1. The Hamiltonian for a driven two-level system is

\[ H = \hbar \omega_0 |2\rangle\langle 2| + \hbar \Omega \cos(\omega t) \left( |1\rangle\langle 2| + |2\rangle\langle 1| \right), \]  

where \( \omega_0 \) is the separation between the bare levels, and \( \omega \) is the driving frequency.

(a) Make a frame transformation generated by the operator \( G = \hbar \omega |2\rangle\langle 2| \), and determine the equation of motion for the state-vector in the new frame, defined by \( |\psi_G(t)\rangle = U_G(t)|\psi(t)\rangle \).

(b) Make the rotating wave approximation (RWA) by assuming that \( \omega \approx \omega_0 \), and dropping any terms that oscillate at or near \( 2\omega_0 \). Write, in terms of the detuning \( \Delta = \omega_0 - \omega \), the effective time-independent Hamiltonian, \( H_G \), that then governs the time evolution of \( |\psi_G(t)\rangle \).

(c) Assume that the system begins at time \( t = 0 \) in the ground-state of \( H_G \), and calculate \( |\psi_G(t)\rangle \). Is this a stationary state in the rotating frame? Now use \( |\psi_S(t)\rangle = U_G^\dagger(t)|\psi_G(t)\rangle \) to see what this state looks like in the Schrödinger picture. Is it a stationary state in the Schrödinger picture?

(d) Assuming the system begins in the ground state of \( H_G \), use second-order time-dependent perturbation theory to treat the fast-oscillating terms that were discarded in the RWA, and compute the probability to find the system in the excited state of \( H_G \), at time \( t > 0 \).

(e) Assume that at time \( t = 0 \), we have \( \Delta > 0 \), \( \Omega = 0 \), and \( |\psi_G(0)\rangle = |1\rangle \). If \( \Omega \) is smoothly increased from zero to \( \Omega_0 \) on time-scale \( T >> 1/\Delta \), what is the state of the system at time \( t = T \)?

2. Consider a system described by the Hamiltonian:

\[ H = -\frac{\hbar \omega}{4} \left( AA + A^\dagger A^\dagger \right), \]  

where \([A, A^\dagger] = 1\). Find and solve the Heisenberg equations of motion for \( A_H(t) \) and \( A_H^\dagger(t) \). Use these solutions to compute the expectation values of \( X \) and \( P \), as well as the variances \( \Delta X \) and \( \Delta P \), as functions of time, for the case where the initial state satisfies \( A|\psi_S(0)\rangle = \alpha|\psi_S(0)\rangle \), where \( \alpha \) is an arbitrary complex number. For the case \( \alpha = 0 \), show that \( \langle X \rangle_t = \langle P \rangle_t = 0 \), but \( \Delta X \) and \( \Delta P \) grow rapidly in time.

Now re-express the Hamiltonian in terms of \( X \) and \( P \). Do your previous answers make sense given this viewpoint?
3. The Hamiltonian for a hydrogen atom is

\[
H = \frac{P_r^2}{2\mu} + \frac{L^2}{2\mu R^2} - \frac{e^2}{4\pi\varepsilon_0 R}.
\]  

(3)

First, use the properties

\[
\langle \vec{r} | P_r^2 | \psi \rangle = -\hbar^2 \frac{1}{r^2} \frac{d}{dr} (r^2 \langle \vec{r} | \psi \rangle),
\]  

(4)

and

\[
\langle \vec{r} | P_r | \psi \rangle = -i\hbar \frac{1}{dr} (\vec{r} | \psi \rangle),
\]  

(5)

to compute the commutators \([P_r^2, R]\) and \([R^{-s}, P_r]\), then use these commutators to derive the Heisenberg equations of motion for \(R\) and \(P_r\).

4. Compute the density of states, \(n(E)\), for a massive particle in a cubic volume of side length \(L\). Then compute the density of states for a two-dimensional massive particle confined to a square area of side length \(L\), and also for a one-dimensional massive particle in an infinite square-well of width \(L\). Then do the same for a photon, whose energy is related to its wavevector by \(E(\vec{k}) = \hbar c|\vec{k}|\).

To determine the density of states at energy \(E\), first determine \(N(E)\), which is the number of quantized \(k\)-values inside a sphere of radius \(k(E)\). Do this by determining the volume in \(k\)-space occupied by a single state, and then divide the volume of the energy-sphere by the single-mode volume. Then compute the density of states via \(n(E) = \frac{d}{dE} N(E)\).

5. Estimate the spontaneous photon-emission rate of an excited atom via Fermi’s golden rule, use the density of states for a photon in a cube of volume \(V\). To estimate the coupling matrix element, use the dipole energy operator \(V = -\vec{d} \cdot \vec{E}\), where \(d = e a_0\) is the atomic dipole moment, and \(\vec{E}\) is the electric field of a single photon in a volume \(V\). To get the value of \(\vec{E}\), take the photons energy to be \(\hbar \omega\), and use the standard energy density of an electromagnetic field \(\frac{u}{2} (E^2 + c^2 B^2)\). Relate \(B\), the magnetic field of the photon, to its electric field via \(\nabla \times \vec{E} = -\frac{d}{dt} \vec{B}\), which from dimensional analysis gives \(k\vec{E} \approx \omega \vec{B}\). For frequencies in the visible spectrum, what is your estimate of \(\Gamma\)?

6. Consider a harmonic oscillator described by \(H^{(0)} = \hbar \Omega (A^\dagger A + 1/2)\). Now consider two possible perturbations \(V_1 = bX^4 \cos(\omega t)\), and \(V_2 = g\delta(X) e^{-\gamma t}\). Assume that the system begins at \(t = 0\), in the ground-state of \(H^{(0)}, |0\rangle\), and use time-dependent perturbation theory to compute the probability to second-order, for the system to be found in the the \(n^{th}\) eigenstate of \(H^{(0)}\) at time \(t\). Consider both \(n = 0\) and \(n \neq 0\) cases.