G|\vec{r}\rangle}. \tag{5}$$

4. Based on Eq. (2), it follows that if G_0 has no negative energy singularities, then the singularities in G must come from the T-matrix. Consider the case of a particle in one dimension with $H_0 = \frac{P^2}{2M}$ and $V = g\delta(X)$, where $g < 0$. Compute the T-matrix, and find it's negative energy singularity, then use Eq. (5) to find the bound-state wavefunction. Does this procedure give the true bound-state energy and wavefunction? Is it necessary to normalize the resulting state by hand, or is it automatically normalized?

5. Follow the same steps as in the previous problem, but for the three-dimensional Huang-Fermi pseudo-potential, defined by $\langle \vec{r}|V|\psi\rangle = g\delta^3(\vec{r})\frac{d}{dr}r\psi(r)$. Show that a single bound state exists only for the repulsive case $g > 0$, and find the bound-state energy and wavefunction.