PHYS852 Quantum Mechanics II, Spring 2010
HOMEWORK ASSIGNMENT 12

Topics covered: Partial waves.

1. Consider S-wave scattering from a hard sphere of radius $a$. First, make the standard s-wave scattering ansatz:
$$\psi(r,\theta,\phi) = \frac{e^{-ikr}}{r} - (1 + 2ikf_0(k))\frac{e^{ikr}}{r}$$
Then, find the value of $f_0(k)$ that satisfies the boundary condition $\psi(a,\theta,\phi) = 0$. What is the partial amplitude $f_0(k)$? What is the s-wave phase-shift $\delta_0(k)$?

2. For P-wave scattering from a hard sphere of radius $a$, make the ansatz
$$\psi(r,\theta) = \left[\left(\frac{1}{kr} - \frac{i}{(kr)^2}\right)e^{-ikr} + (1 + 2ikf_1(k))\left(\frac{1}{kr} + \frac{i}{(kr)^2}\right)e^{ikr}\right]Y_1^0(\theta.$$
Verify that this is an eigenstate of the full Hamiltonian for $r > a$ by showing that it is a linear superposition of two spherical Bessel functions of the third-kind. Again solve for the partial amplitude, $f_1(k)$, by imposing the boundary condition $\psi(a,\theta,\phi) = 0$. What is the phase-shift $\delta_1(k)$? Show that it scales as $(ka)^3$ in the limit $k \to 0$. This is a general result that for small $k$ we have $\delta_\ell(k) \propto k^{2\ell+1}$, called 'threshold behavior. Take the limit as $k \to 0$ and show that $\delta_1(k)$ is negligible compared to $\delta_0(k)$. This is an example of how higher partial waves are 'frozen out' at low energy.

3. Consider S-wave scattering from a spherical potential-well of depth $U_0$ and radius $R$, i.e. $V(r) = -U_0$ for $r < R$, and zero for $r > R$. Make a suitable Ansatz, and determine the s-wave scattering amplitude from the boundary conditions ar $r = R$. What the is the partial amplitude $f_0(k)$? What is the phase-shift $\delta_0(k)$?

   Expand $\delta_0(k)$ in power-series in $k$. The s-wave scattering length $a$ and effective range $r_e$ are defined via:
$$\cot(\delta_0(k)) = -\frac{1}{ka} + \frac{1}{2}kr_e + O(k^2).$$
   Find the scattering length, and show that it is not bound by the radius $R$, but that all values $-\infty < a < \infty$ are possible.

4. Scattering resonances are the scattering analog of tunneling resonances. Consider scattering from the delta-shell potential
$$V(r) = g\delta(r - r_0),$$
   First determine the boundary conditions at $r = 0$ and $r = r_0$, then make a suitable ansatz, apply the necessary boundary conditions, and compute the s-wave scattering amplitude.

   With the coupling strength governed by the dimensionless parameter $\mu = \frac{2Mg}{\hbar^2 k}$, plot the s-wave scattering phase-shift versus $kr_0$ for $\mu = 0.1$ ,1.0, and 10.

   Determine the s-wave bound-states of an infinite spherical well of radius $r_0$. Comment on the relationship between the locations of the delta-barrier resonances and these bound-state energies. What happens to the s-wave scattering length when the incident $k$-value sweeps across the $k$ corresponding to one of these quasi bound states?