PHYS852 Quantum Mechanics II, Spring 2010
HOMEWORK ASSIGNMENT 6: Solutions

Topics covered: Time-independent perturbation theory.

1. [30] **Two-Level System**: Consider the system described by $H = \delta S_z + \Omega S_x$, with $\delta > 0$, where S_x and S_z are components of the spin vector of an $s = 1/2$ particle. Treat the S_z term as the bare Hamiltonian.

   (a) [15] Use perturbation theory to compute the eigenvalues and eigenvectors of H. Compute all terms up to fourth-order in Ω.
   The bare eigenstates are

$$|1^{(0)}\rangle = |\downarrow\rangle \quad (1)$$
$$|2^{(0)}\rangle = |\uparrow\rangle \quad (2)$$

with bare eigenvalues

$$E_1^{(0)} = -\hbar\delta/2 \quad (3)$$
$$E_2^{(0)} = \hbar\delta/2 \quad (4)$$

we therefore need to iterate the following equations up to $j = 4$:

$$E_n^{(j)} = \langle n^{(0)}|V|n^{(j-1)}\rangle - \sum_{k=1}^{j-1} E_n^{(k)}\langle n^{(0)}|n^{(j-k)}\rangle \quad (5)$$

$$|n^{(j)}\rangle = |n^{(0)}\rangle\left[-\frac{1}{2}\sum_{k=1}^{j-1}\langle n^{(k)}|n^{(j-k)}\rangle\right] - \sum_{m\neq n}|m^{(0)}\rangle\left[\frac{\langle m^{(0)}|V|n^{(j-1)}\rangle}{E_{mn}} - \sum_{k=1}^{j-1}\frac{E_n^{(k)}}{E_{mn}}\langle m^{(0)}|n^{(j-k)}\rangle\right] \quad (6)$$

for $j = 1$ we have

$$E_n^{(1)} = \Omega\langle n^{(0)}|S_x|n^{(0)}\rangle = 0 \quad (7)$$

and for the states we have

$$|n^{(1)}\rangle = -\sum_{m\neq n}|m^{(0)}\rangle\frac{\Omega}{E_{mn}}\langle m^{(0)}|S_x|n^{(0)}\rangle = -\sum_{m\neq n}\frac{\hbar\Omega}{2E_{mn}}|m^{(0)}\rangle \quad (8)$$

which gives

$$|1^{(1)}\rangle = -\frac{\Omega}{2\delta}|\uparrow\rangle \quad (9)$$

$$|2^{(1)}\rangle = \frac{\Omega}{2\delta}|\downarrow\rangle \quad (10)$$

for $j = 2$ we have

$$E_n^{(2)} = \Omega\langle n^{(0)}|S_x|n^{(1)}\rangle - E_n^{(1)} = \Omega\langle n^{(0)}|S_x|n^{(1)}\rangle \quad (11)$$

which gives

$$E_1^{(2)} = -\frac{\Omega^2}{2\delta}\langle\downarrow|S_x|\uparrow\rangle = -\frac{\hbar\Omega^2}{4\delta} \quad (12)$$

$$E_2^{(2)} = \frac{\Omega^2}{2\delta}\langle\uparrow|S_x|\downarrow\rangle = \frac{\hbar\Omega^2}{4\delta} \tag{13}$$

and for the states

$$\begin{aligned} |n^{(2)}\rangle &= |n^{(0)}\rangle\left[-\frac{1}{2}\langle n^{(1)}|n^{(1)}\rangle\right] - \sum_{m\neq n}|m^{(0)}\rangle\left[\frac{\Omega}{E_{mn}}\langle m^{(0)}|S_x|n^{(1)}\rangle - \frac{E_n^{(1)}}{E_{mn}}\langle m^{(0)}|n^{(1)}\rangle\right] \\ &= -\frac{\Omega^2}{8\delta^2}|n^{(0)}\rangle \end{aligned} \tag{14}$$

For $j = 3$ we have for the energies

$$E_n^{(3)} = \Omega\langle n^{(0)}|S_x|n^{(2)}\rangle - E_n^{(1)}\langle n^{(0)}|n^{(2)}\rangle - E_n^{(2)}\langle n^{(0)}|n^{(1)}\rangle = 0 \tag{15}$$

and for the states we have

$$\begin{aligned} |n^{(3)}\rangle &= -\frac{1}{2}|n^{(0)}\rangle\left[\langle n^{(1)}|n^{(2)}\rangle + \langle n^{(2)}|n^{(1)}\rangle\right] \\ &- \sum_{m\neq n}|m^{(0)}\rangle\left[\frac{\Omega}{E_{mn}}\langle m^{(0)}|S_x|n^{(2)}\rangle - \frac{E_n^{(1)}}{E_{mn}}\langle m^{(0)}|n^{(2)}\rangle - \frac{E_n^{(2)}}{E_{mn}}\langle m^{(0)}|n^{(1)}\rangle\right] \\ &= \sum_{m\neq n}|m^{(0)}\rangle\left[\frac{\hbar\Omega^3}{16\delta^2 E_{mn}} + \frac{E_n^{(2)}}{E_{mn}}\langle m^{(0)}|n^{(1)}\rangle\right] \end{aligned} \tag{16}$$

which gives

$$|1^{(3)}\rangle = |\uparrow\rangle\left[\frac{\Omega^3}{16\delta^3} + \frac{\Omega^3}{8\delta^3}\right] = \frac{3\Omega^3}{16\delta^3}|\uparrow\rangle \tag{17}$$

$$|2^{(3)}\rangle = |\downarrow\rangle\left[-\frac{\Omega^3}{16\delta^3} - \frac{\Omega^3}{8\delta^3}\right] = -\frac{3\Omega^3}{16\delta^3}|\downarrow\rangle \tag{18}$$

And finally, for $j = 4$, we have

$$\begin{aligned} E_n^{(4)} &= \Omega\langle n^{(0)}|S_x|n^{(3)}\rangle - E_n^{(1)}\langle n^{(0)}|n^{(3)}\rangle - E_n^{(2)}\langle n^{(0)}|n^{(2)}\rangle - E_n^{(3)}\langle n^{(0)}|n^{(1)}\rangle \\ &= \Omega\langle n^{(0)}|S_x|n^{(3)}\rangle + \frac{\Omega^2 E_n^{(2)}}{8\delta^2} \end{aligned} \tag{19}$$

which gives

$$E_1^{(4)} = \frac{3\hbar\Omega^4}{32\delta^3} - \frac{\hbar\Omega^4}{32\delta^3} = \frac{\hbar\Omega^4}{16\delta^3} \tag{20}$$

$$E_2^{(4)} = -\frac{3\hbar\Omega^4}{32\delta^3} + \frac{\hbar\Omega^4}{32\delta^3} = -\frac{\hbar\Omega^4}{16\delta^3} \tag{21}$$

and for the states, we have

$$\begin{aligned} |n^{(4)}\rangle &= -\frac{1}{2}|n^{(0)}\rangle\left[\langle n^{(1)}|n^{(3)}\rangle + \langle n^{(2)}|n^{(2)}\rangle + \langle n^{(3)}|n^{(1)}\rangle\right] \\ &- \sum_{m\neq n}|m^{(0)}\rangle\left[\frac{\Omega}{E_{mn}}\langle m^{(0)}|S_x|n^{(3)}\rangle - \frac{E_n^{(1)}}{E_{mn}}\langle m^{(0)}|n^{(3)}\rangle - \frac{E_n^{(2)}}{E_{mn}}\langle m^{(0)}|n^{(2)}\rangle - \frac{E_n^{(3)}}{E_{mn}}\langle m^{(0)}|n^{(1)}\rangle\right] \\ &= |n^{(0)}\rangle\left[-\langle n^{(1)}|n^{(3)}\rangle - \frac{\Omega^4}{128\delta^4}\right] \end{aligned} \tag{22}$$

which gives

$$|1^{(4)}\rangle = |\downarrow\rangle \left[\frac{3\Omega^4}{32\delta^4} - \frac{\Omega^4}{128\delta^4}\right] = \frac{11\Omega^4}{128\delta^4}|\downarrow\rangle \tag{23}$$

$$|2^{(4)}\rangle = |\uparrow\rangle \left[\frac{3\Omega^4}{32\delta^4} - \frac{\Omega^4}{128\delta^4}\right] = \frac{11\Omega^4}{128\delta^4}|\uparrow\rangle \tag{24}$$

Putting the pieces together gives

$$E_1 \approx -\frac{\hbar\delta}{2} - \frac{\hbar\Omega^2}{4\delta} + \frac{\hbar\Omega^4}{16\delta^3} \tag{25}$$

$$E_2 \approx \frac{\hbar\delta}{2} + \frac{\hbar\Omega^2}{4\delta} - \frac{\hbar\Omega^4}{16\delta^3} \tag{26}$$

and

$$|1\rangle \approx |\downarrow\rangle \left[1 - \frac{\Omega^2}{8\delta^2} + \frac{11\Omega^4}{128\delta^4}\right] + |\uparrow\rangle \left[-\frac{\Omega}{2\delta} + \frac{3\Omega^3}{16\delta^3}\right] \tag{27}$$

$$|2\rangle \approx |\downarrow\rangle \left[\frac{\Omega}{2\delta} - \frac{3\Omega^3}{16\delta^3}\right] + |\uparrow\rangle \left[1 - \frac{\Omega^2}{8\delta^2} + \frac{11\Omega^4}{128\delta^4}\right] \tag{28}$$

(b) [5] Expand the exact eigenvalues and eigenvectors around $\Omega = 0$ and compare to the perturbation theory results.

The exact results are

$$E_1 = -\frac{\hbar}{2}\sqrt{\delta^2 + \Omega^2} \approx -\frac{\hbar\delta}{2} - \frac{\hbar\Omega^2}{4\delta} + \frac{\hbar\Omega^4}{16\delta^3} \tag{29}$$

$$E_2 = \frac{\hbar}{2}\sqrt{\delta^2 + \Omega^2} \approx \frac{\hbar\delta}{2} + \frac{\hbar\Omega^2}{4\delta} - \frac{\hbar\Omega^4}{16\delta^3} \tag{30}$$

$$|1\rangle = \frac{(\delta + \sqrt{(\delta^2 + \Omega^2)})|\downarrow\rangle - \Omega|\uparrow\rangle}{\sqrt{(\delta + \sqrt{\delta^2 + \Omega^2})^2 + \Omega^2}} \approx |\downarrow\rangle \left[1 - \frac{\Omega^2}{8\delta^2} + \frac{11\Omega^4}{128\delta^4}\right] + |\uparrow\rangle \left[-\frac{\Omega}{2\delta} + \frac{3\Omega^3}{16\delta^3}\right] \tag{31}$$

$$|2\rangle = \frac{\omega|\downarrow\rangle + (\delta + \sqrt{\delta^2 + \Omega^2})|\uparrow\rangle}{\sqrt{\Omega^2 + (\delta + \sqrt{\delta^2 + \Omega^2})^2}} \approx |\downarrow\rangle \left[\frac{\Omega}{2\delta} - \frac{3\Omega^3}{16\delta^3}\right] + |\uparrow\rangle \left[1 - \frac{\Omega^2}{8\delta^2} + \frac{11\Omega^4}{128\delta^4}\right] \tag{32}$$

(c) [10] Verify that the states computed in (a) are normalized to unity and orthogonal up to fourth-order.

$$\begin{aligned}\langle 1|1\rangle &= \left[1 - \frac{\Omega^2}{8\delta^2} + \frac{11\Omega^4}{128\delta^4}\right]^2 + \left[-\frac{\Omega}{2\delta} + \frac{3\Omega^3}{16\delta^3}\right]^2 + O(\Omega^5) \\ &= 1 - \frac{\Omega^2}{4\delta^2} + \frac{11\Omega^4}{64\delta^4} + \frac{\Omega^4}{64\delta^4} + \frac{\Omega^2}{4\delta^2} - \frac{3\Omega^4}{16\delta^4} + O(\Omega^5) \\ &= 1 + O(\Omega^5)\end{aligned} \tag{33}$$

$$\begin{aligned}\langle 2|2\rangle &= \left[\frac{\Omega}{2\delta} - \frac{3\Omega^3}{16\delta^3}\right]^2 + \left[1 - \frac{\Omega^2}{8\delta^2} + \frac{11\Omega^4}{128\delta^4}\right]^2 + O(\Omega^5) \\ &= \frac{\Omega^2}{4\delta^2} - \frac{3\Omega^4}{16\delta^4} + 1 - \frac{\Omega^2}{4\delta^2} + \frac{11\Omega^4}{64\delta^4} + \frac{\Omega^4}{64\delta^4} + O(\Omega^5) \\ &= 1 + O(\Omega^5)\end{aligned} \tag{34}$$

It is clear by inspection of Eqs. (27) and (28) that $\langle 1|2\rangle = 0$.

2. [20] **Resonance-frequency shifts**: Consider a system with a 3-dimensional Hilbert space spanned by states $|a\rangle$, $|b\rangle$, and $|c\rangle$. In the basis $\{|a\rangle, |b\rangle, |c\rangle\}$, let the bare Hamiltonian of the system be

$$H_0 = \hbar\Delta \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 3 \end{pmatrix}. \tag{35}$$

For the case where the system is perturbed by the operator

$$V = \hbar\chi \begin{pmatrix} 1 & 1 & 0 \\ 1 & -3 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \tag{36}$$

also given in $\{|a\rangle, |b\rangle, |c\rangle\}$ basis. Calculate the *shifts* in the *resonance frequencies* of the full system relative to those of the unperturbed system, to second-order in χ.

The eigenvalues of H_0 are the solutions to $\det|H_0 - \omega| = 0$, which yields the characteristic equation

$$(1-\omega)\left[(1-\omega)(3-\omega) - 1\right] - \left[-1 + 3 - \omega\right] + \left[1 - (1-\omega)\right] = 0 \tag{37}$$

which simplifies to

$$(1-\omega)\omega(\omega - 4) = 0 \tag{38}$$

so that the eigenvalues of H_0 are $E_1^{(0)} = 0$, $E_2^{(0)} = \hbar\Delta$, and $E_3^{(0)} = 4\hbar\Delta$.
The three bare resonance frequencies are therefore

$$\omega_1^{(0)} = \frac{E_2^{(0)} - E_1^{(0)}}{\hbar} = \Delta, \tag{39}$$

$$\omega_2^{(0)} = \frac{E_3^{(0)} - E_2^{(0)}}{\hbar} = 3\Delta, \tag{40}$$

and

$$\omega_3^{(0)} = \frac{E_3^{(0)} - E_1^{(0)}}{\hbar} = 4\Delta. \tag{41}$$

The eigenvectors satisfy

$$\begin{pmatrix} 1 - \frac{E_n^{(0)}}{\hbar\Delta} & -1 & 1 \\ -1 & 1 - \frac{E_n^{(0)}}{\hbar\Delta} & -1 \\ 1 & -1 & 3 - \frac{E_n^{(0)}}{\hbar\Delta} \end{pmatrix} \begin{pmatrix} \langle a|n^{(0)}\rangle \\ \langle b|n^{(0)}\rangle \\ \langle c|n^{(0)}\rangle \end{pmatrix} = 0 \tag{42}$$

taking $\langle a|n^{(0)}\rangle = 1$ gives for $n = 1$,

$$-\langle b|1^{(0)}\rangle + \langle c|1^{(0)}\rangle = -1 \tag{43}$$

$$-\langle b|1^{(0)}\rangle + 3\langle c|1^{(0)}\rangle = -1 \tag{44}$$

which has the solution $\langle b|1^{(0)}\rangle = 1$ and $\langle c|1^{(0)}\rangle = 0$, so that

$$|1^{(0)}\rangle = \frac{1}{\sqrt{2}}\left[|a\rangle + |b\rangle\right] \tag{45}$$

for $n = 2$ the equations are

$$-\langle b|2^{(0)}\rangle + \langle c|2^{(0)}\rangle = 0 \tag{46}$$

$$-\langle c|2^{(0)}\rangle = 1 \tag{47}$$

so we see that

$$|2^{(0)}\rangle = \frac{1}{\sqrt{3}}\left[|a\rangle - |b\rangle - |c\rangle\right] \tag{48}$$

and for $n = 3$, we have

$$-\langle b|3^{(0)}\rangle + \langle c|3^{(0)}\rangle = 3 \tag{49}$$

$$-3\langle b|3^{(0)}\rangle - \langle c|3^{(0)}\rangle = 1 \tag{50}$$

which gives

$$|3^{(0)}\rangle = \frac{1}{\sqrt{6}}\left[|a\rangle - |b\rangle + 2|c\rangle\right] \tag{51}$$

Switching to the bare-eigenstate basis, we have

$$H_0 = \hbar\Delta \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \tag{52}$$

and

$$V = \begin{pmatrix} \langle 1^{(0)}|V|1^{(0)}\rangle & \langle 1^{(0)}|V|2^{(0)}\rangle & \langle 1^{(0)}|V|3^{(0)}\rangle \\ \langle 2^{(0)}|V|1^{(0)}\rangle & \langle 2^{(0)}|V|2^{(0)}\rangle & \langle 2^{(0)}|V|3^{(0)}\rangle \\ \langle 3^{(0)}|V|1^{(0)}\rangle & \langle 3^{(0)}|V|2^{(0)}\rangle & \langle 3^{(0)}|V|3^{(0)}\rangle \end{pmatrix} \tag{53}$$

we have

$$V|1^{(0)}\rangle = \frac{\hbar\chi}{\sqrt{2}} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \tag{54}$$

$$V|2^{(0)}\rangle = \frac{\hbar\chi}{\sqrt{3}} \begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix} \tag{55}$$

and

$$V|3^{(0)}\rangle = \frac{\hbar\chi}{\sqrt{6}} \begin{pmatrix} 0 \\ 6 \\ 3 \end{pmatrix} \tag{56}$$

from which we can obtain

$$V = \sqrt{\frac{3}{2}}\hbar\chi \begin{pmatrix} 0 & 1 & \sqrt{2} \\ 1 & 0 & -\sqrt{3} \\ \sqrt{2} & -\sqrt{3} & 0 \end{pmatrix} \tag{57}$$

The first-order energy shifts are zero, while the second-order terms are given by

$$E_1^{(2)} = -\frac{|V_{21}|^2}{\hbar\Delta} - \frac{|V_{31}|^2}{4\hbar\Delta} = -\frac{3\hbar\chi^2}{2\Delta} - \frac{6\hbar\chi^2}{8\Delta} = -\frac{9\hbar\chi^2}{4\Delta} \tag{58}$$

$$E_2^{(2)} = \frac{|V_{12}|^2}{\hbar\Delta} - \frac{|V_{32}|^2}{3\hbar\Delta} = \frac{3\hbar\chi^2}{2\Delta} - \frac{9\hbar\chi^2}{6\Delta} = 0 \tag{59}$$

$$E_3^{(2)} = \frac{|V_{13}|^2}{4\hbar\Delta} + \frac{|V_{23}|^2}{3\hbar\Delta} = \frac{6\hbar\chi^2}{8\Delta} + \frac{9\hbar\chi^2}{6\Delta} = \frac{9\hbar\chi^2}{4\Delta} \tag{60}$$

Thus the resonance frequency shifts are therefore

$$\omega_1^{(3)} = \frac{E_2^{(2)} - E_1^{(2)}}{\hbar} = \frac{9\chi^2}{4\Delta}, \tag{61}$$

$$\omega_2^{(2)} = \frac{E_3^{(2)} - E_2^{(2)}}{\hbar} = \frac{9\chi^2}{4\Delta}, \tag{62}$$

and

$$\omega_3 = \frac{E_3^{(2)} - E_1^{(2)}}{\hbar} = \frac{9\chi^2}{2\Delta}. \tag{63}$$

3. [15] Consider a pair of quantum harmonic oscillators, described by the bare Hamiltonian

$$H_0 = \hbar\omega(A^\dagger A + 1/2) + \hbar\Omega(B^\dagger B + 1/2).$$

Assume that $\omega < \Omega < 2\omega$, and determine the three lowest bare energy eigenvalues and eigenvectors. Consider the perturbation

$$V = \hbar g \left(A^\dagger A^\dagger B + B^\dagger A A \right).$$

Show that two of the three lowest levels are exact eigenstates of $H = H_0 + V$. For the remaining bare level, compute the first non-vanishing corrections to the eigenvalue and eigenvector.

The three lowest energy levels are $|1^{(0)}\rangle = |00\rangle$ with energy $E_1^{(0)} = \hbar\frac{\omega+\Omega}{2}$, $|2^{(0)}\rangle = |10\rangle$, with energy $E_2^{(0)} = \hbar\frac{3\omega+\Omega}{2}$, and $|3^{(0)}\rangle = |01\rangle$ with energy $E_3^{(0)} = \hbar\frac{\omega+3\Omega}{2}$.

We have $V|1^{(0)}\rangle = 0$, and $V|2^{(0)}\rangle = 0$ so that $|1^{(0)}\rangle$ and $|2^{(0)}\rangle$ are eigenstates of V with eigenvalue 0. For level $|3\rangle$ we have $V|3^{(0)}\rangle = \hbar\sqrt{2}g|20\rangle$. This means that

$$E_3^{(1)} = 0 \tag{64}$$

so that

$$E_3^{(2)} = -\frac{2\hbar g^2}{(2\omega - \Omega)} \tag{65}$$

and

$$|3^{(1)}\rangle = -|20\rangle\frac{\sqrt{2}g}{(2\omega - \Omega)} \tag{66}$$

are the first non-vanishing corrections.

4. [15] Consider a particle of mass M confined to a 1-dimensional box of length L. Use perturbation theory to caculate the effects of adding a tilt to the box, represented by adding the linear potenital

$$V_{tilt}(x) = \hbar\beta\left(\frac{x}{L} - \frac{1}{2}\right)$$

to the box potential,

$$V_{box}(x) = \begin{cases} 0; & 0 < x < L \\ \infty; & \text{else} \end{cases}$$

Calculate the three lowest perturbed eigenstates to first-order and their corresponding eigenvalues to second-order.

We have

$$E_1^{(0)} = \frac{\hbar^2\pi^2}{2ML^2} \tag{67}$$

$$E_2^{(0)} = \frac{4\hbar^2\pi^2}{2ML^2} \tag{68}$$

and

$$E_3^{(0)} = \frac{9\hbar^2\pi^2}{2ML^2} \tag{69}$$

As well as

$$\langle x|1^{(0)}\rangle = \sqrt{\frac{2}{L}}\sin(\pi x/L) \tag{70}$$

$$\langle x|2^{(0)}\rangle = \sqrt{\frac{2}{L}}\sin(2\pi x/L) \tag{71}$$

and

$$\langle x|3^{(0)}\rangle = \sqrt{\frac{2}{L}}\sin(3\pi x/L) \tag{72}$$

The first-order energy shifts are all zero because $V_{tile}(x)$ is odd with respect to the center of the box. We will need the matrix elements

$$\begin{aligned} V_{mn} &= \frac{2\hbar\beta}{L}\int_0^L dx\,\sin(m\pi x/L)\left(\frac{x}{L} - \frac{1}{2}\right)\sin(n\pi x/L) \\ &= 2\hbar\beta\int_0^1 dy\,\sin(m\pi y)\left(y - \frac{1}{2}\right)\sin(n\pi y) \\ &= \frac{4\hbar\beta mn\left((-1)^{m+n} - 1\right)}{(m^2 - n^2)^2\pi^2} \end{aligned} \tag{73}$$

The first-order states are

$$|1^{(1)}\rangle = -\sum_{n=2}^{\infty}|n^{(0)}\rangle\frac{V_{n1}}{E_n^{(0)} - E_1^{(0)}} = \frac{32\beta ML^2}{\hbar\pi^4}\sum_{m=1}^{\infty}|2m^{(0)}\rangle\frac{m}{((2m)^2 - 1)^3} \tag{74}$$

$$|2^{(1)}\rangle = -\sum_{n\neq 2}|n^{(0)}\rangle\frac{V_{n2}}{E_n^{(0)} - E_2^{(0)}} = \frac{32\beta ML^2}{\hbar\pi^4}\sum_{m=1}^{\infty}|m^{(0)}\rangle\frac{2m - 1}{((2m - 1)^2 - 4)^3} \tag{75}$$

$$|3^{(0)}\rangle = -\sum_{n\neq 3}|n^{(0)}\rangle\frac{V_{n3}}{E_n^{(0)} - E_e^{(0)}} = \frac{96\beta ML^2}{\hbar\pi^4}\sum_{m=1}^{\infty}|m^{(0)}\rangle\frac{m}{(2m)^2 - 9)^3} \tag{76}$$

and the corresponding first-order energies are

$$E_1^{(2)} = \langle 1^{(0)}|V|1^{(1)}\rangle = -\frac{128\beta^2 ML^2}{\pi^6}\sum_{m=1}^{\infty}\frac{(2m)^2}{((2m)^2-1)^5} = -\frac{(15-\pi^2)\beta^2 ML^2}{24\pi^4} \tag{77}$$

$$E_2^{(2)} = \langle 2^{(0)}|V|2^{(1)}\rangle = -\frac{512\beta^2 ML^2}{\pi^6}\sum_{m=1}^{\infty}\frac{(2m-1)^2}{((2m-1)^2-4)^5} = \frac{(4\pi^2-15)\beta^2 ML^2}{384\pi^4} \tag{78}$$

$$E_3^{(2)} = \langle 3^{(0)}|V|3^{(1)}\rangle = -\frac{1152\beta^2 ML^2}{\pi^6}\sum_{m=1}^{\infty}\frac{(2m)^2}{((2m)^2-9)^5} = \frac{(9\pi^2-15)\beta^2 ML^2}{1944\pi^4} \tag{79}$$

The results are then

$$E_1 \approx \frac{\pi^2\hbar^2}{2ML^2} - \frac{(15-\pi^2)\beta^2 ML^2}{24\pi^4} \tag{80}$$

$$E_2 \approx \frac{4\pi^2\hbar^2}{2ML^2} + \frac{(4\pi^2-15)\beta^2 ML^2}{384\pi^4} \tag{81}$$

$$E_3 \approx= \frac{9\pi^2\hbar^2}{2ML^2} + \frac{(9\pi^2-15)\beta^2 ML^2}{1944\pi^4} \tag{82}$$

$$|1\rangle \approx |1^{(0)}\rangle + \frac{32\beta ML^2}{\hbar\pi^4}\sum_{m=1}^{\infty}|2m^{(0)}\rangle\frac{m}{((2m)^2-1)^3} \tag{83}$$

$$|2\rangle \approx |2^{(0)}\rangle + \frac{32\beta ML^2}{\hbar\pi^4}\sum_{m=1}^{\infty}|m^{(0)}\rangle\frac{2m-1}{((2m-1)^2-4)^3} \tag{84}$$

$$|3\rangle \approx |3^{(0)}\rangle + \frac{96\beta ML^2}{\hbar\pi^4}\sum_{m=1}^{\infty}|m^{(0)}\rangle\frac{m}{(2m)^2-9)^3} \tag{85}$$