PHYS852 Quantum Mechanics II, Spring 2010
HOMEWORK ASSIGNMENT 12

Topics covered: Partial waves.

1. Consider S-wave scattering from a hard sphere of radius a. First, make the standard s-wave scattering ansatz:

$$\psi(r,\theta,\phi) = \frac{e^{-ikr}}{r} - (1 + 2ikf_0(k))\,\frac{e^{ikr}}{r}$$

Then, find the value of $f_0(k)$ that satisfies the boundary condition $\psi(a,\theta,\phi) = 0$. What is the partial amplitude $f_0(k)$? What is the s-wave phase-shift $\delta_0(k)$?

Satisfying the required boundary condition at $r = a$ requires

$$0 = e^{-ika} - (1 + 2ikf_0(k))e^{ika}, \tag{1}$$

which gives the s-wave partial amplitude as

$$f_0(k) = -e^{-ika}\frac{\sin(ka)}{k} \tag{2}$$

The phase-shift is related to the partial amplitude via Eqs. (137) or (138) in the lecture notes, which give

$$\delta_0(k) = -ka \tag{3}$$

From Eq. (137) in the notes, it then follows that the scattering length is a. Thus we can interpret the scattering length of a particular scatterer as the radius of the hard-sphere whose scattering amplitude matches that of the scatterer in the low energy ($k \to 0$) limit.

2. For P-wave scattering from a hard sphere of radius a, make the ansatz

$$\psi(r,\theta) = \left[\left(\frac{1}{kr} - \frac{i}{(kr)^2}\right)e^{-ikr} + (1+2ikf_1(k))\left(\frac{1}{kr} + \frac{i}{(kr)^2}\right)e^{ikr}\right]Y_1^0(\theta.$$

Verify that this is an eigenstate of the full Hamiltonian for $r > a$ by showing that it is a linear superposition of two spherical Bessel functions of the third-kind. Again solve for the partial amplitude, $f_1(k)$, by imposing the boundary condition $\psi(a,\theta,\phi) = 0$. What is the phase-shift $\delta_1(k)$? Show that it scales as $(ka)^3$ in the limit $k \to 0$. This is a general result that for small k we have $\delta_\ell(k) \propto k^{2\ell+1}$, called 'threshold behavior. Take the limit as $k \to 0$ and show that $\delta_1(k)$ is negligible compared to $\delta_0(k)$. This is an example of how higher partial waves are 'frozen out' at low energy.

The spherical Bessel functions of the third-kind are defined via

$$h_\ell(\rho) = -i\rho^\ell\left(-\frac{1}{\rho}\frac{d}{d\rho}\right)^\ell \frac{e^{i\rho}}{\rho} \tag{4}$$

For $\ell = 1$ and $\rho = kr$, this gives

$$h_1(kr) = -\left[\frac{1}{kr} + \frac{i}{(kr)^2}\right]e^{ikr} \tag{5}$$

The other Bessel function of the third-kind is $h_1^*(kr)$, so that we have

$$\psi(r,\theta) = -\left[h_1(kr) + (1+2ikf_1(k))\,h_1^*(kr)\right]Y_0^1(\theta) \tag{6}$$

which is therefore a solution of the free-space Hamiltonian.

The boundary condition $\psi(a,\theta) = 0$ becomes

$$\left[1 - \frac{i}{ka}\right]e^{-ika} + (1+2ikf_1(k))\left[1 + \frac{i}{ka}\right]e^{ika} = 0,$$

Solving for $(1+2ikf_1(k))$ gives

$$\begin{aligned}(1-2ikf_1(k)) &= -\frac{(1-i/ka)e^{-ika}}{(1+i/ka)e^{ika}} \\ &= \frac{(1+ika)}{(1-ika)}e^{-i2ka}.\end{aligned} \tag{7}$$

For any complex number $Z = x + iy$ we have

$$\frac{Z}{Z^*} = \frac{re^{i\theta}}{re^{-i\theta}} = e^{i2\theta} = e^{i2\arctan(y/x)}.$$

This shows that

$$(1-2ikf_1(k)) = e^{2i(\arctan(ka)-ka)}$$

from which we can read off the p-wave phase-shift

$$\delta_1(k) = \arctan(ka) - ka.$$

Expanding the r.h.s. in power series gives

$$\delta_1(k) = -\frac{(ka)^3}{3} + O(a^5),$$

which verifies the 'threshold behavior'.

In the limit $ka \to 0$, we have

$$\frac{\delta_1(k)}{\delta_0(k)} = \frac{(ka)^3}{3ka} = \frac{(ka)^2}{3},$$

which shows that $\delta_1(k)$ is negligible compared to $\delta_0(k)$ as long as $k \ll 1/a$.

3. Consider S-wave scattering from a spherical potential-well of depth U_0 and radius R, i.e. $V(r) = -U_0$ for $r < R$, and zero for $r > R$. Make a suitable Ansatz, and determine the s-wave scattering amplitude from the boundary conditions ar $r = R$. What the is the partial amplitude $f_0(k)$? What is the phase-shift $\delta_0(k)$?

   Expand $\delta_0(k)$ in power-series in k. The s-wave scattering length a and effective range r_e are defined via:
$$\cot(\delta_0(k)) = -\frac{1}{ka} + \frac{1}{2}kr_e + O(k^2).$$
   Find the scattering length, and show that it is not bound by the radius R, but that all values $-\infty < a < \infty$ are possible.

   For s-waves, our Ansatz should be

$$u(r) = \begin{cases} e^{-ikr} - (1 + 2ikf_0(k))e^{ikr}; & r > R \\ A \sin Kr; & r < R \end{cases}, \tag{8}$$

   where $K = \sqrt{k^2 + k_0^2}$, with $k_0 = \sqrt{2MU_0}/\hbar$. The boundary conditions are then

$$u(R^-) = u(R^+) \tag{9}$$
$$u'(R^-) = u'(R^+) \tag{10}$$

   which gives us
$$A \sin(KR) = e^{-ikR} - (1 + 2ikf_0(k))e^{ikR}, \tag{11}$$
   and
$$AK \cos(KR) = -ik\left[e^{-ikR} + (1 + 2ikf_0(k))e^{ikR}\right]. \tag{12}$$
   we can divide the two equations to eliminate A, giving us

$$K \cot(KR) = -ik\frac{e^{-ikR} + (1 + 2ikf_0(k))e^{ikR}}{e^{-ikR} - (1 + 2ikf_0(k))e^{ikR}}. \tag{13}$$

   solving for $1 + 2ikf_0(k)$ gives us

$$1 + 2ikf_0(k) = \frac{K + ik\tan(KR)}{K - ik\tan(KR)}e^{-2ikR}, \tag{14}$$

   hence the s-wave partial amplitude is

$$f_0(k) = \frac{1}{2i}\left[\frac{K + ik\tan(KR)}{K - ik\tan(KR)}e^{-2ikR} - 1\right] \tag{15}$$

   The s-wave phase-shift is

$$\delta_0(K) = -kR + \tan^{-1}\left(\frac{k\tan(KR)}{K}\right) \tag{16}$$

   which has the expansion
$$k\cot(\delta_0(k) = -\frac{1}{a} + \frac{1}{2}r_e k^2 + \dots \tag{17}$$

where

$$a = R - \frac{\tan(k_0 R)}{k_0} \tag{18}$$

and

$$r_e = R\left(1 - \frac{k_0^2 R^2}{3(\tan(k_0 R) - k_0 R)^2}\right) + \frac{1}{(\tan(k_0 R) - k_0 R)}\frac{1}{k0} \tag{19}$$

4. Scattering resonances are the scattering analog of tunneling resonances. Consider scattering from the delta-shell potential

$$V(r) = g\delta(r - r_0),$$

First determine the boundary conditions at $r = 0$ and $r = r_0$, then make a suitable ansatz, apply the necessary boundary conditions, and compute the s-wave scattering amplitude.

With the coupling strength governed by the dimensionless parameter $\mu = \frac{2Mg}{\hbar^2 k}$, plot the s-wave scattering phase-shift versus kr_0 for $\mu = 0.1$,1.0, and 10.

Determine the s-wave bound-states of an infinite spherical well of radius r_0. Comment on the relationship between the locations of the delta-barrier resonances and these bound-state energies. What happens to the s-wave scattering length when the incident k-value sweeps across the k corresponding to one of these quasi bound states?

We start by integrating the radial wave equation from $r_0 - \epsilon$ to $r_0 + \epsilon$,

$$\int_{r_0-\epsilon}^{r_0+\epsilon} dr \left(E + \frac{\hbar^2}{2M}\frac{d^2}{dr^2} + g\delta(r - r_0) \right) u(r) = 0$$

which becomes

$$2\epsilon E u(r_0) + \frac{\hbar^2}{2M}(u'(r_0 + \epsilon) - u'(r_0 - \epsilon)) + g u(r_0) = 0$$

taking $\epsilon \longrightarrow 0$ then gives

$$-\frac{\hbar^2}{2M}(u_1'(r_0) - u_2'(r_0)) + g u(r_0) = 0,$$

where region 1 corresponds to $r > r_0$, and region 2 is the inner region. The boundary condition is therefore

$$u_1'(r_0) = u_2'(r_0) + \frac{2Mg}{\hbar^2} u(r_0)$$

Now for the outer region, we must choose

$$u_1(r) = e^{-ikr} - (1 + 2ikf_0(k))e^{ikr}.$$

while for the inner region, we need

$$u_2(r) = A\sin(kr)$$

Note that in this case it is the same k in both regions.

From $u_1(r_0) = u_2(r_0)$, we get

$$A\sin(kr_0) = e^{-ikr_0} - (1 + 2ikf_0(k))e^{ikr_0}$$

while the delta-function boundary condition gives

$$Ak\cos(kr_0) + \frac{2Mg}{\hbar^2} A\sin(kr_0) = -ik\left(e^{-ikr_0} + (1 + 2ikf_0(k))e^{ikr_0}\right)$$

Dividing the second equation by the first gives

$$k\cot(kr_0) + \frac{2Mg}{\hbar^2} = -ik\frac{e^{-ikr_0} + (1+2ikf_0(k))e^{ikr_0}}{e^{-ikr_0} - (1+2ikf_0(k))e^{ikr_0}}$$

Solving for $(1+2ikf_0(k)$ gives

$$1+2ikf_0(k) = \frac{\cot(kr_0)+\mu+i}{\cot(kr_0)+\mu-i}e^{-i2kr_0}$$

Solving for $f_0(k)$ gives

$$\begin{aligned} f_0(k) &= -\frac{1}{2ik}\left[1-\frac{\cot(kr_0)+\mu+i}{\cot(kr_0)+\mu-i}\right] \\ &= \frac{1}{k(\cot(kr_0)+\mu-i)} \end{aligned}$$

Going back to

$$1+2ikf_0(k) = \frac{\cot(kr_0)+\mu+i}{\cot(kr_0)+\mu-i}e^{-i2kr_0}$$

we can see that the s-wave phase-shift is

$$\delta_0(k) = \tan^{-1}\left(\frac{1}{\mu+\cot(kr_0)}\right) - kr_0$$

Now if $y = \tan^{-1}\left(\frac{1}{x}\right)$, it follows that $\tan y = \frac{1}{x}$. Since $\frac{1}{\tan y} = \cot y$, we then have $\cot y = x$, or $y = \cot^{-1}(x)$ Thus can express the phase-shift as

$$\delta_0(k) = \cot^{-1}\left(\cot(kr_0)+\mu\right) - kr_0.$$

Now if we hold μ fixed and vary r_0, we can plot δ_0 vs. kr_0. If we restrict ourselves to $-\pi < \delta_0 \le \pi$, the plot looks like The spherical infinite well, has solutions of the form

$$u(r) = A\sin(kr)$$

it needs to vanish at $r = r_0$, which leads to $kr_0 = n\pi$ as the bound-state condition. This matches up with the resonances for large μ

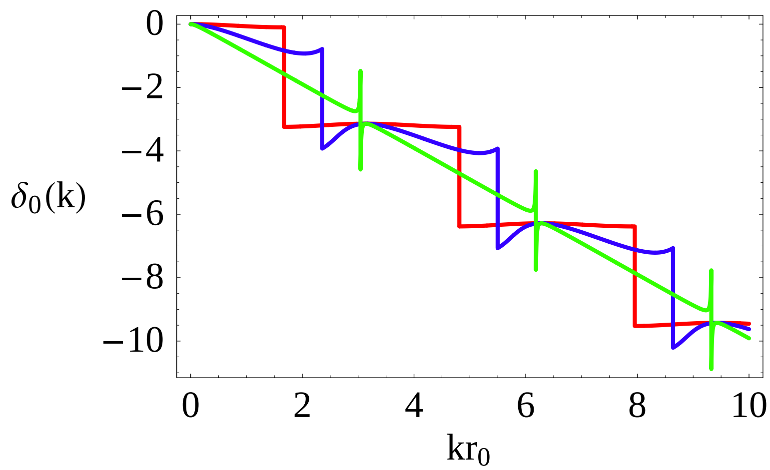

Figure 1: The s-wave phase-shift, $\delta_0(k)$ versus kr_0 for $\mu = 0.1$ (red), $\mu = 1.0$ (blue), and $\mu = 10.0$ (green). We see that there are resonances, that start out broad, and get very narrow, whose locations are moving towards integer multiples of π as μ increases.