1. **Symmetry:** A quantum system is said to possess a ‘symmetry’ if the Hamiltonian operator, $H$, is invariant under the associated transformation. In other words, if $H' = H$, where $H' := U^\dagger H U$.

   (a) [5] Show that $H' = H$ is equivalent to $[H, U] = 0$

   (b) [5] Any hermitian operator can be used to generate a unitary operator via $U = e^{-iG\phi}$, where $G^\dagger = G$ is the ‘generator’ of the symmetry transformation, and $\phi$ is a free parameter. Show that $[H, G] = 0$ is necessary and sufficient for $H$ to be symmetric under $U$.

   (c) [5] Show that when $[H, G] = 0$, the probability distribution over the eigenvalues of $G$ does not change in time. In QM this means that $G$ is a ‘constant of motion’. Must a QM constant of motion have a well-defined value?

   (d) [5] What operator is the ‘generator’ of translation? If a system possesses ‘translational symmetry’ what operator is a constant of motion?

   (e) [5] Consider a particle described by the Hamiltonian

   \[ H = \frac{P^2}{2M} + V(X). \]  

What operator is the generator of translation? Show that $H$ has translational symmetry only if $V(x) = V_0$.

2. Consider a system described by the Hamiltonian

\[ H = \frac{P^2}{2M} + \frac{1}{2}M\omega^2X^2 + MgX, \]  

where $g$ has units of acceleration.

   (a) [5] Show that $U_T^\dagger(d)XU_T(d) = X + d$ and $U_T^\dagger(d)PU_T(d) = P$.

   (b) [5] Solve for $d$ and $E_0$ such that $H' := U_T^\dagger(d)H U_T(d)$ satisfies

\[ H' = E_0 + \frac{P^2}{2M} + \frac{1}{2}M\omega^2X^2 \]  

(c) [5] Let $|\phi'_n\rangle$, $n = 0, 1, 2, \ldots$ be the $n^{th}$ eigenstate of $H'$, with corresponding eigenvalue $E'_n$. What are $E'_n$ and $\phi'_n(x) = \langle x | \phi'_n \rangle$?

   (d) [5] Show that $|\phi'_n\rangle := U_T(d)|\phi'_n\rangle$ is an eigenstate of $H$ with eigenvalue $E_n$. What is the relationship between $E_n$ and $E'_n$?

   (e) [5] What is the relationship between $\phi_n(x) := \langle x | \phi_n \rangle$ and $\phi'_n(x)$? What is $\phi_n(x)$?

3. [10] Show explicitly that the momentum operator of a particle $\vec{P}$ is a vector operator with respect to rotation. Show that the operator $P^2 = \vec{P} \cdot \vec{P}$ is invariant under rotation about any axis (hint: chose a coordinate system where the axis of rotation is the z-axis).
4. \[\text{[40/35]}\] Consider an infinitesimal rotation about an arbitrary axis, described by the unitary operator

\[U_R(\vec{\epsilon}) = e^{-\frac{i}{\hbar}\vec{L}\vec{\epsilon}} = 1 - \frac{i}{\hbar} L_1 \epsilon_1 - \frac{i}{\hbar} L_2 \epsilon_2 - \frac{i}{\hbar} L_3 \epsilon_3. \tag{4}\]

where \(\vec{L} = \sum_j L_j \vec{e}_j\) and \(\vec{\epsilon} = \sum_j \epsilon_j \vec{e}_j\), with \(\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}\) being a right-handed set of orthogonal unit vectors. Using this notation, the angular momentum components are given by \(L_j = \sum_{k, \ell} \epsilon_{j, k, \ell} R_k P_\ell\), with \(\epsilon_{j, k, \ell}\) being the totally antisymmetric Levi-Cevita tensor.

The components of \(\vec{R}\) and \(\vec{P}\) satisfy the commutation relation \([R_j, P_k] = i\hbar \delta_{jk}\).

(a) \[\text{[10]}\] Evaluate \(R_j' = U_R^1(\vec{\epsilon}) R_j U_R(\vec{\epsilon})\) for each component of the position operator \(\vec{R} = \sum_j R_j \vec{e}_j\), and use this to deduce the \(3 \times 3\) matrix, \(M(\vec{\epsilon})\) that rotates an ordinary vector by the infinitesimal angle \(\vec{\epsilon}\).

(b) \[\text{[5]}\] Show that \(M(-\vec{\epsilon}) = M^T(\vec{\epsilon})\), then show that \(M^T(\vec{\epsilon}) M(\vec{\epsilon}) = I\).

(c) \[\text{[5]}\] Now consider a finite rotation by \(\vec{\delta} = \sum_j \delta_j \vec{e}_j\), described by the \(3 \times 3\) matrix \(M(\vec{\delta})\). Clearly we must have \(M(\vec{\delta}) = M^N(\vec{\delta}/N)\). Take the limit as \(N \to \infty\), and use your result to part (a) to show that we can put \(M(\vec{\delta})\) into the form:

\[M(\vec{\delta}) = \lim_{N \to \infty} \left(1 - \frac{1}{N} \Lambda(\vec{\delta})\right)^N = e^{-\Lambda(\vec{\delta})} \tag{6}\]

where \(\Lambda(\vec{\delta})\) is a \(3 \times 3\) antisymmetric matrix, whose components are given by \(\Lambda_{j,k}(\vec{\delta}) = \sum_\ell \epsilon_{j, k, \ell} \delta_\ell\).

(d) \[\text{[5]}\] Show that the eigenvalues of \(\Lambda(\vec{\delta})\) are \(\omega_0 = 0\), and \(\omega_\pm = \pm i \delta\), where \(\delta = |\vec{\delta}|\).

(e) Show that the eigenvectors of \(\Lambda(\vec{\delta})\) are

\[\vec{u}_0 = \frac{\vec{\delta}}{\delta} \tag{7}\]

\[\vec{u}_\pm = \frac{(\delta_1 \delta_2 \pm i \delta \delta_3) \vec{e}_1 + (\delta_3^2 - \delta^2) \vec{e}_2 + (\delta_2 \delta_3 \mp i \delta \delta_1) \vec{e}_3}{\sqrt{2 \delta^2 (\delta^2 - \delta_2^2)}} \tag{8}\]

(f) \[\text{[5]}\] Based on your result to part (e), show that

\[M(\vec{\delta}) \vec{V} = \vec{u}_0 (\vec{u}_0 \cdot \vec{V}) + \vec{u}_- e^{i\delta}(\vec{u}_+ \cdot \vec{V}) + \vec{u}_+ e^{-i\delta}(\vec{u}_- \cdot \vec{V}) \tag{9}\]

where \(\vec{V}\) is an arbitrary vector.

(g) \[\text{[5+5 bonus]}\] Based on your results to parts (e) and (f), show that

\[\vec{V}' = U_R^1(\vec{\delta}) \vec{V} U_R(\vec{\delta}) = M(\vec{\delta}) \vec{V} = \frac{\vec{\delta} \cdot \vec{V}}{\delta^2} + \left[\vec{V} - \frac{\vec{\delta} \cdot \vec{V}}{\delta^2}\right] \cos(\delta) + \frac{\vec{\delta} \times \vec{V}}{\delta} \sin(\delta) \tag{10}\]