PHYS852 Quantum Mechanics II, Spring 2010
HOMEWORK ASSIGNMENT 13

Topics covered: Hilbert-space Frame Transformations, Time-Dependent Perturbation Theory

1. The Hamiltonian for a driven two-level system is

$$H = \hbar\omega_0|2\rangle\langle 2| + \hbar\Omega\cos(\omega t)\left(|1\rangle\langle 2| + |2\rangle\langle 1|\right), \tag{1}$$

   where ω_0 is the separation between the bare levels, and ω is the driving frequency.

   (a) Make a frame transformation generated by the operator $G = \hbar\omega|2\rangle\langle 2|$, and determine the equation of motion for the state-vector in the new frame, defined by $|\psi_G(t)\rangle = U_G(t)|\psi(t)\rangle$.

   (b) Make the rotating wave approximation (RWA) by assuming that $\omega \approx \omega_0$, and dropping any terms that oscillate at or near $2\omega_0$. Write, in terms of the detuning $\Delta = \omega_0 - \omega$, the effective time-independent Hamiltonian, H_G, that then governs the time evolution of $|\psi_G(t)\rangle$.

   (c) Assume that the system begins at time $t = 0$ in the ground-state of H_G, and calculate $|\psi_G(t)\rangle$. Is this a stationary state in the rotating frame? Now use $|\psi_S(t)\rangle = U_G^\dagger(t)|\psi_G(t)\rangle$ to see what this state looks like in the Scrödinger picture. Is it a stationary state in the Schrödinger picture?

   (d) Assuming the system begins in the ground state of H_G, use second-order time-dependent perturbation theory to treat the fast-oscillating terms that were discarded in the RWA, and compute the probability to find the system in the excited state of H_G, at time $t > 0$.

   (e) Assume that at time $t = 0$, we have $\Delta > 0$, $\Omega = 0$, and $|\psi_G(0)\rangle = |1\rangle$. If Ω is smoothly increased from zero to Ω_0 on time-scale $T >> 1/\Delta$, what is the state of the system at time $t = T$?

2. Consider a system described by the Hamiltonian:

$$H = -\frac{\hbar\omega}{4}\left(AA + A^\dagger A^\dagger\right), \tag{2}$$

   where $[A, A^\dagger] = 1$. Find and solve the Heisenberg equations of motion for $A_H(t)$ and $A_H^\dagger(t)$. Use these solutions to compute the expectation values of X and P, as well as the variances ΔX, and ΔP, as functions of time, for the case where the initial state satisfies $A|\psi_S(0)\rangle = \alpha|\psi_S(0)\rangle$, where α is an arbitrary complex number. For the case $\alpha = 0$, show that $\langle X\rangle_t = \langle P\rangle_t = 0$, but ΔX and ΔP grow rapidly in time.

   Now re-express the Hamiltonian in terms of X and P. Do your previous answers make sense given this viewpoint?

3. The Hamiltonian for a hydrogen atom is

$$H = \frac{P_r^2}{2\mu} + \frac{L^2}{2\mu R^2} - \frac{e^2}{4\pi\epsilon_0 R}. \tag{3}$$

First, use the properties

$$\langle\vec{r}|P_r^2|\psi\rangle = -\hbar^2 \frac{1}{r^2}\frac{d}{dr} r^2 \frac{d}{dr}\langle\vec{r}|\psi\rangle, \tag{4}$$

and

$$\langle\vec{r}|P_r|\psi\rangle = -i\hbar\frac{d}{dr}\langle\vec{r}|\psi\rangle, \tag{5}$$

to compute the commutators $[P_r^2, R]$ and $[R^{-s}, P_r]$, then use these commutators to derive the Heisenberg equations of motion for R and P_r.

4. Compute the density of states, $n(E)$, for a massive particle in a cubic volume of side length L. Then compute the density of states for a two-dimensional massive particle confined to a square area of side length L, and also for a one-dimensional massive particle in an infinite square-well of width L. Then do the same for a photon, whose energy is related to its wavevector by $E(\vec{k}) = \hbar c|k|$.

   To determine the density of states at energy E, first determine $N(E)$, which is the number of quantized k-values inside a sphere of radius $k(E)$. Do this by determining the volume in k-space occupied by a single state, and then divide the volume of the energy-sphere by the single-mode volume. Then compute the density of states via $n(E) = \frac{d}{dE}N(E)$.

5. Estimate the spontaneous photon-emission rate of an excited atom via Fermi's golden rule, use the density of states for a photon in a cube of volume V. To estimate the coupling matrix element, use the dipole energy operator $V = -\vec{d}\cdot\vec{\mathcal{E}}$, where $d = ea_0$ is the atomic dipole moment, and $\mathcal{E}$ is the electric field of a single photon in a volume V. To get the value of $\mathcal{E}$, take the photons energy to be $\hbar\omega$, and use the standard energy density of an electromagnetic field $u = \frac{\epsilon_0}{2}\left(\mathcal{E}^2 + c^2\mathcal{B}^2\right)$. Relate $\mathcal{B}$, the magnetic field of the photon, to its electric field via $\nabla\times\vec{\mathcal{E}} = -\frac{d}{dt}\vec{\mathcal{B}}$, which from dimensional analysis gives $k\mathcal{E} \approx \omega\mathcal{B}$. For frequencies in the visible spectrum, what is your estimate of Γ?

6. Consider a harmonic oscillator described by $H^{(0)} = \hbar\Omega(A^\dagger A + 1/2)$. Now consider two possible perturbations $V_1 = bX^4\cos(\omega t)$, and $V_2 = g\delta(X)e^{-\gamma t}$. Assume that the system begins at $t = 0$, in the ground-state of $H^{(0)}$, $|0\rangle$, and use time-dependent perturbation theory to compute the probability to second-order, for the system to be found in the the n^{th} eigenstate of $H^{(0)}$ at time t. Consider both $n = 0$ and $n \neq 0$ cases.