PHYS852 Quantum Mechanics II, Spring 2010
HOMEWORK ASSIGNMENT 3

Topics covered: Unitary transformations, translation, rotation, vector operators

1. [25]**Symmetry**: A quantum system is said to posses a 'symmetry' if the Hamiltonian operator, H, is invariant under the associated transformation. In other words, if $H' = H$, where $H' := U^\dagger H U$.

   (a) [5] Show that $H' = H$ is equivalent to $[H, U] = 0$
   Start with $U^\dagger H U = H$ Hit from the left with U and use $UU^\dagger = I$ to get $HU = UH$. Put both terms on the l.h.s. to get $HU - UH = 0$ or equivalently $[H, U] = 0$.

   (b) [5] Any hermitian operator can be used to generate a unitary operator via $U = e^{-iG\phi}$, where $G^\dagger = G$ is the 'generator' of the symmetry transformation, and ϕ is a free parameter. Show that $[H, G] = 0$ is necessary and sufficient for H to be symmetric under U.
   If $[H, G] = 0$ then it follows that $[H, f(G)] = 0$ for any single-variable function $f(x)$. As U is of this form, it follows that $[H, U] = 0$ so that H is symmetric with respect to the U.

   (c) [5] Show that when $[H, G] = 0$, the probability distribution over the eigenvalues of G does not change in time. In QM this means that G is a 'constant of motion'. Must a QM constant of motion have a well-defined value?
   Since $[H, G] = 0$ it follows that simultaneous eigenstates of H and G exist. We can label them $|n, g\rangle$ so that $H|n, g\rangle = E_n|n, g\rangle$ and $G|n, g\rangle = g|n, g\rangle$. The most general state is then $|\psi(t)\rangle = \sum_n \sum_g c_{n,g}(t)|n, g\rangle$. From Schrödinger's equation we find that

$$\begin{aligned}\frac{d}{dt}c_{n,g}(t) &= \frac{d}{dt}\langle n, g|\psi(t)\rangle \\ &= -\frac{i}{\hbar}\langle n, g|H|\psi(t)\rangle \\ &= -i\omega_n c_{n,g} \end{aligned} \qquad (1)$$

   so that $c_{n,g}(t) = e^{-i\omega_n t}c_{n,g}(0)$, which gives

$$|\psi(t)\rangle = \sum_n \sum_g c_{n,m}(0)e^{-i\omega_n t}|n, g\rangle \qquad (2)$$

   The projector onto the subspace with eigenvalue g is $I(g) = \sum_n |n, g\rangle\langle n, g|$, so that the probability for the system to be in an eigenstate of G with eigenvalue g is

$$\begin{aligned}P(g, t)) &= \langle\psi(t)|I(g)|\psi(t)\rangle \\ &= \sum_{n,n',n''}\sum_{g',g''} c^*_{n',g'}(0)e^{i\omega_{n'}t}c_{n'',g''}(0)e^{-i\omega_{n''}t}\langle n', g'|n, g\rangle\langle n, g|n'', g''\rangle \\ &= \sum_{n,n',n''}\sum_{g',g''} c^*_{n',g'}(0)e^{i\omega_{n'}t}c_{n'',g''}(0)e^{-i\omega_{n''}t}\delta_{n',n}\delta_{g',g}\delta_{n'',n}\delta_{g'',g} \\ &= \sum_n |c_{n,g}(0)|^2 \end{aligned} \qquad (3)$$

   which we see is independent of time.

   (d) [5] If a system possesses 'translational symmetry' what operator is a constant of motion?
   The translation operator is $U_T(d) = e^{-\frac{i}{\hbar}dP}$ so that P is the generator of translation. Thus in a system with translational symmetry, momentum will be conserved.

(e) [5] Consider a particle described by the Hamiltonian

$$H = \frac{P^2}{2M} + V(X). \tag{4}$$

What operator is the generator of translation? Show that H has translational symmetry only if $V(x) = V_0$.

For H to possess translational symmetry required $[H, P] = 0$. This then requires $[V(X), P] = 0$. We know that $[V(X), P] = i\hbar V'(X)$, so translational symmetry requires $\frac{d}{dx}V(x) = 0$, or equivalently $V(x) = V_0$.

2. [25] Consider a system described by the Hamiltonian

$$H = \frac{P^2}{2M} + \frac{1}{2}M\omega^2 X^2 + MgX, \tag{5}$$

where g has units of acceleration.

(a) [5] Show that $U_T^\dagger(d) X U_T(d) = X + d$ and $U_T^\dagger(d) P U_T(d) = P$.
We have

$$\begin{aligned} X' &= U_T^\dagger(d) X U_T(d) \\ &= \int dx U_T^\dagger(d)|x\rangle x\langle x|U_T(d) \\ &= \int dx |x-d\rangle x \langle x-d| \\ &= \int dx |x\rangle (x+d) \langle x| \\ &= \int dx |x\rangle x \langle x| + d \int dx |x\rangle\langle x| \\ &= X + d \end{aligned} \tag{6}$$

We know that $P' = P$ because $[U_T(d), P] = 0$, given that $U_T(d)$ is a function of P.

(b) [5] Solve for d and E_0 such that $H' := U_T^\dagger(d) H U_T(d)$ satisfies

$$H' = E_0 + \frac{P^2}{2M} + \frac{1}{2}M\omega^2 X^2 \tag{7}$$

We start from

$$\begin{aligned} H' &= U_T^\dagger(d) H U_T(d) \\ &= \frac{U_T^\dagger(d) P U_T(d) U_T^\dagger(d) P U_T(d)}{2M} + \frac{1}{2}M\omega^2 U_T^\dagger(d) X U_T(d) U_T^\dagger(d) X U_T(d) + Mg U_T^\dagger(d) X U_T(d) \\ &= \frac{P'^2}{2M} + \frac{1}{2}M\omega^2 X'^2 + MgX' \\ &= \frac{P^2}{2M} + \frac{1}{2}M\omega^2 (X+d)^2 + MG(X+d) \\ &= \frac{P^2}{2M} + \frac{1}{2}M\omega^2 X^2 + (M\omega^2 d + MG)X + \frac{1}{2}M\omega^2 d^2 + MGd \end{aligned} \tag{8}$$

Therefore the linear term will cancel for

$$d = -\frac{G}{\omega^2} \tag{9}$$

giving

$$\begin{aligned} H' &= \frac{P^2}{2M} + \frac{1}{2}M\omega^2 X^2 + \frac{1}{2}\frac{MG^2}{\omega^2} - \frac{MG^2}{\omega^2} \\ &= \frac{P^2}{2M} + \frac{1}{2}M\omega^2 X^2 - \frac{MG^2}{2\omega^2} \end{aligned} \tag{10}$$

so we see that

$$E_0 = -\frac{MG^2}{2\omega^2} \tag{11}$$

(c) [5] Let $|\phi'_n\rangle$, $n = 0, 1, 2, \dots$ be the n^{th} eigenstate of H', with corresponding eigenvalue E'_n. What are E'_n and $\phi'_n(x) = \langle x|\phi'_n\rangle$?

We have

$$H'|\phi'_n\rangle = E'_n|\phi'_n\rangle \tag{12}$$

Because H' is just an SHO plus a constant, we know that

$$E'_n = \hbar\omega\left(n + \frac{1}{2}\right) - \frac{MG^2}{2\omega^2} \tag{13}$$

As a constant shift in the zero-point energy doesn't change the shape of the wavefunction, we have

$$\phi'_n(x) = \frac{1}{\sqrt{\sqrt{\pi}2^n n!\lambda}} H_n\left(x/\lambda\right) e^{-\frac{1}{2}(x/\lambda)^2} \tag{14}$$

where $\lambda = \sqrt{\frac{\hbar}{M\omega}}$.

(d) [5] Show that $|\phi_n\rangle := U_T(d)|\phi'_n\rangle$ is an eigenstate of H with eigenvalue E_n. What is the relationship between E_n and E'_n?

$$\begin{aligned} H|\phi_n\rangle &= HU_T(d)|\phi'_n\rangle \\ &= U_T(d)U_T^\dagger(d)HU_T(d)|\phi'_n\rangle \\ &= U_T(d)H'|\phi'_n\rangle \\ &= U_T(d)E'_n|\phi'_n\rangle \\ &= E'_nU_T(d)|\phi'_n\rangle \\ &= E'_n|\phi_n\rangle \end{aligned} \tag{15}$$

with the definition $H|\phi_n\rangle = E_n|\phi_n\rangle$ we see that $E_n = E'_n$.

(e) [5] What is the relationship between $\phi_n(x) := \langle x|\phi_n\rangle$ and $\phi'_n(x)$? What is $\phi_n(x)$?

$$\begin{aligned} \phi_n(x) &= \langle x|\phi_n\rangle \\ &= \langle x|U_T(d)|\phi'_n\rangle \\ &= \langle x - d|\phi'_n\rangle \\ &= \phi'_n(x - d) \end{aligned} \tag{16}$$

this gives

$$\phi_n(x) = \frac{1}{\sqrt{\sqrt{\pi}2^n n!\lambda}} H_n\left((x + G/\omega^2)/\lambda\right) e^{-\frac{1}{2}\left((x+G/\omega^2)/\lambda\right)^2} \tag{17}$$

3. [10] Show explicitly that the momentum operator of a particle $\vec{P}$ is a vector operator with respect to rotation. Show that the operator $P^2 = \vec{P} \cdot \vec{P}$ is invariant under rotation about any axis (hint: chose a coordinate system where the axis of rotation is the z-axis).
To show that $\vec{P}$ is a vector, we can consider an infinitesimal rotation about the z axis.

$$\begin{aligned} P'_x &= e^{\frac{i}{\hbar}\epsilon L_z} P_x e^{-\frac{i}{\hbar}\epsilon L_z} \\ &= \left(1 + \frac{i}{\hbar}\epsilon(XP_y - YP_x)\right) P_x \left(1 - \frac{i}{\hbar}\epsilon(ZP_y - YP_x)\right) \\ &= P_x + \frac{i}{\hbar}\epsilon\,[XP_y - YP_x, P_x] \\ &= P_x - \epsilon P_y \end{aligned} \tag{18}$$

$$\begin{aligned} P'_y &= e^{\frac{i}{\hbar}\epsilon L_z} P_y e^{-\frac{i}{\hbar}\epsilon L_z} \\ &= \left(1 + \frac{i}{\hbar}\epsilon(XP_y - YP_x)\right) P_y \left(1 - \frac{i}{\hbar}\epsilon(ZP_y - YP_x)\right) \\ &= P_y + \frac{i}{\hbar}\epsilon\,[XP_y - YP_x, P_y] \\ &= P_y + \epsilon P_x \end{aligned} \tag{19}$$

$$\begin{aligned} P'_z &= e^{\frac{i}{\hbar}\epsilon L_z} P_z e^{-\frac{i}{\hbar}\epsilon L_z} \\ &= \left(1 + \frac{i}{\hbar}\epsilon(XP_y - YP_x)\right) P_z \left(1 - \frac{i}{\hbar}\epsilon(ZP_y - YP_x)\right) \\ &= P_z + \frac{i}{\hbar}\epsilon\,[XP_y - YP_x, P_z] \\ &= P_z \end{aligned} \tag{20}$$

so that

$$\begin{pmatrix} P'_x \\ P'_y \\ P'_z \end{pmatrix} = \begin{pmatrix} 1 & -\epsilon & 0 \\ \epsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix} \tag{21}$$

for a finite rotation this becomes

$$\begin{pmatrix} P'_x \\ P'_y \\ P'_z \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix} \tag{22}$$

which is the same as

$$\vec{P}' = M(\theta)\vec{P} \tag{23}$$

$$\begin{aligned} P^{2\prime} &= {P'_x}^2 + {P'_y}^2 + {P'_z}^2 \\ &= (\cos\theta P_x - \sin\theta P_y)^2 + (\cos\theta P_y + \sin\theta P_x)^2 + P_z^2 \\ &= \cos^2\theta P_x^2 - \cos\theta\sin\theta(P_xP_y + P_yP_x) + \sin^2\theta P_y^2 + \cos^2\theta P_y^2 + \cos\theta\sin\theta(P_yP_x + P_xP_y) \\ &+ \sin^2\theta P_x^2 + P_z^2 \\ &= (\cos^2\theta + \sin^2\theta)(P_x^2 + P_y^2) + P_z^2 \\ &= P_x^2 + P_y^2 + P_z^2 \\ &= P^2 \end{aligned} \tag{24}$$

4. [40/35] Consider an infinitesimal rotation about an arbitrary axis, described by the unitary operator

$$U_R(\vec{\epsilon}) = e^{-\frac{i}{\hbar}\vec{L}\cdot\vec{\epsilon}} = 1 - \frac{i}{\hbar}L_1\epsilon_1 - \frac{i}{\hbar}L_2\epsilon_2 - \frac{i}{\hbar}L_3\epsilon_3. \tag{25}$$

where $\vec{L} = \sum_j L_j\vec{e}_j$ and $\vec{\epsilon} = \sum_j \epsilon_j\vec{e}_j$, with $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ being a right-handed set of orthogonal unit vectors. Using this notation, the angular momentum components are given by $L_j = \sum_{k,\ell} \epsilon_{j,k,\ell} R_k P_\ell$, with $\epsilon_{j,k,\ell}$ being the totally antisymmetric Levi-Cevita tensor,

$$\epsilon_{jk\ell} = \begin{cases} 0; & \text{any index repeated} \\ 1; & \text{cyclic permutations of } \{j,k,\ell\} = \{1,2,3\} \\ -1; & \text{cyclic permutations of } \{j,k,\ell\} = \{3,2,1\} \end{cases} . \tag{26}$$

The components of $\vec{R}$ and $\vec{P}$ satisfy the commutation relation $[R_j, P_k] = i\hbar\delta_{j,k}$.

(a) [10] Evaluate $R'_j = U_R^\dagger(\vec{\epsilon})R_jU_R(\vec{\epsilon})$ for each component of the position operator $\vec{R} = \sum_j R_j\vec{e}_j$, and use this to deduce the 3×3 matrix, $M(\vec{\epsilon})$ that rotates an ordinary vector by the infinitesimal angle $\vec{\epsilon}$.

In the infinitesimal limit, we have $U_R(\vec{\epsilon}) = 1 - \frac{i}{\hbar}\sum_j \epsilon_j L_j$, so that

$$\begin{aligned} R'_j &= \left(1 + \frac{i}{\hbar}\sum_k \epsilon_k L_k\right) R_j \left(1 + \frac{i}{\hbar}\sum_\ell \epsilon_\ell L_\ell\right) \\ &= R_j + \frac{i}{\hbar}\sum_k \epsilon_k \left[L_k, R_j\right] \\ &= R_j + \frac{i}{\hbar}\sum_{k\ell m} \epsilon_k \epsilon_{k\ell m} \left[R_\ell P_m, R_j\right] \\ &= R_j + \frac{i}{\hbar}\sum_{k\ell m} \epsilon_k \epsilon_{k\ell m} R_\ell \left(-i\hbar\right) \delta_{m,j} \\ &= R_j + \sum_{k\ell} \epsilon_k \epsilon_{k\ell j} R_\ell \\ &= \sum_\ell \left(\delta_{j,\ell} + \sum_k \epsilon_k \epsilon_{k\ell j}\right) R_\ell \end{aligned} \tag{27}$$

This tells us that

$$\begin{aligned} M_{jk} &= \delta_{jk} + \sum_\ell \epsilon_{\ell k j}\epsilon_\ell \\ &= \delta_{jk} - \sum_\ell \epsilon_{jk\ell}\epsilon_\ell \end{aligned} \tag{28}$$

so that

$$M(\vec{\epsilon}) = \begin{pmatrix} 1 & -\epsilon_3 & \epsilon_2 \\ \epsilon_3 & 1 & -\epsilon_1 \\ -\epsilon_2 & \epsilon_1 & 1 \end{pmatrix} \tag{29}$$

(b) [5] Show that $M(-\vec{\epsilon}) = M^T(\vec{\epsilon})$, then show that $M^T(\vec{\epsilon}) = M^{-1}(\vec{\epsilon})$ by showing that $M^T(\vec{\epsilon})M(\vec{\epsilon}) = I$.

$$\begin{aligned} M_{jk}(-\vec{\epsilon}) &= \delta_{jk} + \sum_\ell \epsilon_{jk\ell}\epsilon_\ell \\ &= \delta_{kj} - \sum_\ell \epsilon_{kj\ell}\epsilon_\ell \\ &= M_{kj}(\vec{\epsilon}) \\ &= M^T_{jk}(\vec{\epsilon}) \end{aligned} \tag{30}$$

$$\begin{aligned} \left(M^T(\vec{\epsilon})M(\vec{\epsilon})\right)_{jk} &= \sum_\ell M^T_{j\ell}(\vec{\epsilon})M_{\ell k}(\vec{\epsilon}) \\ &= \sum_\ell \left(\delta_{j\ell} + \sum_m \epsilon_{j\ell m}\epsilon_m\right)\left(\delta_{\ell k} - \sum_n \epsilon_{\ell kn}\epsilon_n\right) \\ &= \sum_\ell \delta_{j\ell}\delta_{\ell k} - \sum_{\ell n} \delta_{j\ell}\epsilon_{\ell kn}\epsilon_n + \sum_{\ell m} \epsilon_{j\ell m}\epsilon_m\delta_{\ell k} \\ &= \delta_{jk} - \sum_n \epsilon_{jkn}\epsilon_n + \sum_m \epsilon_{jkm}\epsilon_m \\ &= \delta_{jk} \end{aligned} \tag{31}$$

(c) [5] Now consider a finite rotation by $\vec{\delta} = \sum_j \delta_j \vec{e}_j$, described by the 3×3 matrix $M(\vec{\delta})$. Clearly we must have $M(\vec{\delta}) = M^N(\vec{\delta}/N)$. Take the limit as $N \to \infty$, and use your result to part (a) to show that we can put $M(\vec{\delta})$ into the form:

$$M(\vec{\delta}) = \lim_{N\to\infty}\left(1 - \frac{1}{N}\Lambda(\vec{\delta})\right)^N = e^{-\Lambda(\delta)} \tag{32}$$

where $\Lambda(\vec{\delta})$ is a 3×3 antisymmetric matrix, whose components are given by $\Lambda_{j,k}(\vec{\delta}) = \sum_\ell \epsilon_{j,k,\ell}\delta_\ell$.

$$M(\vec{\delta}) = \left(M\left(\vec{\delta}/N\right)\right)^N \tag{33}$$

where

$$\begin{aligned} M_{jk}(\vec{\delta}/N) &= \delta_{jk} - \sum_\ell \epsilon_{jk\ell}\frac{\delta_\ell}{N} \\ &= \delta_{jk} - \frac{1}{N}\sum_\ell \epsilon_{jk\ell}\delta_\ell \\ &= \delta_{jk} - \frac{1}{N}\Lambda_{jk}(\vec{\delta}) \end{aligned} \tag{34}$$

(d) [5] Show that the eigenvalues of $\Lambda(\vec{\delta})$ are $\omega_0 = 0$, and $\omega_\pm = \pm i\delta$, where $\delta = |\vec{\delta}|$.
The eigenvalues of $\Lambda(\vec{\delta})$ are solutions to

$$\det \begin{vmatrix} -\omega & \delta_3 & -\delta_2 \\ -\delta_3 & -\omega & \delta_1 \\ \delta_2 & -\delta_1 & -\omega \end{vmatrix} = 0 \tag{35}$$

which gives the characteristic equation

$$-\omega^3 - \omega\delta_1^2 + \delta_1\delta_2\delta_3 - \omega\delta_3^2 - \delta_1\delta_2\delta_3 - \omega\delta_2^2 = 0 \tag{36}$$

which simplifies to

$$\omega\left(\omega^2 + \delta^2\right) = 0 \tag{37}$$

so that the solutions are $\omega_0 = 0$, and $\omega_\pm = \pm i\delta$.

(e) Show that the eigenvectors of $\Lambda(\vec{\delta})$ are

$$\begin{aligned} \vec{u}_0 &= \frac{\vec{\delta}}{\delta} && (38) \\ \vec{u}_\pm &= \frac{(\delta_1\delta_2 \pm i\delta\delta_3)\vec{e}_1 + (\delta_2^2 - \delta^2)\vec{e}_2 + (\delta_2\delta_3 \mp i\delta\delta_1)\vec{e}_3}{\sqrt{2\delta^2(\delta^2 - \delta_2^2)}} && (39) \end{aligned}$$

$$\begin{aligned} \left(\Lambda(\vec{\delta})\vec{u}_0 - \omega_0\vec{u}_0\right)_j &= \sum_k \Lambda_{jk}(\vec{\delta})u_{0,k} \\ &= \sum_{k\ell} \epsilon_{jk\ell}\delta_\ell u_{0,k} \\ &= \frac{1}{\delta}\sum_{k\ell} \epsilon_{jk\ell}\delta_\ell\delta_k \\ &= \left(\vec{\delta} \times \vec{\delta}\right)_j \\ &= 0 \end{aligned} \tag{40}$$

$$\begin{aligned} \left(\Lambda(\vec{\delta})\vec{u}_\pm - \omega_\pm\vec{u}_\pm\right)_j &= \sum_k \left(\Lambda_{jk} - \omega_\pm\delta_{jk}\right)u_{\pm,k} \\ &= \sum_{k\ell} \epsilon_{jk\ell}u_{\pm,k}\delta_\ell \mp i\delta u_{\pm,j} \\ &= \left(\vec{u}_\pm \times \vec{\delta} \mp i\delta\vec{u}_\pm\right)_j \\ &= 0 \end{aligned} \tag{41}$$

where we have used

$$\begin{aligned} \vec{u}_\pm \times \vec{\delta} &= \vec{e}_1\left(u_{\pm,2}\delta_3 - \delta_2 u_{\pm,3}\right) + \vec{e}_2\left(u_{\pm,3}\delta_1 - \delta_3 u_{\pm,1}\right) + \vec{e}_3\left(u_{\pm,1}\delta_2 - \delta_1 u_{\pm,2}\right) \\ &= \vec{e}_1\left(\delta_2^2\delta_3 - \delta^2\delta_3 - \delta_2^2\delta_3 \pm i\delta\delta_1\delta_2\right) + \vec{e}_2\left(\delta_1\delta_2\delta_3 \mp i\delta\delta_1^2 - \delta_1\delta_2\delta_3 \mp i\delta\delta_3^2\right) \\ &+ \vec{e}_3\left(\delta_1\delta_2^2 \pm i\delta\delta_2\delta_3 - \delta_1\delta_2^2 + \delta^2\delta_1\right) \\ &= \pm i\delta\left[\vec{e}_1\left(\delta_1\delta_2 \pm i\delta\delta_3\right) + \vec{e}_2\left(-\delta_1^2 - \delta_3^2\right) + \vec{e}_3\left(\delta_2\delta_3 \mp i\delta\delta_1\right)\right] \\ &= \pm i\delta\vec{u}_\pm \end{aligned} \tag{42}$$

(f) [5] Based on your result to part (e), show that

$$M(\vec{\delta})\vec{V} = \vec{u}_0(\vec{u}_0 \cdot \vec{V}) + \vec{u}_- e^{i\delta}(\vec{u}_+ \cdot \vec{V}) + \vec{u}_+ e^{-i\delta}(\vec{u}_- \cdot \vec{V}) \tag{43}$$

where $\vec{V}$ is an arbitrary vector.
We start from

$$M(\vec{\delta}) = e^{-\Lambda(\vec{\delta})} \tag{44}$$

Using Dirac notation, we have

$$\Lambda(\vec{\delta}) = |u_0\rangle\omega_0\langle u_0| + |u_+\rangle\omega_+\langle u_+| + |u_-\rangle\omega_-\langle u_-| \tag{45}$$

so that

$$M(\vec{\delta}) = |u_0\rangle e^{-\omega_0}\langle u_0| + |u_+\rangle e^{-\omega_+}\langle u_+| + |u_-\rangle e^{-\omega_-}\langle u_-| \tag{46}$$

switching back to standard notation, this gives

$$M(\vec{\delta})\vec{V} = \vec{u}_0\left(\vec{u}_0^* \cdot \vec{V}\right) + \vec{u}_+ e^{-i\delta}\left(\vec{u}_+^* \cdot \vec{V}\right) + \vec{u}_- e^{i\delta}\left(\vec{u}_-^* \cdot \vec{V}\right) \tag{47}$$

noting that $\vec{u}_0^* = \vec{u}_0$, and $\vec{u}_\pm^* = \vec{u}_\mp$, this gives

$$M(\vec{\delta})\vec{V} = \vec{u}_0\left(\vec{u}_0 \cdot \vec{V}\right) + \vec{u}_+ e^{-i\delta}\left(\vec{u}_- \cdot \vec{V}\right) + \vec{u}_- e^{i\delta}\left(\vec{u}_+ \cdot \vec{V}\right) \tag{48}$$

(g) [5+5 bonus] Based on your results to parts (e) and (f), show that

$$\vec{V}' = U_R^\dagger(\vec{\delta})\vec{V}U_R(\vec{\delta}) = M(\vec{\delta})\vec{V} = \frac{\vec{\delta}(\vec{\delta}\cdot\vec{V})}{\delta^2} + \left[\vec{V} - \frac{\vec{\delta}(\vec{\delta}\cdot\vec{V})}{\delta^2}\right]\cos(\delta) + \frac{\vec{\delta}\times\vec{V}}{\delta}\sin(\delta) \tag{49}$$

$$\begin{aligned}
\vec{V}' &= M(\vec{\delta})\vec{V} \\
&= \frac{\vec{\delta}\left(\vec{\delta}\cdot\vec{V}\right)}{\delta^2} + \cos(\delta)\left[\vec{u}_+\left(\vec{u}_-\cdot\vec{V}\right) + \vec{u}_-\left(\vec{u}_+\cdot\vec{V}\right)\right] \\
&- i\sin(\delta)\left[\vec{u}_+\left(\vec{u}_-\cdot\vec{V}\right) - \vec{u}_-\left(\vec{u}_+\cdot\vec{V}\right)\right]
\end{aligned} \tag{50}$$

$$\vec{u}_+\cdot\vec{V} = \frac{(\delta_1\delta_2 + i\delta\delta_3)V_1 + (\delta_2^2 - \delta^2)V_2 + (\delta_2\delta_3 - i\delta\delta_1)V_3}{\sqrt{2\delta^2(\delta^2 - \delta_2^2)}} \tag{51}$$

$$\vec{u}_-\cdot\vec{V} = \frac{(\delta_1\delta_2 - i\delta\delta_3)V_1 + (\delta_2^2 - \delta^2)V_2 + (\delta_2\delta_3 + i\delta\delta_1)V_3}{\sqrt{2\delta^2(\delta^2 - \delta_2^2)}} \tag{52}$$

$$\begin{aligned}
\left[\vec{u}_+\left(\vec{u}_-\cdot\vec{V}\right) + \vec{u}_-\left(\vec{u}_+\cdot\vec{V}\right)\right]_1 &= \frac{(\delta_1\delta_2 + i\delta\delta_3)\left[(\delta_1\delta_2 - i\delta\delta_3)V_1 + (\delta_2^2 - \delta^2)V_2 + (\delta_2\delta_3 + i\delta\delta_1)V_3\right]}{2\delta^2(\delta^2 - \delta_2^2)} \\
&+ \frac{(\delta_1\delta_2 - i\delta\delta_3)\left[(\delta_1\delta_2 + i\delta\delta_3)V_1 + (\delta_2^2 - \delta^2)V_2 + (\delta_2\delta_3 - i\delta\delta_1)V_3\right]}{2\delta^2(\delta^2 - \delta_2^2)} \\
&= \frac{(\delta_1^2\delta_2^2 + \delta^2\delta_3^2)}{\delta^2(\delta_1^2 + \delta_3^2)}V_1 - \frac{\delta_1\delta_2}{\delta^2}V_2 + \frac{\delta_1\delta_2^2\delta_3 - \delta_1\delta^2\delta_3}{\delta^2(\delta^2 - \delta_2^2)}V_3 \\
&= \frac{\delta^2(\delta_1^2 + \delta_3^2) - \delta_1^2(\delta_1^2 + \delta_3^2)}{\delta^2(\delta_1^2 + \delta_3^2)}V_1 - \frac{\delta_1\delta_2}{\delta^2}V_2 - \frac{\delta_1\delta_3}{\delta^2}V_3 \\
&= V_1 - \frac{\delta_1}{\delta^2}\left(\vec{\delta}\cdot\vec{V}\right)
\end{aligned} \tag{53}$$

$$\left[\vec{u}_+\left(\vec{u}_-\cdot\vec{V}\right)-\vec{u}_-\left(\vec{u}_+\cdot\vec{V}\right)\right]_1 = \frac{(\delta_1\delta_2+i\delta\delta_3)\left[(\delta_1\delta_2-i\delta\delta_3)V_1+(\delta_2^2-\delta^2)V_2+(\delta_2\delta_3+i\delta\delta_1)V_3\right]}{2\delta^2(\delta^2-\delta_2^2)}$$
$$- \frac{(\delta_1\delta_2-i\delta\delta_3)\left[(\delta_1\delta_2+i\delta\delta_3)V_1+(\delta_2^2-\delta^2)V_2+(\delta_2\delta_3-i\delta\delta_1)V_3\right]}{2\delta^2(\delta^2-\delta_2^2)}$$
$$= i\frac{\delta_3(\delta_2^2-\delta^2)}{\delta(\delta^2-\delta_2^2)}V_2+i\frac{\delta_1^2\delta_2+\delta_2\delta_3^2}{\delta(\delta^2-\delta_2^2)}V_3$$
$$= -i\frac{\delta_3V_2}{\delta}+i\frac{\delta_2V_3}{\delta}$$
$$= \frac{i}{\delta}\left(\vec{\delta}\times\vec{V}\right)_1 \quad (54)$$

$$\left[\vec{u}_+\left(\vec{u}_-\cdot\vec{V}\right)+\vec{u}_-\left(\vec{u}_+\cdot\vec{V}\right)\right]_2 = -\frac{(\delta^2-\delta_2^2)\left[(\delta_1\delta_2-i\delta\delta_3)V_1+(\delta_2^2-\delta^2)V_2+(\delta_2\delta_3+i\delta\delta_1)V_3\right]}{2\delta^2(\delta^2-\delta_2^2)}$$
$$- \frac{(\delta^2-\delta_2^2)\left[(\delta_1\delta_2+i\delta\delta_3)V_1+(\delta_2^2-\delta^2)V_2+(\delta_2\delta_3-i\delta\delta_1)V_3\right]}{2\delta^2(\delta^2-\delta_2^2)}$$
$$= -\frac{\delta_1\delta_2}{\delta^2}V_1+\frac{\delta^2-\delta_2^2}{\delta^2}V_2-\frac{\delta_2\delta_3}{\delta^2}V_3$$
$$= V_2-\frac{\delta_2}{\delta^2}\left(\vec{\delta}\cdot\vec{V}\right) \quad (55)$$

$$\left[\vec{u}_+\left(\vec{u}_-\cdot\vec{V}\right)-\vec{u}_-\left(\vec{u}_+\cdot\vec{V}\right)\right]_2 = -\frac{(\delta^2-\delta_2^2)\left[(\delta_1\delta_2-i\delta\delta_3)V_1+(\delta_2^2-\delta^2)V_2+(\delta_2\delta_3+i\delta\delta_1)V_3\right]}{2\delta^2(\delta^2-\delta_2^2)}$$
$$+ \frac{(\delta^2-\delta_2^2)\left[(\delta_1\delta_2+i\delta\delta_3)V_1+(\delta_2^2-\delta^2)V_2+(\delta_2\delta_3-i\delta\delta_1)V_3\right]}{2\delta^2(\delta^2-\delta_2^2)}$$
$$= i\frac{\delta_3V_1}{\delta}-i\frac{\delta_1V_3}{\delta}$$
$$= \frac{i}{\delta}\left(\vec{\delta}\times\vec{V}\right)_2 \quad (56)$$

$$\left[\vec{u}_+\left(\vec{u}_-\cdot\vec{V}\right)+\vec{u}_-\left(\vec{u}_+\cdot\vec{V}\right)\right]_3 = \frac{(\delta_1\delta_2-i\delta\delta_3)\left[(\delta_1\delta_2-i\delta\delta_3)V_1+(\delta_2^2-\delta^2)V_2+(\delta_2\delta_3+i\delta\delta_1)V_3\right]}{2\delta^2(\delta^2-\delta_2^2)}$$
$$+ \frac{(\delta_1\delta_2+i\delta\delta_3)\left[(\delta_1\delta_2+i\delta\delta_3)V_1+(\delta_2^2-\delta^2)V_2+(\delta_2\delta_3-i\delta\delta_1)V_3\right]}{2\delta^2(\delta^2-\delta_2^2)}$$
$$= V_3-\frac{\delta_3}{\delta^2}\left(\vec{\delta}\cdot\vec{V}\right) \quad (57)$$

$$\left[\vec{u}_+\left(\vec{u}_-\cdot\vec{V}\right)-\vec{u}_-\left(\vec{u}_+\cdot\vec{V}\right)\right]_3 = \frac{(\delta_1\delta_2-i\delta\delta_3)\left[(\delta_1\delta_2-i\delta\delta_3)V_1+(\delta_2^2-\delta^2)V_2+(\delta_2\delta_3+i\delta\delta_1)V_3\right]}{2\delta^2(\delta^2-\delta_2^2)}$$
$$- \frac{(\delta_1\delta_2+i\delta\delta_3)\left[(\delta_1\delta_2+i\delta\delta_3)V_1+(\delta_2^2-\delta^2)V_2+(\delta_2\delta_3-i\delta\delta_1)V_3\right]}{2\delta^2(\delta^2-\delta_2^2)}$$
$$= \frac{i}{\delta}\left(\vec{\delta}\times\vec{V}\right)_3 \quad (58)$$

Putting these pieces together gives

$$\vec{V}' = \frac{\vec{\delta}\left(\vec{\delta} \cdot \vec{V}\right)}{\delta^2} + \left[\vec{V} - \frac{\vec{\delta}\left(\vec{\delta} \cdot \vec{V}\right)}{\delta^2}\right] \cos(\delta) + \frac{\vec{\delta} \times \vec{V}}{\delta} \sin(\delta) \tag{59}$$