PHYS852 Quantum Mechanics II, Spring 2010
HOMEWORK ASSIGNMENT 13: Solutions

Topics covered: Hilbert-space Frame Transformations, Time-Dependent Perturbation Theory

1. The Hamiltonian for a driven two-level system is

$$H = \hbar\omega_0|2\rangle\langle 2| + \hbar\Omega\cos(\omega t)\left(|1\rangle\langle 2| + |2\rangle\langle 1|\right), \tag{1}$$

where ω_0 is the separation between the bare levels, and ω is the driving frequency.

(a) Make a frame transformation generated by the operator $G = \hbar\omega|2\rangle\langle 2|$, and determine the equation of motion for the state-vector in the new frame, defined by $|\psi_G(t)\rangle = U_G(t)|\psi(t)\rangle$.

First, we note that

$$U_{GS}(t) = e^{iGt/\hbar} = |1\rangle\langle 1| + e^{i\omega t}|2\rangle\langle 2| \tag{2}$$

(the $|1\rangle\langle 1|$ term must be there in order to satisfy $U_{GS}(0) = I$).

Following the general theory in the lecture notes, we have

$$\begin{aligned} H_G(t) &= U_{GS}(t)H_S(t)U_{SG}(t) - G \\ &= \left(|1\rangle\langle 1| + e^{-i\omega t}|2\rangle\langle 2|\right) H \left(|1\rangle\langle 1| + e^{i\omega t}|2\rangle\langle 2|\right) - \hbar\omega|2\rangle\langle 2| \\ &= \hbar(\omega_0 - \omega)|2\rangle\langle 2| + \hbar\Omega\cos(\omega t)\left(|e^{i\omega t}|1\rangle\langle 2| + e^{-i\omega t}|2\rangle\langle 1|\right) \\ &= \hbar\Delta|2\rangle\langle 2| + \frac{\hbar\Omega}{2}\left(|1\rangle\langle 2| + |2\rangle\langle 1|\right) + \frac{\hbar\Omega}{2}\left(e^{i2\omega t}|1\rangle\langle 2| + e^{-i2\omega t}|2\rangle\langle 1|\right). \end{aligned} \tag{3}$$

Hence the equation of motion for $|\psi_G(t)\rangle$ is:

$$\frac{d}{dt}|\psi_G(t)\rangle = -\frac{i}{\hbar}H_G(t)|\psi_G(t)\rangle \tag{4}$$

(b) Make the rotating wave approximation (RWA) by assuming that $\omega \approx \omega_0$, and dropping any terms that oscillate at or near $2\omega_0$. Write, in terms of the detuning $\Delta = \omega_0 - \omega$, the effective time-independent Hamiltonian, H_G, that then governs the time evolution of $|\psi_G(t)\rangle$.

Dropping the terms rotating at $\pm 2\omega$ gives:

$$H_{G,RWA} = \hbar\Delta|2\rangle\langle 2| + \frac{\hbar\Omega}{2}\left(|1\rangle\langle 2| + |2\rangle\langle 1|\right), \tag{5}$$

In terms of the Rabi Hamiltonian, $H_{Rabi} = \Delta S_z + \Omega S_x$, we have

$$H_{G,RWA} = H_{Rabi} + \frac{\hbar\Delta}{2}. \tag{6}$$

(c) Assume that the system begins at time $t = 0$ in the ground-state of H_G, and calculate $|\psi_G(t)\rangle$. Is this a stationary state in the rotating frame? Now use $|\psi_S(t)\rangle = U_{SG}(t)|\psi_G(t)\rangle$ to see what this state looks like in the Schrödinger picture. Is it a stationary state in the Schrödinger picture?

The eigenstates of H_G are those of H_{Rabi}, and the eigenvalues of H_G, are those of H_{Rabi} plus the constant shift $\frac{\hbar\Delta}{2}$.

The ground state of H_G is therefore

$$|\omega_{-G}\rangle = \frac{\left(\Omega + \sqrt{\Omega^2 + \Delta^2}\right)|1\rangle - \Delta|2\rangle}{\sqrt{2\left(\Omega^2 + \Delta^2 + \Omega\sqrt{\Omega^2 + \Delta^2}\right)}}, \tag{7}$$

which is a stationary state in the rotating frame (Recall that a stationary state evolves as $|\phi(t)\rangle = e^{-i\omega t}|\phi(0)\rangle$).

In the Schrödinger picture, this state becomes

$$|\omega_{-S}\rangle = \frac{\left(\Omega + \sqrt{\Omega^2 + \Delta^2}\right)|1\rangle - \Delta e^{-i\omega t}|2\rangle}{\sqrt{(\Omega + \sqrt{\Omega^2 + \Delta^2})^2 + \Delta^2}}, \tag{8}$$

which is not a stationary state. The point is the eigenstates of H_G are not energy eigenstates, but the problem can be solved in the rotating frame, and then transformed at the end to get the solution in the Schrödinger picture. Note that for time-dependent V, energy is not a conserved quantity. Up to a tiny correction due to the dropped terms in the RWA, $H_{G,RWA}$ is a constant of motion. This constant is often referred to as the 'quasi energy'.

(d) Assuming the system begins in the ground state of H_G, use second-order time-dependent perturbation theory to treat the fast-oscillating terms that were discarded in the RWA, and compute the probability to find the system in the excited state of H_G, at time $t > 0$.

Here we will use the tools of TDPT applied within the rotating frame:
Thus we have

$$H_0 = H_{G,RWA} \tag{9}$$

and

$$V = \frac{\hbar\Omega}{2}\left(e^{i2\omega t}|1\rangle\langle 2| + e^{-i2\omega t}|2\rangle\langle 1|\right). \tag{10}$$

The amplitude to be in state in $|\omega_{+,g}\rangle$ is given by

$$\begin{aligned} u_{+-}(t) &= -\frac{i}{\hbar}\int_0^t dt_1\, e^{-i\omega_{+G}(t-t_1)}V_{+-}(t_1)e^{-i\omega_{-G}t_1}, \\ &= -\frac{i}{\hbar}e^{-i\omega_{+G}t}\int_0^t dt_1\, e^{i(\omega_{+G}-\omega_{-G})t_1}V_{+-}(t_1). \end{aligned} \tag{11}$$

We have

$$\omega_{+G} - \omega_{-G} = \sqrt{\Omega^2+\Delta^2}, \tag{12}$$

and

$$|\omega_{+G}\rangle = \frac{\Delta|1\rangle + \left(\Omega + \sqrt{\Omega^2+\Delta^2}\right)|2\rangle}{\sqrt{(\Omega+\sqrt{\Omega^2+\Delta^2})^2+\Delta^2}}, \tag{13}$$

so that

$$\begin{aligned} V_{+-}(t) &= \frac{\hbar\Omega}{2}\left(e^{i2\omega t}\langle\omega_{+G}|1\rangle\langle 2|\omega_{-G}\rangle + e^{-i2\omega t}\langle\omega_{+G}|2\rangle\langle 1|\omega_{-G}\rangle\right) \\ &= -\frac{\hbar\Omega}{2}\frac{\Delta^2 e^{i2\omega t} - (\Omega+\sqrt{\Omega^2+\Delta^2})^2 e^{-i2\omega t}}{\Delta^2+(\Omega+\sqrt{\Omega^2+\Delta^2})^2} \end{aligned} \tag{14}$$

Thus we have

$$u_{+-}(t) = i\frac{\Omega}{2}e^{-i\sqrt{\Omega^2+\Delta^2}t/2}\int_0^t dt_1\, e^{i\sqrt{\Omega^2+\Delta^2}t_1}\frac{\Delta^2 e^{i2\omega t_1} - (\Omega+\sqrt{\Omega^2+\Delta^2})^2 e^{-i2\omega t_1}}{\Delta^2+(\Omega+\sqrt{\Omega^2+\Delta^2})^2} \tag{15}$$

The key approximation here is to assume $\omega \gg \sqrt{\Omega^2+\Delta^2}$, which gives

$$\begin{aligned} u_{+-}(t) &\approx i\frac{\Omega}{2}e^{-i\sqrt{\Omega^2+\Delta^2}t/2}\int_0^t dt_1\, \frac{\Delta^2 e^{i2\omega t_1} - (\Omega+\sqrt{\Omega^2+\Delta^2})^2 e^{-i2\omega t_1}}{\Delta^2+(\Omega+\sqrt{\Omega^2+\Delta^2})^2} \\ &\approx i\frac{\Omega}{2\omega}e^{-i\sqrt{\Omega^2+\Delta^2}t/2}\sin(\omega t)\frac{\Delta^2 e^{i\omega t} - (\Omega+\sqrt{\Omega^2+\Delta^2})^2 e^{-i\omega t}}{\Delta^2+(\Omega+\sqrt{\Omega^2+\Delta^2})^2} \end{aligned} \tag{16}$$

The transition probability is then

$$\begin{aligned} P_{+-}(t) &\approx |u_{+-}(t)|^2 \\ &\approx \frac{\Omega^2}{4\omega^2}\sin^2(\omega t)\frac{\Delta^4 - 2\Delta^2(\Omega+\sqrt{\Omega^2+\Delta^2})\cos(2\omega t) + (\Omega+\sqrt{\Omega^2+\Delta^2})^4}{\Delta^4 + 2\Delta^2(\Omega+\sqrt{\Omega^2+\Delta^2})^2 + (\Omega+\sqrt{\Omega^2+\Delta^2})^4} \end{aligned} \tag{17}$$

Time-averaging on a scale small compared to $1/\sqrt{\Omega^2+\Delta^2}$ but large compared to $1/\omega$ then gives

$$P_{+-(t)} \approx \frac{\Omega^2}{8\omega^2} \frac{\Delta^4 + (\Omega + \sqrt{\Omega^2+\Delta^2})^4}{\Delta^4 + 2\Delta^2(\Omega + \sqrt{\Omega^2+\Delta^2})^2 + (\Omega + \sqrt{\Omega^2+\Delta^2})^4} \tag{18}$$

The scaling as $\left(\frac{\Omega}{\omega}\right)^2$ is the reason why the rotating wave approximation is very accurate for the case $\Omega \ll \omega$.

(e) Assume that at time $t = 0$, we have $\Delta > 0$, $\Omega = 0$, and $|\psi_G(0)\rangle = |1\rangle$. If Ω is smoothly increased from zero to Ω_0 on time-scale $T >> 1/\Delta$, what is the state of the system at time $t = T$?

Assuming that the RWA valid, then for $\Delta > 0$ and $\Omega = 0$ the ground state of H_G is $|1\rangle$, and the energy-gap is $\hbar\Delta$.
Thus if Ω is smoothly increased on a time-scale $T \gg 1/\Delta = \hbar/E_{gap}$, then according to the Adiabatic theorem, the system will stay in the ground-state of H_G, which changes as Ω is varied. Thus the state of the system at time T is given in the rotating frame by Eq. (7), and in the Schrödinger picture, by Eq. (8).

This is interesting, because the adiabatic following was in the rotating frame, so that in the non-rotating frame, where V is oscillating in time, the system adiabatically follows a non-stationary state.

2. Consider a system described by the Hamiltonian:

$$H = -\frac{\hbar\omega}{4}\left(AA + A^\dagger A^\dagger\right), \tag{19}$$

where $[A, A^\dagger] = 1$. Find and solve the Heisenberg equations of motion for $A_H(t)$ and $A_H^\dagger(t)$. Use these solutions to compute the expectation values of X and P, as well as the variances ΔX, and ΔP, as functions of time, for the case where the initial state satisfies $A|\psi_S(0)\rangle = \alpha|\psi_S(0)\rangle$, where α is an arbitrary complex number. For the case $\alpha = 0$, show that $\langle X\rangle_t = \langle P\rangle_t = 0$, but ΔX and ΔP grow rapidly in time.

Now re-express the Hamiltonian in terms of X and P. Do your previous answers make sense given this viewpoint?

We start from

$$\frac{d}{dt}A_H = \frac{i}{\hbar}[H_H, A_H] = -i\frac{\omega}{4}[A_H^\dagger A_H^\dagger, A_H] = i\frac{\omega}{2}A_H^\dagger \tag{20}$$

Taking the Hermitian Conjugate then gives

$$\frac{d}{dt}A_H^\dagger = -i\frac{\omega}{2}A_H \tag{21}$$

Using these, we can compute the second derivative of A_H,

$$\frac{d^2}{dt^2}A_H = \frac{\omega^2}{4}A_H \tag{22}$$

The solution is therefore

$$A_H(t) = Be^{\omega t/2} + Ce^{-\omega t/2} \tag{23}$$

or equivalently

$$A_H(t) = B\cosh(\omega t/2) + C\sinh(\omega t/2) \tag{24}$$

Choosing the later form, we have the initial conditions $A_H(0) = A_S$ and $A_H^\dagger(0) = A_S^\dagger$, which gives

$$B = A_S \tag{25}$$

To satisfy Eq. (20), we need

$$C = iA_S^\dagger \tag{26}$$

so we have

$$A_H(t) = A_S\cosh(\omega t/2) + iA_S^\dagger\sinh(\omega t/2) \tag{27}$$

$$A_H^\dagger(t) = A_S^\dagger\cosh(\omega t/2) - iA_S\sinh(\omega t/2) \tag{28}$$

Now $\langle A^\dagger A\rangle = \langle\psi_S(0)|A_H^\dagger(t)A_H(t)|\psi_S(0)\rangle$ giving

$$\begin{aligned}\langle A^\dagger A\rangle &= \langle A_S^\dagger A_S\rangle\cosh^2(\omega t/2) + \langle A_S A_S^\dagger\rangle\sinh^2(\omega t/2)\\ &+ i\Big(\langle A_S^\dagger A_S^\dagger\rangle - \langle A_S A_S\rangle\Big)\cosh(\omega t/2)\sinh(\omega t/2)\end{aligned} \tag{29}$$

which with $A_s|\psi_S(0)\rangle = \alpha|\psi_S(0)\rangle$ and $A_S A_S^\dagger = 1 + A_S^\dagger A_S$, gives

$$\langle A^\dagger A\rangle = |\alpha|^2\cosh(\omega t) + 2\mathrm{Im}\{\alpha^2\}\cosh(\omega t/2)\sinh(\omega t/2) + \sinh^2(\omega t/2) \tag{30}$$

The solution in terms of X and P are

$$\begin{aligned} X_H(t) &= \frac{\lambda}{\sqrt{2}}\left(A_H(t)+A_H^\dagger(t)\right) \\ &= \frac{\lambda}{\sqrt{2}}\left[(A_S+A_S^\dagger)\cosh(\omega t/2)-i(A_S-A_S^\dagger)\sinh(\omega t/2)\right] \end{aligned} \tag{31}$$

and

$$\begin{aligned} P_H(t) &= \frac{\hbar}{\sqrt{2}i\lambda}\left(A_H(t)-A_H^\dagger(t)\right) \\ &= \frac{\hbar}{\sqrt{2}i\lambda}\left[(A_S-A_S^\dagger)\cosh(\omega t/2)+i(A_S+A_S^\dagger)\sinh(\omega t/2)\right] \end{aligned} \tag{32}$$

So that for the coherent state, we have

$$\langle X\rangle = \sqrt{2}\lambda\left[\mathrm{Re}\{\alpha\}\cosh(\omega t/2)+\mathrm{Im}\{\alpha\}\sinh(\omega t/2)\right] \tag{33}$$

and

$$\langle P\rangle = \sqrt{2}\frac{\hbar}{\lambda}\left[\mathrm{Im}\{\alpha\}\cosh(\omega t/2)+\mathrm{Re}\{\alpha\}\sinh(\omega t/2)\right] \tag{34}$$

The variances are given for the coherence state by

$$\begin{aligned} \Delta X &= \sqrt{\langle X^2\rangle-\langle X\rangle^2} \\ &= \frac{\lambda}{2}\sqrt{\cosh(\omega t)} \end{aligned} \tag{35}$$

and

$$\begin{aligned} \Delta P &= \sqrt{\langle P^2\rangle-\langle P\rangle^2} \\ &= \frac{\hbar}{2\lambda}\sqrt{\cosh(\omega t)} \end{aligned} \tag{36}$$

Thus for $\alpha = 0$, we have $\langle X\rangle = \langle P\rangle = 0$, but the variances still grow exponentially in time.

Using $A = \frac{1}{\sqrt{2}}\left(\bar{X}+i\bar{P}\right)$ and $A^\dagger = \frac{1}{\sqrt{2}}\left(\bar{X}-i\bar{P}\right)$ the Hamiltonian becomes:

$$H = \frac{\hbar\omega}{4}\left(\bar{P}^2-\bar{X}^2\right)$$

which is essentially an inverted harmonic oscillator potential. Thus the exponential growth of the operators is consistent with the unstable equilibrium. Even when the mean-values of $\bar{X}$ and $\bar{P}$ vanish, a wave-packet spreads exponentially due to the instability.

3. The Hamiltonian for a hydrogen atom is

$$H = \frac{P_r^2}{2\mu} + \frac{L^2}{2\mu R^2} - \frac{e^2}{4\pi\epsilon_0 R}. \tag{37}$$

First, use the properties

$$\langle \vec{r}|P_r^2|\psi\rangle = -\hbar^2 \frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}\langle \vec{r}|\psi\rangle, \tag{38}$$

and

$$\langle \vec{r}|P_r|\psi\rangle = -i\hbar\frac{d}{dr}\langle \vec{r}|\psi\rangle, \tag{39}$$

to compute the commutators $[P_r^2, R]$ and $[R^{-s}, P_r]$, then use these commutators to derive the Heisenberg equations of motion for R and P_r.

Since R commutes with L^2 and R^s, we have

$$\frac{d}{dt}R = \frac{i}{2\mu\hbar}[P_r^2, R] \tag{40}$$

let $[P_r^2, R] = M$, then we have

$$\begin{aligned}
\langle r|M|\psi\rangle &= \langle r|P_r^2 R|\psi\rangle - \langle r|RP_r^2|\psi\rangle \\
&= -\hbar^2\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\langle r|R|\psi\rangle - r\langle r|P_r^2|\psi\rangle \\
&= -\hbar^2\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}r - \frac{1}{r}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\right]\langle r|\psi\rangle \\
&= -2\hbar^2\left[\frac{1}{r} + \frac{\partial}{\partial r}\right]\langle r|\psi\rangle \\
&= -2\hbar^2\langle r|\left(R^{-1} + \frac{i}{\hbar}P_r\right)|\psi\rangle
\end{aligned} \tag{41}$$

so we see that

$$M = -\frac{2\hbar^2}{R} - 2i\hbar P_r \tag{42}$$

which gives

$$\frac{d}{dt}R = -\frac{i\hbar}{\mu}\frac{1}{R} + \frac{1}{\mu}P_r \tag{43}$$

Let $M_s = [R^{-s}, P_r]$, then

$$\begin{aligned}
\langle r|M_s|\psi\rangle &= \langle r|R^{-s}P_r|\psi\rangle - \langle r|P_r R^{-s}|\psi\rangle \\
&= r^{-s}\langle r|P_r|\psi\rangle + i\hbar\frac{\partial}{\partial r}\langle r|R^{-s}|\psi\rangle \\
&= -i\hbar r^{-s}\frac{\partial}{\partial r}\langle r|\psi\rangle + i\hbar\frac{\partial}{\partial r}r^{-s}\langle r|\psi\rangle \\
&= -sr^{-s-1}i\hbar\langle r|\psi\rangle
\end{aligned} \tag{44}$$

so we see that

$$M_s = i\hbar(-sR^{-s-1}), \tag{45}$$

as expected. With this, we find:

$$\begin{aligned} \frac{d}{dt}P_r &= \frac{iL^2}{2m\hbar}[R^{-2}, P_r] - \frac{ie^2}{4\pi\epsilon_0\hbar}[R^{-1}, P_r] \\ &= \frac{L^2}{m}R^{-3} - \frac{e^2}{4\pi\epsilon_0}R^{-2} \end{aligned} \tag{46}$$

Something is wrong with this result, as the equation of motion for R is not real-valued. The answer lies in our choice of P_r.

Obviously our definition of $\langle r|P_r^2|\psi\rangle$ is correct, because the radial component of the kinetic energy is well known, so the problem must lie in our treatment of $\langle r|P_r|\psi\rangle$. Now the usual radial component of the gradient is $\frac{\partial}{\partial r}$, so that if we define $P_r = \vec{e}_r \cdot \vec{P} = -i\hbar\vec{e}_r \cdot \vec{\nabla}$, we get the result $P_r = -i\hbar\frac{\partial}{\partial r}$. This is encouraging because it leads to $[R, P_r] = i\hbar$. However, as we see, it gives non-sense in the Heisenberg picture.

Another possible definition is $P_r = -i\hbar\frac{1}{r}\frac{\partial}{\partial r}r$. This is intriguing because it also satisfies $[R, P_r] = i\hbar$, as well as gives the same result for $\langle r|P_r^2|\psi\rangle$. Note that with this definition, we have

$$\langle r|P_r|\psi\rangle = -i\hbar\frac{1}{r}\frac{\partial}{\partial r}r\langle r|\psi\rangle = -i\hbar\left[\frac{1}{r} + \frac{\partial}{\partial r}\right]\langle r|\psi\rangle. \tag{47}$$

Compare this to the second-to-last line in Eq. (10), and we see that with the new definition, we have

$$\frac{d}{dt}R = \frac{1}{\mu}P_r \tag{48}$$

as expected.

Recomputing the result $[R^{-s}, P_r]$, we find

$$\begin{aligned} \langle r|[R^{-s}, P_r]|\psi\rangle &= -i\hbar\left(r^{-s-1}\frac{\partial}{\partial r}r - r^{-1}\frac{\partial}{\partial r}r^{-s+1}\right)\langle r|\psi\rangle \\ &= -i\hbar\left(r^{-s-1} + r^{-s}\frac{\partial}{\partial r} - (-s+1)r^{-s-1} - r^{-s}\frac{\partial}{\partial r}\right)\langle r|\psi\rangle \\ &= i\hbar(-sr^{-s-1})\langle r|\psi\rangle \end{aligned} \tag{49}$$

so that we have

$$[R^{-s}, P_r] = i\hbar(-sR^{-s-1}) \tag{50}$$

which is the same result as with our first definition, so that Eq. (15) is still correct.

Thus it is only by looking at the equation of motion that we discovered the correct form of P_r. Both $P_r \to -i\hbar\frac{\partial}{\partial r}$ and $P_r \to -i\hbar\frac{1}{r}\frac{\partial}{\partial r}r$ give the expectated commutation relation with R, and match the known result for the radial kinetic energy.

4. Compute the density of states, $n(E)$, for a massive particle in a cubic volume of side length L. Then compute the density of states for a two-dimensional massive particle confined to a square area of side length L, and also for a one-dimensional massive particle in an infinite square-well of width L. Then do the same for a photon, whose energy is related to its wavevector by $E(\vec{k}) = \hbar c|k|$.

To determine the density of states at energy E, first determine $N(E)$, which is the number of quantized k-values inside a sphere of radius $k(E)$. Do this by determining the volume in k-space occupied by a single state, and then divide the volume of the energy-sphere by the single-mode volume. Then compute the density of states via $n(E) = \frac{d}{dE}N(E)$.

For a massive particle, we have

$$k(E) = \frac{\sqrt{2ME}}{\hbar} \tag{51}$$

while for a photon we have

$$k(E) = \frac{E}{\hbar c} \tag{52}$$

For both particles, we can use periodic boundary conditions, so that $\vec{k} = \sum_{j=1}^{d} \frac{2\pi}{L} m_j$, where d is the dimension of the system, and m_j is any integer on $(-\infty, \infty)$.
This gives a k-space volume per state of

$$V_s = \left(\frac{2\pi}{L}\right)^d \tag{53}$$

The volume of the energy-shell is given by

$$V(E) = c_d \left(k(E)\right)^d, \tag{54}$$

where $c_1 = 2$, $c_2 = \pi$, and $c_3 = \frac{4}{3}\pi$. The total number of states with energy below E is then given by

$$N(E) = \frac{V(E)}{V_s} = c_d \left(\frac{L}{2\pi}\right)^d \left(k(E)\right)^d. \tag{55}$$

so that the density of states is

$$n(E) = \frac{d}{dE}N(E) = dc_d \left(\frac{L}{2\pi}\right)^d \left(k(E)\right)^{d-1} \frac{dk(E)}{dE} \tag{56}$$

For massive (non-relativistic) particles, we have

$$\frac{dk(E)}{dE} = \frac{1}{2E}k(E), \tag{57}$$

so that

$$n(E) = dc_d \left(\frac{\sqrt{2M}L}{2\pi\hbar}\right)^d E^{\frac{d-2}{2}} \tag{58}$$

For $d = 1, 2, 3$, we have

$$n(E) = \frac{2L}{(2\pi)\hbar}\frac{\sqrt{2M}}{\sqrt{E}}, \frac{2\pi L^2}{(2\pi)^2\hbar^2}(\sqrt{2M})^2, \frac{4\pi L^3}{(2\pi)^3\hbar^3}(\sqrt{2M})^3\sqrt{E} \tag{59}$$

which shows that in 1d, the density of states decreases with increasing E, while in 2d it is constant, and in 3d increases with E.

For a photon, we have

$$\frac{dk(E)}{dE} = \frac{1}{\hbar c} \tag{60}$$

so that

$$n(E) = dc_d\left(\frac{L}{2\pi\hbar c}\right)^d E^{d-1} \tag{61}$$

For $d = 1, 2, 3$, this gives

$$n(E) = \frac{2L}{(2\pi)\hbar}\frac{1}{c}, \frac{2\pi L^2}{(2\pi)^2\hbar^2}\frac{E}{c^2}, \frac{4\pi L^3}{(2\pi)^3\hbar^3}\frac{E^2}{c^3} \tag{62}$$

which is quite different from that of a massive particle.

5. Estimate the spontaneous photon-emission rate of an excited atom via Fermi's golden rule, use the density of states for a photon in a cube of volume V. To estimate the coupling matrix element, use the dipole energy operator $V = -\vec{d} \cdot \vec{\mathcal{E}}$, where $d = ea_0$ is the atomic dipole moment, and $\mathcal{E}$ is the electric field of a single photon in a volume V. To get the value of $\mathcal{E}$, take the photons energy to be $\hbar\omega$, and use the standard energy density of an electromagnetic field $u = \frac{\epsilon_0}{2}(\mathcal{E}^2 + c^2\mathcal{B}^2)$. Relate $\mathcal{B}$, the magnetic field of the photon, to its electric field via $\nabla \times \vec{\mathcal{E}} = -\frac{d}{dt}\vec{\mathcal{B}}$, which from dimensional analysis gives $k\mathcal{E} \approx \omega\mathcal{B}$. For frequencies in the visible spectrum, what is your estimate of Γ?

Fermi's Golden Rule tells us

$$\Gamma = \frac{2\pi}{\hbar}|V|^2 n(E) \tag{63}$$

We can estimate $|V|^2$ as $d^2\mathcal{E}^2$, where $\mathcal{E}$ is the electric field of a single-photon. The energy of an electromagnetic field is

$$E = \frac{1}{2}\left(\epsilon_0\mathcal{E}^2 + \frac{1}{\mu_0}\mathcal{B}^2\right)V, \tag{64}$$

where V is the volume The energy of a photon is $\hbar\omega$, while the magnetic field is related to the E-field by $\nabla \times \mathcal{E} = -\frac{d}{dt}\mathcal{B}$, which by units gives $k\mathcal{E} = \omega\mathcal{B}$. With $\omega = ck$, this gives

$$\hbar\omega = \frac{1}{2}\left(\epsilon_0 + \frac{1}{\mu_0 c^2}\right)\mathcal{E}^2 = \epsilon_0\mathcal{E}^2 \tag{65}$$

which leads to

$$\mathcal{E} = \sqrt{\frac{\hbar\omega}{\epsilon_0 V}} \tag{66}$$

putting this in Fermi's Golden rule, together with the density of states, $n(E) = \frac{4\pi\omega^2 V}{(2\pi)^3\hbar c^3}$, gives

$$\Gamma = \frac{2\pi}{d^2\hbar}\frac{\hbar\omega}{\epsilon_0 V}\frac{4\pi\omega^2 V}{(2\pi)^3\hbar c^3} = \frac{1}{4\pi\epsilon_0}\frac{4e^2a_0^2\omega^3}{\hbar c^3} \tag{67}$$

Frequencies in the visible spectrum are on the order $\omega \sim 10^{15}$, with $e = 1.6 \times 10^{-19}$C, $a_0 = 5.3 \times 10^{-11}$m, $\hbar = 1.0 \times 10^{-34}$J, and $\epsilon_0 = 8.9 \times 10^{-12}\frac{C}{Jm}$, we find that $\Gamma \sim 10^7$, which is what is observed experimentally. To improve upon this estimate, we have to take into account that there are two polarizations (doubles the density of states) and average over the dipole emission pattern. This leads to a correction factor of 1/3.

6. Consider a harmonic oscillator described by $H^{(0)} = \hbar\Omega(A^\dagger A + 1/2)$. Now consider two possible perturbations $V_1 = bX^4\cos(\omega t)$, and $V_2 = g\delta(X)e^{-\gamma t}$. Assume that the system begins at $t = 0$, in the ground-state of $H^{(0)}$, $|0\rangle$, and use time-dependent perturbation theory to compute the probability to second-order, for the system to be found in the the n^{th} eigenstate of $H^{(0)}$ at time t. Consider both $n = 0$ and $n \neq 0$ cases.

The formula for the first-order transition amplitude is

$$u_{fi}(t) = -\frac{i}{\hbar}\int_0^t dt_1 e^{-i\omega_f(t-t_1)} V_{fi}(t_1) e^{i\omega_i t_1}. \tag{68}$$

In the first example, we have

$$V_{n0} = b\frac{\lambda^4}{4}\langle n|(A + A^\dagger)^4|0\rangle\cos(\omega t) \tag{69}$$

For the matrix-elements, we see that the only non-zero terms are $n = 0, 2, 4$, because there are at most 4 creation operators, and to reach an odd state would require an odd number of $A^\dagger$'s plus equal numbers of additional A's and $A^\dagger$'s (which cancel each other), so that the total number of terms would need to be odd, whereas we have always four terms. For $n = 4$ and $n = 2$, we then have

$$\langle 4|(A + A^\dagger)^4|0\rangle = \langle 4|(A^\dagger)^4|0\rangle = \sqrt{4!} = 2\sqrt{6}. \tag{70}$$

$$\begin{aligned} \langle 2|(A + A^\dagger)^4|0\rangle &= \langle 2|A^\dagger A^\dagger A^\dagger A + A^\dagger A^\dagger A A^\dagger + A^\dagger A A^\dagger A^\dagger + A A^\dagger A^\dagger A^\dagger|0\rangle \\ &= 6\langle 2|A^\dagger A^\dagger|0\rangle \\ &= 6\sqrt{2} \end{aligned} \tag{71}$$

which gives us

$$\begin{aligned} u_{40}(t) &= -\frac{\sqrt{3}ib\lambda^4}{\sqrt{2}\hbar}e^{-9i\Omega t/2}\int_0^t dt_1\, e^{i4\Omega t_1}\cos(\omega t) \\ &= -\frac{\sqrt{3}ib\lambda^4}{\sqrt{2}\hbar}e^{-9i\Omega t/2}\left[e^{i(4\Omega+\omega)t/2}\frac{\sin((4\Omega+\omega)t/2)}{4\Omega+\omega} + e^{i(4\Omega-\omega)t/2}\frac{\sin((4\Omega-\omega)t/2)}{4\Omega-\omega}\right] \end{aligned} \tag{72}$$

and

$$\begin{aligned} u_{20}(t) &= -\frac{3ib\lambda^4}{\sqrt{2}\hbar}e^{-5i\Omega t/2}\int_0^t dt_1\, e^{i2\Omega t_1}\cos(\omega t) \\ &= -\frac{3ib\lambda^4}{\sqrt{2}\hbar}e^{-5i\Omega t/2}\left[e^{i(2\Omega+\omega)t/2}\frac{\sin((2\Omega+\omega)t/2)}{2\Omega+\omega} + e^{i(2\Omega-\omega)t/2}\frac{\sin((2\Omega-\omega)t/2)}{2\Omega-\omega}\right] \end{aligned} \tag{73}$$

Thus we have

$$\begin{aligned} P_{40}(t) &= |u_{40}(t)|^2 \\ &= \frac{3b^2\lambda^8}{2\hbar^2}\left[\frac{\sin^2((4\Omega+\omega)t/2)}{(4\Omega+\omega)^2} + 2\cos(\omega t/2)\frac{\sin((4\Omega+\omega)t/2)\sin((4\Omega-\omega)t/2)}{(4\Omega+\omega)(4\Omega-\omega)}\right. \\ &+ \left.\frac{\sin^2((4\Omega-\omega)t/2}{(4\Omega-\omega)^2}\right] \end{aligned} \tag{74}$$

and

$$\begin{aligned} P_{20}(t) &= |u_{20}(t)|^2 \\ &= \frac{9b^2\lambda^8}{2\hbar^2}\left[\frac{\sin^2((4\Omega+\omega)t/2)}{(4\Omega+\omega)^2} + 2\cos(\omega t/2)\frac{\sin((2\Omega+\omega)t/2)\sin((2\Omega-\omega)t/2)}{(2\Omega+\omega)(2\Omega-\omega)}\right. \\ &+ \left.\frac{\sin^2((2\Omega-\omega)t/2}{(2\Omega-\omega)^2}\right] \end{aligned} \tag{75}$$

and we can then take

$$P_{00}(t) = 1 - P_{20}(t) - P_{40}(t) \tag{76}$$

For the second problem, we have

$$V_{n0} = g\langle n|\delta(X)|0\rangle e^{-\gamma t} = g\phi_n(0)\phi_0(0)e^{-\gamma t} \tag{77}$$

which gives

$$\begin{aligned} u_{n0}(t) &= -\frac{ig\phi_n(0)\phi_0(0)}{\hbar}e^{-i(2n+1)\Omega t/2}\int_0^t dt_2\, e^{in\Omega t_1}e^{-\gamma t_1} \\ &= -\frac{ig\phi_n(0)\phi_0(0)}{\hbar}e^{-i(2n+1)\Omega t/2}\frac{e^{(in\Omega-\gamma)t}-1}{in\Omega-\gamma} \end{aligned} \tag{78}$$

and then

$$P_{n0}(t) = \frac{|g|^2|\phi_n(0)|^2|\phi_0(0)|^2}{\hbar^2}\frac{1-2\cos(n\Omega t)e^{-\gamma t}+e^{-2\gamma t}}{(n\Omega)^2+\gamma^2} \tag{79}$$

with again

$$P_{00}(t) = 1 - \sum_{n\neq 0} P_{n0}(t) \tag{80}$$

For large enough γ, we can take the limit $\gamma t \gg 1$, where we find

$$P_{00}(t) = 1 - \frac{|g|^2|\phi_0|^2}{\hbar^2}\sum_{n\neq 0}\frac{|\phi_n(0)|^2}{(n\Omega)^2+\gamma^2} \tag{81}$$