PHYS852 Quantum Mechanics II, Spring 2010
HOMEWORK ASSIGNMENT 4

Topics covered: rotation with spin, exchange symmmetry

1. A vector pointing in the θ, ϕ direction, can be formed by starting with a vector pointing along $\vec{e}_z$, then applying an active rotation by θ about the y-axis, followed by a rotation by ϕ about the z-axis.

   (a) Verify this for an ordinary vector, by starting with the vector $(0, 0, 1)^T$ and using

$$R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}; \qquad R_z(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

   (b) Thus for a spin-1/2 system, the spin-up state with respect to the θ, ϕ direction can be found in the basis of S_z eigenstates, by starting with the spin-up state along $\vec{e}_z$, and applying unitary rotation operators, i.e.

$$|\uparrow_{\theta\phi}\rangle = e^{-\frac{i}{\hbar}\phi S_z} e^{-\frac{i}{\hbar}\theta S_y} |\uparrow_z\rangle. \tag{2}$$

   In this way, find the states $|\uparrow_{\theta\phi}\rangle$ and $|\downarrow_{\theta\phi}\rangle$.

   (c) Compute the operator $S_{\theta\phi}$ using unitary rotation operators to transform S_z, and compare it to the result using the 3×3 rotation matrices.

   (d) Using your results from parts (b) and (c), show explicitly that $S_{\theta\phi}|\uparrow_{\theta\phi}\rangle = \frac{\hbar}{2}|\uparrow_{\theta\phi}\rangle$ and $S_{\theta\phi}|\downarrow_{\theta\phi}\rangle = -\frac{\hbar}{2}|\downarrow_{\theta\phi}\rangle$.

2. **The Bloch Sphere**: The most-general spin-1/2 state is

$$|\psi\rangle = c_\uparrow|\uparrow_z\rangle + c_\downarrow|\downarrow_z\rangle, \tag{3}$$

   where $c_\uparrow$ and $c_\downarrow$ are c-numbers. This state is subject to the constraint $|c_\uparrow|^2 + |c_\downarrow|^2 = 1$, and is defined only up to a global phase-factor. This means that it only requires two real numbers to specify a spin-1/2 state. The state $|\uparrow_{\theta\phi}\rangle$ from problem 1 has two free real-valued parameters. This means that every possible spin-1/2 state must be spin-up with respect to some axis. Determine the axis angles, (θ, ϕ), for a state of the form (3).

   The dynamical evolution of a spin-1/2 state can therefore be viewed as the motion of a single point on a sphere of unit radius, known as the Bloch sphere, i.e. the state $|\psi(t)\rangle = c_\uparrow(t)|\uparrow_z\rangle + c_\downarrow(t)|\downarrow_z\rangle$ maps onto the coordinate $(\theta(t), \phi(t))$. Describe the trajectory on the Bloch sphere of an arbitrary initial state, subject to the Hamiltonian

$$H = \omega S_z. \tag{4}$$

   In addition, find the constant of motion, and express it in the form $f(\theta(t), \phi(t)) = f(\theta(0), \phi(0))$.

3. Consider two identical spin-1/2 particles in a one-dimensional Harmonic oscillator potential, so that

$$H = H_1 + H_2 \tag{5}$$

with

$$H_j = \frac{P_j^2}{2M} + \frac{1}{2}M\omega^2 X_j^2 \tag{6}$$

(a) Show that H, H_1 and H_2 form a set of 3 commuting observables, so that simultaneous eigenstates of H, H_1 and H_2 exist. Label these states as $|n_1, n_2\rangle$ where $H_j|n_1, n_2\rangle = E_{n_j}|n_1, n_2\rangle$, and $H|n_1, n_2\rangle = (E_{n_1} + E_{n_2})|n_1, n_2\rangle$. Does the set $\{|n_1, n_1\rangle\}$ form a complete basis for the two-particle orbital Hilbert space?

(b) Switch to relative and center-of-mass coordinates, by expressing the operators X_1, X_2, P_1, and P_2, in terms of the operators X_{CM}, X, P_{CM} and P. Show that H separates as $H = H_{CM}(X_{CM}, P_{CM}) + H_r(X, P)$. Show that H, H_{CM} and H_r all commute, so that simultaneous eigenvalues of H, H_{CM} and H_r exist. Label these states as $|N, n\rangle$, where $H_{CM}|N, n\rangle = E_N|N, n\rangle$, $H_r|N, n\rangle = E_n|N, n\rangle$, and $H|N, n\rangle = (E_N + E_n)|N, n\rangle$. Does the set of states $\{|N, n\rangle\}$ form a compete basis for the two-particle orbital Hilbert space?

(c) Let $X_j|x_1, x_2\rangle = x_j|x_1, x_2\rangle$, $X_{CM}|x_{CM}, x\rangle = x_{CM}|x_{CM}, x\rangle$, and $X|x_{CM}, x\rangle = x|x_{CM}, x\rangle$. The exchange operator is defined by $P_{1,2}|x_1, x_2\rangle = |x_2, x_1\rangle$. What is $P_{12}|x_{CM}, x\rangle$?

(d) Show that the states $|n_1, n_2\rangle$ are in general not eigenstates of the exchange operator, but that the states $|N, n\rangle$ are. What is the exchange eigenvalue of the state $|N, n\rangle$?

(e) If the two particles are in a spin-singlet state, which of the $|N, n\rangle$ states are forbidden? Which are forbidden for the spin-triplet state?

(f) Assume a zero-range interaction of the form $V(x_1, x_2) = g\delta(x_1 - x_2)$. For the 'repulsive' case, $g > 0$ will the true ground state be a singlet or triplet state? What about for 'attractive' interactions, i.e. $g < 0$?