PHYS852 Quantum Mechanics II, Spring 2010
HOMEWORK ASSIGNMENT 7: Solutions

Topics covered: addition of three angular momenta, degenerate perturbation theory

1. Consider a system of two spin-1/2 particles, described by $\vec{S}_1$ and $\vec{S}_2$, and one spin-1 particle, described by $\vec{S}_3$. Let $\vec{S}' = \vec{S}_1 + \vec{S}_2$. What are the allowed values of s'? For each allowed value, give the allowed m' values. Use Clebsch Gordan coefficients to express the $|s'm'm_3\rangle$ states in terms of the $|m_1m_2m_3\rangle$ states.

   Let $\vec{S} = \vec{S}_1 + \vec{S}_2 + \vec{S}_3$. What are the allowed values of the quantum number s? For each s value, list the alllowed m-values. Express the states $|s', s, m\rangle$ as linear superpositions of the $|s'm'm_3\rangle$ states, and then as linear superpositions of the $|m_1m_2m_3\rangle$ states.

   Notation:
   For $s = 2$, we use $|\Uparrow\rangle$ for $m = 2$, $|\uparrow\rangle$ for $m = 1$, $|0\rangle$ for $m = 0$, $|\downarrow\rangle$ for $m = -1$, and $|\Downarrow\rangle$ for $m = -2$;
   For $s = 1$, we use $|\uparrow\rangle$ for $m = 1$, $|0\rangle$ for $m = 0$, and $|\downarrow\rangle$ for $m = -1$;
   For $s = 1/2$, we use $|\uparrow\rangle$ for $m = 1/2$, and $|\downarrow\rangle$ for $m = -1/2$.

   The allowed values of s' are $0, 1$
   For $s' = 0$, the only allowed value for m' is 0.
   For $s' = 1$, the allowed values for m' are $-1, 0, 1$.
   The $|s'm'm_3\rangle$ states follow the standard singlet/triplet forms:

$$|s'm'm_3\rangle = |m_1m_2m_3\rangle \tag{1}$$
$$|1\uparrow m_3\rangle = |\uparrow\uparrow m_3\rangle \tag{2}$$
$$|10m_3\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow m_3\rangle + |\downarrow\uparrow m_3\rangle) \tag{3}$$
$$|1\downarrow m_3\rangle = |\downarrow\downarrow m_3\rangle \tag{4}$$
$$|00m_3\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow m_3\rangle - |\downarrow\uparrow m_3\rangle) \tag{5}$$

   For $s' = 0$, the allowed value of s is 1.
   For $s' = 1$, the allowed values of s are $0, 1, 2$.
   For $s = 0$, the allowed value of m is 0.
   For $s = 1$, the allowed values of m are $-1, 0, 1$.
   For $s = 2$, the allowed values of m are $-2, -1, 0, 1, 2$.

$$|s'sm\rangle = |s'm'm_3\rangle = |m_1m_2m_3\rangle \tag{6}$$
$$|12\Uparrow\rangle = |1\uparrow\uparrow\rangle = |\uparrow\uparrow\uparrow\rangle \tag{7}$$
$$|12\uparrow\rangle = \tfrac{1}{\sqrt{2}}(|1\uparrow 0\rangle + |10\uparrow\rangle) = \frac{1}{2}(\sqrt{2}|\uparrow\uparrow 0\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle) \tag{8}$$
$$|120\rangle = \tfrac{1}{\sqrt{6}}(|1\uparrow\downarrow\rangle + 2|100\rangle + |1\downarrow\uparrow\rangle) = \frac{1}{\sqrt{6}}(|\uparrow\uparrow\downarrow\rangle + \sqrt{2}|\uparrow\downarrow 0\rangle + \sqrt{2}|\downarrow\uparrow 0\rangle + |\downarrow\downarrow\uparrow\rangle) \tag{9}$$
$$|12\downarrow\rangle = \tfrac{1}{\sqrt{2}}(|10\downarrow\rangle + |1\downarrow 0\rangle) = \frac{1}{2}(|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + \sqrt{2}|\downarrow\downarrow 0\rangle) \tag{10}$$
$$|12\Downarrow\rangle = |1\downarrow\downarrow\rangle = |\downarrow\downarrow\downarrow\rangle \tag{11}$$
$$|11\uparrow\rangle = \tfrac{1}{\sqrt{2}}(|1\uparrow 0\rangle - |10\uparrow\rangle) = \frac{1}{2}(\sqrt{2}|\uparrow\uparrow 0\rangle - |\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle) \tag{12}$$
$$|110\rangle = \tfrac{1}{\sqrt{2}}(|1\uparrow\downarrow\rangle - |1\downarrow\uparrow\rangle) = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\downarrow\rangle - |\downarrow\downarrow\uparrow\rangle) \tag{13}$$
$$|11\downarrow\rangle = \tfrac{1}{\sqrt{2}}(|10\downarrow\rangle - |1\downarrow 0\rangle) = \frac{1}{2}(|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle - \sqrt{2}|\downarrow\downarrow 0\rangle) \tag{14}$$
$$|100\rangle = \tfrac{1}{\sqrt{3}}(|1\uparrow\downarrow\rangle - |100\rangle + |1\downarrow\uparrow\rangle) = \frac{1}{\sqrt{6}}(\sqrt{2}|\uparrow\uparrow\downarrow\rangle - |\uparrow\downarrow 0\rangle - |\downarrow\uparrow 0\rangle + \sqrt{2}|\downarrow\downarrow\uparrow\rangle) \tag{15}$$
$$|01\uparrow\rangle = |00\uparrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle) \tag{16}$$
$$|010\rangle = |000\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow 0\rangle - |\downarrow\uparrow 0\rangle) \tag{17}$$
$$|01\downarrow\rangle = |00\downarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\downarrow\rangle - |\downarrow\uparrow\downarrow\rangle) \tag{18}$$

Calculations:

$$S_-|12\Uparrow\rangle = (S'_- + S_{3-})|1\uparrow\uparrow\rangle = (S_{1-} + S_{2-} + S_{3-})|\uparrow\uparrow\uparrow\rangle \tag{19}$$

$$\sqrt{2\cdot 3 - 2\cdot\uparrow}|12\uparrow\rangle = \sqrt{1\cdot 2 - 1\cdot 0}|10\uparrow\rangle + \sqrt{1\cdot 2 - 1\cdot 0}|1\uparrow 0\rangle = |\downarrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\rangle + \sqrt{1\cdot 2 - 1\cdot 0}|\uparrow\uparrow 0\rangle \tag{20}$$

$$2|12\uparrow\rangle = \sqrt{2}(|10\uparrow\rangle + |1\uparrow 0\rangle) = |\downarrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\rangle + \sqrt{2}|\uparrow\uparrow 0\rangle \tag{21}$$

$$|12\uparrow\rangle = \frac{1}{\sqrt{2}}(|10\uparrow\rangle + |1\uparrow 0\rangle) = \frac{1}{2}(|\downarrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\rangle + \sqrt{2}|\uparrow\uparrow 0\rangle). \tag{22}$$

$$S_-|12\uparrow\rangle = (S'_- + S_{3-})\frac{1}{\sqrt{2}}(|10\uparrow\rangle + |1\uparrow 0\rangle) = (S_{1-} + S_{2-} + S_{3-})\frac{1}{2}(|\downarrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\rangle + \sqrt{2}|\uparrow\uparrow 0\rangle \tag{23}$$

$$\sqrt{6}|120\rangle = |1\downarrow\uparrow\rangle + 2|100\rangle + |10\downarrow\rangle = (|\downarrow\downarrow\uparrow\rangle + \sqrt{2}|\downarrow\uparrow 0\rangle + \sqrt{2}|\uparrow\downarrow 0\rangle + |\uparrow\uparrow\downarrow\rangle) \tag{24}$$

$$|120\rangle = \frac{1}{\sqrt{6}}(|1\downarrow\uparrow\rangle + 2|100\rangle + |10\downarrow\rangle)) = \frac{1}{\sqrt{6}}(|\downarrow\downarrow\uparrow\rangle + \sqrt{2}|\downarrow\uparrow 0\rangle + \sqrt{2}|\uparrow\downarrow 0\rangle + |\uparrow\uparrow\downarrow\rangle) \tag{25}$$

$$S_-|11\uparrow\rangle = (S'_- + S_{3-})\frac{1}{\sqrt{2}}(|1\uparrow 0\rangle - |10\uparrow\rangle) = (S_{1-} + S_{2-} + S_{3-})\frac{1}{2}(\sqrt{2}|\uparrow\uparrow 0\rangle - |\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle) \tag{26}$$

$$\sqrt{2}|110\rangle = |1\uparrow\downarrow\rangle - |1\downarrow\uparrow\rangle = |\uparrow\uparrow\downarrow\rangle - |\downarrow\downarrow\uparrow\rangle \tag{27}$$

$$|110\rangle = \frac{1}{\sqrt{2}}(|1\uparrow\downarrow\rangle - |1\downarrow\uparrow\rangle) = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\downarrow\rangle - |\downarrow\downarrow\uparrow\rangle) \tag{28}$$

$$S_-|110\rangle = (S'_- + S_{3-})\frac{1}{\sqrt{2}}(|1\uparrow\downarrow\rangle - |1\downarrow\uparrow\rangle) = (S_{1-} + S_{2-} + S_{3-})\frac{1}{\sqrt{2}}(|\uparrow\uparrow\downarrow\rangle - |\downarrow\downarrow\uparrow\rangle) \tag{29}$$

$$\sqrt{2}|11\downarrow\rangle = |10\downarrow\rangle - |1\downarrow 0\rangle = \frac{1}{\sqrt{2}}(|\downarrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle - \sqrt{2}|\downarrow\downarrow 0\rangle) \tag{30}$$

$$|11\downarrow\rangle = \frac{1}{\sqrt{2}}(|10\downarrow\rangle - |1\downarrow 0\rangle) = \frac{1}{2}(|\downarrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle - \sqrt{2}|\downarrow\downarrow 0\rangle) \tag{31}$$

2. Derive general expressions for the energy-shifts up to third-order and state vectors up to second-order in degenerate perturbation theory.

As described in the lecture notes, solving the first- and second-order equations gives:

$$E_{nm} = E_n^{(0)} + \lambda v_{nm} - \lambda^2 \sum_{\substack{n'\neq n \\ m'}} \frac{V_{nmn'm'}V_{n'm'nm}}{\Delta_{n'n}} \tag{32}$$

$$\begin{aligned}
|nm\rangle \;=\; & |nm^{(0)}\rangle \left[1 - \frac{\lambda^2}{2} \sum_{\substack{n'\neq n \\ m'}} \frac{V_{nmn'm'}}{\Delta_{n'n}} \left(\sum_{m''\neq m} \sum_{\substack{n'''\neq n \\ m'''}} \frac{V_{n'm'nm''}V_{nm''n'''m'''}V_{n'''m'''nm}}{v^2_{nm''m}\Delta_{n'''n}} + V_{n'm'nm} \right)\right] \\
&+ \sum_{m'\neq m} |nm'^{(0)}\rangle \left[\lambda \sum_{\substack{n''\neq n \\ m''}} \frac{V_{nm'n''m''}V_{n''m''nm}}{v_{nm'm}\Delta_{n''n}}\right] + \sum_{\substack{n'\neq n \\ m'}} |n'm'^{(0)}\rangle \Bigg[-\lambda \frac{V_{n'm'nm}}{\Delta_{n'n}} \left(1 + \lambda \frac{v_{nm}}{\Delta_{n'n}}\right) \\
&+ \lambda^2 \sum_{\substack{n''\neq n \\ m''}} \left(V_{n'm'n''m''} - \sum_{m'''\neq m} \frac{V_{n'm'nm'''}V_{nm'''n''m''}}{v_{nm'''m}} \right) \frac{V_{n''m''nm}}{\Delta_{n'n}\Delta_{n''n}} \Bigg]
\end{aligned} \tag{33}$$

To proceed, we need to use the third-order equations to find the third-order energy-shift and the second-order correction to the $|nm'^{(0)}\rangle$ states.
The third-order equation is:

$$(H_0 - E_n^{(0)})|nm^{(3)}\rangle = -V|nm^{(2)}\rangle + E^{(3)}_{nm}|nm^{(0)}\rangle + E^{(2)}_{nm}|nm^{(1)}\rangle + E^{(1)}_{nm}|nm^{(2)}\rangle \tag{34}$$

Hitting from the left with $\langle nm^{(0)}|$ and solving for $E^{(3)}_{nm}$ gives

$$\begin{aligned}
E^{(3)}_{nm} &= \langle nm^{(0)}|V|nm^{(2)}\rangle - E^{(2)}_{nm}\langle nm^{(0)}|nm^{(1)}\rangle - E^{(1)}_{nm}\langle nm^{(0)}|nm^{(2)}\rangle \\
&= \sum_{\substack{n'\neq n \\ m'}} V_{nmn'm'}\langle n'm'^{(0)}|nm^{(2)}\rangle \\
&= \sum_{\substack{n'\neq n \\ m'}} \sum_{\substack{n''\neq n,n' \\ m''}} \frac{V_{nmn'm'}V_{n'm'n''m''}V_{n''m''nm}}{\Delta_{n'n}\Delta_{n''n}} \\
&+ \sum_{\substack{n'\neq n \\ m'}} \frac{V_{nmn'm'}v_{n'm'}V_{n'm'nm}}{\Delta^2_{n'n}} - \sum_{\substack{n'\neq n \\ m'}} \frac{V_{nmn'm'}V_{n'm'nm}v_{nm}}{\Delta^2_{n'n}} \\
&- \sum_{\substack{n'\neq n \\ m'}} \sum_{\substack{n''\neq n \\ m''}} \sum_{m'''\neq m} \frac{V_{nmn'm'}V_{n'm'nm'''}V_{nm'''n''m''}V_{n''m''nm}}{v_{nm'''m}\Delta_{n'n}\Delta_{n''n}}
\end{aligned} \tag{35}$$

Hitting from the left with $\langle nm'^{(0)}|$ and solving for $\langle nm'^{(0)}|nm^{(2)}\rangle$ gives

$$\begin{aligned}
\langle nm'^{(0)}|nm^{(2)}\rangle &= \frac{\langle nm'^{(0)}|V|nm^{(2)}\rangle}{v_{nm}} - \frac{E_{nm}}{v_{nm}}^{(2)}\langle nm'^{(0)}|nm^{(1)}\rangle \\
&= -\sum_{\substack{n''\neq n \\ m''}} \frac{V_{nm'n''m''}V_{n''m''nm}}{\Delta^2_{n''n}} \\
&+ \sum_{\substack{n''\neq n \\ m''}} \left[\sum_{\substack{n'''\neq n,n'' \\ m'''}} \frac{V_{nm'n''m''}V_{n''m''n'''m'''}V_{n'''m'''nm}}{v_{nm}\Delta_{n''n}\Delta_{n'''n}} + \frac{V_{nm'n''m''}v_{n''m''}V_{n''m''nm}}{v_{nm}\Delta^2_{n''n}} \right] \\
&- \sum_{\substack{n''\neq n \\ m''}} \sum_{\substack{n'''\neq n \\ m'''}} \sum_{m''''\neq m} \frac{V_{nm'n''m''}V_{n''m''nm''''}V_{nm''''n'''m'''}V_{n'''m'''nm}}{v_{nm}v_{nm''''m}\Delta_{n''n}\Delta_{n'''n}} \\
&- \sum_{\substack{n''\neq n \\ m''}} \sum_{\substack{n'''\neq n \\ m'''}} \frac{V_{nm'n''m''}V_{n''m''nm}V_{nmn'''m'''}V_{n'''m'''nm}}{v_{nm}\Delta_{n''n}v_{nm'm}\Delta_{n'''n}} \qquad (36)
\end{aligned}$$

3. Consider a system described in the basis $\{|1\rangle, |2\rangle, |3\rangle, |4\rangle\}$ by the bare Hamiltonian

$$H_0 = \hbar\omega_0 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \tag{37}$$

Let the system be perturbed by the operator

$$V = \hbar g \begin{pmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{pmatrix} \tag{38}$$

First, determine the 'good basis' for degenerate perturbation theory, then compute the eigenvalues and eigenvectors of $H = H_0 + V$ to second-order in perturbation theory.

The bare Hamiltonian has two degenerate subspaces. The set $\{|1\rangle, |2\rangle\}$ spans the space corresponding to $E_1^{(0)} = \hbar\omega_0$, and the set $\{|3\rangle, |4\rangle\}$ spans the space corresponding to $E_2^{(0)} = 2\hbar\omega_0$.
By inspection, we see that

$$V_1 = V_2 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \tag{39}$$

There fore the good basis states are

$$|11^{(0)}\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle) \tag{40}$$

$$|12^{(0)}\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle) \tag{41}$$

$$|21^{(0)}\rangle = \frac{1}{\sqrt{2}}(|3\rangle - |4\rangle) \tag{42}$$

$$|22^{(0)}\rangle = \frac{1}{\sqrt{2}}(|3\rangle + |4\rangle) \tag{43}$$

with first-order energies given by

$$v_{11} = -2\hbar g \tag{44}$$

$$v_{12} = 2\hbar g \tag{45}$$

$$v_{21} = -2\hbar g \tag{46}$$

$$v_{22} = 2\hbar g \tag{47}$$

In this basis we have

$$V = 2\hbar g \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \tag{48}$$

From this, we see that states $|11^{(0)}\rangle$ and $|21^{(0)}\rangle$ are not coupled to any other states by V. Thus, beyond the first-order energy-shift, they are not perturbed by V.

Thus the problem maps onto a single non-degenerate problem in the $\{|12^{(0)}\rangle, |22^{(0)}\rangle\}$ Hilbert space, for which we have

$$H_0 = \hbar\omega_0 \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \tag{49}$$

and

$$V = 2\hbar g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{50}$$

The states to second-order are therefore

$$|11\rangle = |11^{(0)}\rangle \tag{51}$$

$$|12\rangle \approx |12^{(0)}\rangle \left(1 - \frac{2g^2}{\omega_0^2}\right) - |22^{(0)}\rangle \frac{2g}{\omega_0} \tag{52}$$

$$|21\rangle = |21^{(0)}\rangle \tag{53}$$

$$|22\rangle \approx |12^{(0)}\rangle \frac{2g}{\omega_0} + |22^{(0)}\rangle \left(1 - \frac{2g^2}{\omega_0^2}\right) \tag{54}$$

The energies to second-order are

$$E_{11} = \hbar\omega_0 - 2\hbar g \tag{55}$$

$$E_{12} \approx \hbar\omega_0 + 2\hbar g - \frac{4\hbar g^2}{\omega_0} \tag{56}$$

$$E_{21} = 2\hbar\omega_0 - 2\hbar g \tag{57}$$

$$E_{22} \approx 2\hbar\omega_0 + 2\hbar g + \frac{4\hbar g^2}{\omega_0} \tag{58}$$

4. Prove that in order to predict the resonance-frequency shifts of a perturbed quantum system to second-order, one needs only to compute the energy levels to second-order, and that this implicitly requires computing the eigenstates only to first-order.

   Then show that in order to compute the shift in the expectation-value of an arbitrary operator to second-order, one must compute the eigenstates to second-order.

   The resonance frequency shifts are the differences between the energy eigenvalues, which means they can be computed to second-order directly from the second-order energies. The second-order energy corrections are given in terms of the first-order eigenstates via $E_n^{(2)} = -\langle n^{(0)}|V|n^{(1)}\rangle$.

   To compute the expectation value of an operator to second-order, we need to evaluate

$$\begin{aligned} \langle A\rangle &= \langle\psi(t)|A|\psi(t)\rangle \\ &= \sum_{mn} c_m^* c_n \langle m|A|n\rangle e^{i(E_m-E_n)t/\hbar} \end{aligned} \tag{59}$$

   expanding to 2nd order gives

$$\begin{aligned} \langle A\rangle &= \sum_m n c_m^* c_n \Big[\langle m^{(0)}|A|n^{(0)}\rangle + \lambda\left(\langle m^{(1)}|A|n^{(0)}\rangle + \langle m^{(0)}|A|n^{(1)}\rangle\right) \\ &\quad +\lambda^2\left(\langle m^{(0)}|A|n^{(2)}\rangle + \langle m^{(1)}|A|n^{(1)}\rangle + \langle m^{(2)}|A|n^{(0)}\rangle\right)\Big] \\ \times &\quad e^{i\left[E_m^{(0)}-E_n^{(0)}+\lambda\left(E_m^{(1)}-E_n^{(1)}\right)+\lambda^2\left(E_m^{(2)}-E_n^{(2)}\right)\right]t/\hbar} \end{aligned} \tag{60}$$

   which shows that the second-order eigenstates are required.

5. Consider a pair of identical harmonic oscillators, described by the Hamiltonian

$$H_0 = \hbar\omega(A_1^\dagger A_1 + A_2^\dagger A_2 + 1). \tag{61}$$

Determine the energy eigenvalues and degeneracies of the two lowest energy levels of H_0.

Let the system be perturbed by the operator

$$V = \hbar\lambda(A_1^\dagger A_2^\dagger + A_1^\dagger A_2 + A_2^\dagger A_1 + A_1 A_2). \tag{62}$$

For the states comprising the two lowest bare energy levels, determine the full energy eigenvalues of $H = H_0 + V$ to second-order and the eigenstates to first-order in λ.

The lowest energy level is the ground state, $E_1^{(0)} = \hbar\omega$, with non-degenerate eigenstate

$$|1^{(0)}\rangle = |00\rangle, \tag{63}$$

The first excited energy level is $E_2^{(0)} = 2\hbar\omega$, with $d_2 = 2$. The degenerate subspace is spanned by the states $\{|01\rangle, |10\rangle\}$.

in basis $\{|01\rangle, |10\rangle\}$, we have

$$V_2 = \hbar\lambda \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{64}$$

For the ground-state, we have

$$E_1^{(1)} = 0 \tag{65}$$

and

$$|1^{(1)}\rangle = -\sum_{\substack{n_1,n_2=0 \\ (n_1,n_2)\neq(0,0)}}^{\infty} |n_1 n_2\rangle \frac{\langle n_1 n_2|V|00\rangle}{\hbar\omega(n_1+n_2)} \tag{66}$$

with

$$\begin{aligned} \langle n_2 n_2|V|m_1 m_2\rangle \;=\; & \hbar\lambda\Big(\sqrt{(m_1{+}1)(m_2{+}1)}\delta_{n_1,m_1+1}\delta_{n_2,m_2+1} + \sqrt{m_2(m_1{+}1)}\delta_{n_1,m_1+1}\delta_{n_2,m_2-1} \\ & + \sqrt{m_1(m_2{+}1)}\delta_{n_1,m_1+1}\delta_{n_2,m_2-1} + \sqrt{m_1 m_2}\delta_{n_1,m_1-1}\delta_{n_2,m_2-1}\Big) \end{aligned} \tag{67}$$

this becomes

$$|1^{(1)}\rangle = -\frac{\lambda}{2\omega}|11\rangle \tag{68}$$

with the second-order energy then given by

$$E_1^{(2)} = -\frac{\lambda}{2\omega}\langle 00|V|11\rangle = -\frac{\hbar\lambda^2}{2\omega} \tag{69}$$

For the $n = 2$ subspace, the 'good basis' is

$$|21^{(0)}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \tag{70}$$

$$|22^{(0)}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \tag{71}$$

The first-order energies are

$$E_{21}^{(1)} = -\hbar\lambda \tag{72}$$

$$E_{22}^{(1)} = \hbar\lambda \tag{73}$$

The first-order eigenstates are

$$\begin{aligned} |21^{(1)}\rangle &= -|12\rangle\frac{\langle 12|V|21^{(0)}\rangle}{2\hbar\omega} - |21\rangle\frac{\langle 21|V|21^{(0)}\rangle}{2\hbar\omega} \\ &= -\frac{\lambda}{2\omega}\left(|12\rangle - |21\rangle\right) \\ |22^{(1)}\rangle &= -\frac{\lambda}{2\omega}\left(|12\rangle + |21\rangle\right) \end{aligned} \tag{74}$$

so that the second-order energies are

$$\begin{aligned} E_{21}^{(2)} &= \langle 21^{(0)}|V|21^{(1)}\rangle \\ &= -\frac{\hbar\lambda^2}{2\sqrt{2}\omega}(\langle 01| - \langle 10|)A_1A_2|(|12\rangle - |21\rangle) \\ &= -\frac{\hbar\lambda^2}{\omega} \end{aligned} \tag{75}$$

$$\begin{aligned} E_{22}^{(2)} &= \langle 22^{(0)}|V|22^{(1)}\rangle \\ &= -\frac{\hbar\lambda^2}{\omega} \end{aligned} \tag{76}$$

We have then

$$E_1 \approx \hbar\omega - \frac{\hbar\lambda^2}{2\omega} \tag{77}$$

$$E_{21} \approx 2\hbar\omega - \hbar\lambda - \frac{\hbar\lambda^2}{\omega} \tag{78}$$

$$E_{22} \approx 2\hbar\omega + \hbar\lambda - \frac{\hbar\lambda^2}{\omega} \tag{79}$$

and

$$|1\rangle \approx |00\rangle - \frac{\lambda}{2\omega}|11\rangle \tag{80}$$

$$|21\rangle \approx \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) - \frac{\lambda}{2\omega}(|12\rangle - |21\rangle) \tag{81}$$

$$|22\rangle \approx \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) - \frac{\lambda}{2\omega}(|12\rangle + |21\rangle) \tag{82}$$