PHYS852 Quantum Mechanics II, Spring 2010
HOMEWORK ASSIGNMENT 9

Topics covered: hydrogen hyper-fine structure, Wigner-Ekert theorem, Zeeman effect

1. **Relations between $\vec{V}$ and $\vec{J}$**: For a rotation by ϕ about the z-axis, we have $U^\dagger V_z U = V_z$, $U^\dagger V_x U = \cos\phi V_x - \sin\phi V_y$, and $U^\dagger V_y U = \sin\phi V_x + \cos\phi V_y$, where $U = e^{-(i/\hbar)\phi J_z}$.

   (a) Consider an infinitesimal rotation by $\delta\phi$, and use these expressions to show:
   $$[J_z, V_z] = 0, \tag{1}$$
   $$[J_z, V_x] = i\hbar V_y, \tag{2}$$
   $$[J_z, V_y] = -i\hbar V_x. \tag{3}$$
   Write out the six additional commutators generated by cyclic permutation of the indices.

   b.) Use the results from (a) to show:
   $$[J_z, V_\pm] = \pm\hbar V_\pm \tag{4}$$
   $$[J_\pm, V_\pm] = 0 \tag{5}$$
   $$[J_\pm, V_\mp] = \pm 2\hbar V_z \tag{6}$$
   where $V_\pm = V_x \pm iV_y$.

2. **Derivation of Wigner-Ekert theorem**: Verify Eqs. (108)-(127) in the Atomic Physics lecture notes.

3. **Applying the Wigner-Ekert theorem**: Let $\vec{L} = \vec{L}_1 + \vec{L}_2$. Use the Wigner-Eckert theorem to show that
   $$\langle \ell_1\ell_2\ell m_\ell | L_{1z} | \ell_1\ell_2\ell m_\ell\rangle = g m_\ell \tag{7}$$
   and calculate the g-factor, $g = g(\ell_1, \ell_2, \ell)$.

   Do the same for $\langle \ell_1\ell_2\ell m_\ell | L_{2z} | \ell_1\ell_2\ell m_\ell\rangle$, and then show that you get the correct result for
   $$\langle \ell_1\ell_2\ell m_\ell | (L_{1z} + L_{2z}) | \ell_1\ell_2\ell m_\ell\rangle \tag{8}$$

4. **Strong-field Zeeman Effect**: for the case $\hbar\omega_0 \gg |E_1^{(0)}|\alpha^2$, give the energies and Zeeman sub-levels of the $n = 3$ level in terms of the Larmor frequency, $\omega_0 = \frac{|e|B}{2M_3}$.

   Verify for $n = 3$ that there are $2_n + 1 - \delta_{n,1}$ Zeeman sublevels, each separated by $\hbar\omega_0$, and that the degeneracy of the m^{th} sublevel ($m \in \{-n, -n+1, \ldots, n\}$, with $m = 0$ excluded for $n = 1$) is $d_{n,m} = 2(n - |m|) + \delta_{|m|,n} - 2\delta_{m,0}$.

5. **Weak-field Zeeman Effect**: for the case $\hbar\omega_0 \ll |E_1^{(0)}|\alpha^2 \frac{M_e}{M_p}$, compute the energies and degeneracies of the Zeeman sub-levels for both the $n = 3$, $j = 3/2$ and $n = 3$, $j = 5/2$ levels.