PHYS852 Quantum Mechanics II, Spring 2010
HOMEWORK ASSIGNMENT 9: Solutions

Topics covered: hydrogen hyper-fine structure, Wigner-Ekert theorem, Zeeman effect

1. **Relations between $\vec{V}$ and $\vec{J}$**: For a rotation by ϕ about the z-axis, we have $U^\dagger V_z U = V_z$, $U^\dagger V_x U = \cos\phi V_x - \sin\phi V_y$, and $U^\dagger V_y U = \sin\phi V_x + \cos\phi V_y$, where $U = e^{-(i/\hbar)\phi J_z}$.

   (a) Consider an infinitesimal rotation by $\delta\phi$, and use these expressions to show:

$$[J_z, V_z] = 0, \tag{1}$$

$$[J_z, V_x] = i\hbar V_y, \tag{2}$$

$$[J_z, V_y] = -i\hbar V_x. \tag{3}$$

   Write out the six additional commutators generated by cyclic permutation of the indices.

   For an infinitesimal rotation, we can expand U as $U \approx 1 - \frac{i}{\hbar}\phi J_z$, so that keeping terms up to first-order in ϕ gives

$$V_x + \frac{i}{\hbar}\phi[J_z, V_x] = V_x - \phi V_y \tag{4}$$

$$V_y + \frac{i}{\hbar}[J_z, V_y] = \phi V_x + V_y \tag{5}$$

$$V_z + \frac{i}{\hbar}[J_,V_z] = V_z \tag{6}$$

   from which we can read off:

$$[J_z, V_x] = i\hbar V_y \tag{7}$$

$$[J_z, V_y] = -i\hbar V_x \tag{8}$$

$$[J_z, V_z] = 0 \tag{9}$$

   Cyclic permutation of indices then gives:

$$[J_x, V_y] = i\hbar V_z \qquad [J_y, V_z] = i\hbar V_x \tag{10}$$

$$[J_x, V_z] = -i\hbar V_y \qquad [J_y, V_x] = -i\hbar V_z \tag{11}$$

$$[J_x, V_x] = 0 \qquad [J_y, V_y] = 0 \tag{12}$$

   b.) Use the results from (a) to show:

$$[J_z, V_\pm] = \pm\hbar V_\pm \tag{13}$$

$$[J_\pm, V_\pm] = 0 \tag{14}$$

$$[J_\pm, V_\mp] = \pm 2\hbar V_z \tag{15}$$

where $V_\pm = V_x \pm iV_y$.

$$\begin{aligned} [J_z, V_\pm] &= [J_z, V_x] \pm i[J_z, V_y] \\ &= i\hbar V_y \pm \hbar V_x \\ &= \pm\hbar(V_x \pm iV_y) \\ &= \pm\hbar V_\pm \end{aligned} \tag{16}$$

$$\begin{aligned} [J_\pm, V_\pm] &= [J_x, V_x] \pm i[J_x, V_y] \pm i[J_y, V_x] - [J_y, V_y] \\ &= \mp\hbar V_z \pm \hbar V_z \\ &= 0 \end{aligned} \tag{17}$$

$$\begin{aligned} [J_\pm, V_\mp] &= [J_x, V_x] \mp i[J_x, V_y] \pm i[J_y, V_x] + [J_y, V_y] \\ &= \pm\hbar V_z \pm \hbar V_z \\ &= \pm 2\hbar V_z \end{aligned} \tag{18}$$

2. **Derivation of Wigner-Ekert theorem**: Verify Eqs. (108)-(127) in the Atomic Physics lecture notes.
   Eq. (108):

$$\begin{aligned} [J_z, V_z] &= 0 \\ [J_z, V_z]|kjm\rangle &= 0 \\ J_z(V_z|kjm\rangle) &= V_z J_z|kjm\rangle \\ J_z(V_z|kjm\rangle) &= \hbar m V_z|kjm\rangle \end{aligned} \tag{19}$$

   Eq. (109):

$$\begin{aligned} [J_z, V_\pm] &= \pm\hbar V_\pm \\ [J_z, V_\pm]|kjm\rangle &= \pm\hbar V_\pm|kjm\rangle \\ J_z(V_\pm|kjm\rangle) &= V_\pm(J_z \pm \hbar)|kjm\rangle \\ J_z(V_\pm|kjm\rangle &= \hbar(m \pm 1)V_\pm|kjm\rangle \end{aligned} \tag{20}$$

   Eqs. (110-114):

$$\begin{aligned} [J_\pm, V_\pm] &= 0 \\ \langle kj(m\pm2)|[J_\pm, V_\pm]|kjm\rangle &= 0 \\ \langle kj(m\pm2)|J_\pm V_\pm|kjm\rangle &= \langle kj(m\pm2)|V_\pm J_\pm|kjm\rangle \\ \sqrt{j(j+1)-(m\pm2)(m\pm1)}\langle kj(m\pm1)|V_\pm|kjm\rangle &= \sqrt{j(j+1)-m(m\pm1)}\langle kj(m|pm2)|V_\pm|kj(m\pm1)\rangle \\ \langle kj(m\pm2)|J_\pm|kj(m\pm1)\rangle\langle kj(m\pm1)|V_\pm|kjm\rangle &= \langle kj(m\pm1)|J_\pm|kjm\rangle\langle kj(m|pm2)|V_\pm|kj(m\pm1)\rangle \\ \frac{\langle kj(m\pm1)|V_\pm|kjm\rangle}{\langle kj(m\pm1)|J_\pm|kjm\rangle} &= \frac{\langle kj(m\pm2)|V_\pm|kj(m\pm1)\rangle}{\langle kj(m\pm2)|J_\pm|kj(m\pm1)\rangle} \\ a_\pm(k, j, m) &= a_\pm(k, j, m+1) \end{aligned} \tag{21}$$

Eqs. (115-117): We can start from the conclusion, and check that it is equivalent to the starting equation:

$$I_{jk}V_{\pm}I_{kj} = a_{\pm}(k,j)I_{kj}J_{\pm}I_{kj} \tag{22}$$

This is an operator-valued equation, so it must be true element-by-element, which means:

$$\begin{aligned}
\langle kjm|I_{kj}V_{\pm}I_{kj}|kjm'\rangle &= a_{\pm}(k,j)\langle kjm|I_{kj}J_{\pm}I_{kj}|kjm'\rangle \\
\langle kjm|V_{\pm}|kjm'\rangle &= a_{\pm}(k,j)\langle kjm|J_{\pm}|kjm'\rangle \\
\langle kjm|V_{\pm}|kj(m\mp1)\rangle\delta_{m'\pm1,m} &= a_{\pm}(k,j)\langle kjm|J_{\pm}|kj(m\mp1)\rangle\delta_{m'\pm1,m} \\
\langle kjm|V_{\pm}|kj(m\mp1)\rangle &= a_{\pm}(k,j)\langle kjm|J_{\pm}|kj(m\mp1)\rangle
\end{aligned} \tag{23}$$

Eqs. (118-123):

$$\begin{aligned}
[J_{\mp},V_{\pm}] &= \mp2\hbar V_z \\
\langle kjm|[J_{\mp},V_{\pm}]|kjm'\rangle &= \mp2\hbar\langle kjm|V_z|kjm'\rangle \\
\langle kjm|J_{\mp}V_{\pm}|kjm'\rangle - \langle kjm|V_{\pm}J_{\mp}|kjm'\rangle &= \mp2\hbar\langle kjm|V_z|kjm'\rangle \\
\sqrt{j(j{+}1){-}m(m{\pm}1)}\langle kj(m{\pm}1)|V_{\pm}|kjm'\rangle - \sqrt{j(j{+}1{-}m'(m'{\mp}1)}\langle kjm|V_{\pm}|kj(m'{\mp}1)\rangle &= \mp2\langle kjm|V_z|kjm'\rangle
\end{aligned} \tag{24}$$

with (117), the r.h.s. becomes:

$$\begin{aligned}
& a_{\pm}(k,j)\left(\sqrt{j(j{+}1){-}m(m{\pm}1)}\langle kj(m{\pm}1)|J_{\pm}|kjm'\rangle - \sqrt{j(j{+}1{-}m'(m'{\mp}1)}\langle kjm|J_{\pm}|kj(m'{\mp}1)\rangle\right) \\
&= a_{\pm}(k,j)\hbar\left(\sqrt{j(j{+}1){-}m(m{\pm}1)}\sqrt{j(j{+}1){-}m'(m'{\pm}1)} - \sqrt{j(j{+}1{-}m'(m'{\mp}1)}\sqrt{j(j+1)-(m'{\mp}1)m'}\right)\delta_{m,m'} \\
& a_{\pm}(k,j)\hbar\left(j(j{+}1) - m(m{\pm}1) - j(j{+}1) + m(m{\mp}1)\right)\delta_{m,m'} \\
& a_{\pm}(k,j)\hbar\left(-m^2 \mp m + m^2 \mp m\right)\delta_{m,m'} \\
& \mp2\hbar m\, a_{\pm}(k,j)\delta_{m,m'}
\end{aligned} \tag{25}$$

so we end up with

$$\begin{aligned}
\langle kjm|V_z|kjm'\rangle &= a_{\pm}(k,j)\,\hbar m\delta_{m,m'} \\
&= a_{\pm}(k,j)\,\langle kjm|J_z|kjm'\rangle
\end{aligned} \tag{26}$$

as the l.h.s. is the same for both '+' and '-', we can take $a_{\pm}(k,j) \to a(k,j)$. With this equation, together with (117), it follows that

$$\vec{V}_{kj} = a(k,j)\vec{J}_{kj}. \tag{27}$$

Eqs. (124-127): with $|\psi_{kj}\rangle$ being an arbitrary state in the I_{jk} subspace, we have

$$\begin{aligned}
\langle\psi_{kj}|\vec{J}\cdot\vec{V}|\psi_{kj}\rangle &= a(k,j)\langle\psi_{kj}|J^2|\psi_{kj}\rangle \\
&= a(k,j)\hbar^2 j(j+1)
\end{aligned} \tag{28}$$

so that

$$a(k,j) = \frac{\langle\vec{J}\cdot\vec{V}\rangle_{kj}}{\hbar^2 j(j+1)}. \tag{29}$$

3. **Applying the Wigner-Ekert theorem**: Let $\vec{L} = \vec{L}_1 + \vec{L}_2$. Use the Wigner-Eckert theorem to show that

$$\langle \ell_1 \ell_2 \ell m_\ell | L_{1z} | \ell_1 \ell_2 \ell m_\ell \rangle = g m_\ell \tag{30}$$

and calculate the g-factor, $g = g(\ell_1, \ell_2, \ell)$.
With respect to the subspace of fixed ℓ_1, ℓ_2, and ℓ, we can use the Wigner-Ekert theorem to replace $\vec{L_1}$ with $g_1 \vec{L}$, where

$$g_1 = \frac{\langle \vec{L}_1 \cdot \vec{L} \rangle_{\ell_1 \ell_2 \ell}}{\hbar^2 \ell(\ell+1)} \tag{31}$$

Using $\vec{L} = \vec{L}_1 + \vec{L_2}$ and $\vec{L}_1 \cdot \vec{L}2 = \frac{1}{2}(L^2 - L_1^2 - L_2^2)$, this becomes

$$\begin{aligned} g_1 &= \frac{\langle L_1^2 + \frac{1}{2}(L^2 - L_1^2 - L_2^2) \rangle}{\hbar^2 \ell(\ell+1)} \\ &= \frac{\langle Ł^2 + L_1^2 - L_2^2 \rangle}{2\hbar^2 \ell(\ell+1)} \\ &= \frac{1}{2} + \frac{\ell_1(\ell_1+1) - \ell_2(\ell_2+1)}{\ell(\ell+1)} \end{aligned} \tag{32}$$

Do the same for $\langle \ell_1 \ell_2 \ell m_\ell | L_{2z} | \ell_1 \ell_2 \ell m_\ell \rangle$, and then show that you get the correct result for

$$\langle \ell_1 \ell_2 \ell m_\ell | (L_{1z} + L_{2z}) | \ell_1 \ell_2 \ell m_\ell \rangle \tag{33}$$

For $\vec{L}_2$, we can swap indices to get

$$\begin{aligned} g_2 &= \frac{1}{2} + \frac{\ell_2(\ell_2+1) - \ell_1(\ell_1+1)}{\ell(\ell+1)} \\ &= 1 - g_1 \end{aligned} \tag{34}$$

This gives us

$$\begin{aligned} \langle \ell_1 \ell_2 \ell m_\ell | (L_{1z} + L_{2z}) | \ell_1 \ell_2 \ell m_\ell \rangle &= (g_1 + g_2) \langle L_z \rangle \\ &= \hbar m_\ell \end{aligned} \tag{35}$$

which agrees with the results obtained more directly as

$$\begin{aligned} \langle \ell_1 \ell_2 \ell m_\ell | (L_{1z} + L_{2z}) | \ell_1 \ell_2 \ell m_\ell \rangle &= \langle L_z \rangle \\ &= \hbar m_\ell \end{aligned} \tag{36}$$

4. **Strong-field Zeeman Effect**: for the case $\hbar\omega_0 \gg |E_1^{(0)}|\alpha^2$, give the energies and Zeeman sub-levels of the $n = 3$ level in terms of the Larmor frequency, $\omega_0 = \frac{|e|B}{2M_3}$.
Verify for $n = 3$ that there are $d_n = 2n + 1 - \delta_{n,1}$ Zeeman sublevels, each separated by $\hbar\omega_0$, and that the degeneracy of the m^{th} sublevel ($m \in \{-n, -n+1, \ldots, n\}$, with $m = 0$ excluded for $n = 1$) is $d_{n,m} = 2(n - |m|) + \delta_{|m|,n} - 2\delta_{m,0}$.

In this regime, we can neglect the fine-structure, so that

$$V = \omega_0(L_z + 2S_z) \tag{37}$$

For $n = 3$, we have $\ell = 0, 1, 2$, and $s = 1/2$.
For the $\ell = 0$ level, we have

m_ℓ	m_s	ΔE
0	$\frac{1}{2}$	$\hbar\omega_0$
0	$-\frac{1}{2}$	$-\hbar\omega_0$

For the $\ell = 1$ level, we have

m_ℓ	m_s	ΔE
1	$\frac{1}{2}$	$2\hbar\omega_0$
1	$-\frac{1}{2}$	0
0	$\frac{1}{2}$	$\hbar\omega_0$
0	$-\frac{1}{2}$	$-\hbar\omega_0$
-1	$\frac{1}{2}$	0
-1	$-\frac{1}{2}$	$-2\hbar\omega_0$

Lastly, for $\ell = 2$, we have

m_ℓ	m_s	ΔE
2	$\frac{1}{2}$	$3\hbar\omega_0$
2	$-\frac{1}{2}$	$\hbar\omega_0$
1	$\frac{1}{2}$	$2\hbar\omega_0$
1	$-\frac{1}{2}$	0
0	$\frac{1}{2}$	$\hbar\omega_0$
0	$-\frac{1}{2}$	$-\hbar\omega_0$
-1	$\frac{1}{2}$	0
-1	$-\frac{1}{2}$	$-2\hbar\omega_0$
-2	$\frac{1}{2}$	$-\hbar\omega_0$
-2	$-\frac{1}{2}$	$-3\hbar\omega_0$

The number of sub-levels is 7, which agrees with $d_3 = 2 \cdot 3 + 1 - \delta_{3,1} = 7$, with degeneracies:

ΔE	$\lvert m\rvert$	$d_{counted}$	$d_{formula}$
$3\hbar\omega_0$	3	1	1
$2\hbar\omega_0$	2	2	2
$\hbar\omega_0$	1	4	4
0	0	4	4
$-\hbar\omega_0$	1	4	4
$-2\hbar\omega_0$	2	2	2
$-3\hbar\omega_0$	3	1	1

5. **Weak-field Zeeman Effect**: for the case $\hbar\omega_0 \ll |E_1^{(0)}|\alpha^2 \frac{M_e}{M_p}$, compute the energies and degeneracies of the Zeeman sub-levels for both the $n = 3$, $j = 3/2$ and $n = 3$, $j = 5/2$ levels.

In this regime, we consider Zeeman as a perturbation on the hyperfine structure. The the good quantum numbers are j, f, and m_f. We can therefore use the Wigner-Ekert theorem to convert $\vec{L} + 2\vec{S}$ into $\vec{F}$ in a two stage process, first we have $\vec{L} + 2\vec{S} = g_J \vec{J}$, where

$$\begin{aligned} g_J(\ell, s) &= \frac{\langle(\vec{L} + 2\vec{S}) \cdot \vec{J}\rangle}{\hbar^2 j(j+1)} \\ &= \frac{\langle J^2 + \vec{S} \cdot \vec{J}\rangle}{\hbar^2 j(j+1)} \\ &= \frac{\langle J^2 + S^2 + \vec{S} \cdot \vec{L}\rangle}{\hbar^2 j(j+1)} \\ &= \frac{\langle 2J^2 + 2S^2 + J^2 - L^2 - S^2\rangle}{2\hbar^2 j(j+1)} \\ &= \frac{\langle 3J^2 + S^2 - L^2\rangle}{2\hbar^2 j(j+1)} \\ &= \frac{12j(j+1) + 3 - 4\ell(\ell+1)}{8j(j+1)} \end{aligned} \tag{38}$$

We can then use $\vec{J} = g_F \vec{F}$, where

$$\begin{aligned} g_F &= \frac{\langle \vec{J} \cdot \vec{F}\rangle}{\hbar^2 f(f+1)} \\ &= \frac{\langle J^2 + \vec{J} \cdot \vec{I}\rangle}{\hbar^2 f(f+1)} \\ &= \frac{\langle F^2 + J^2 - I^2\rangle}{\hbar^2 f(f+1)} \\ &= \frac{4f(f+1) + 4j(j+1) - 3}{8f(f+1)} \end{aligned} \tag{39}$$

so that

$$\langle \ell s i j f m_f | V_Z | \ell s i j f m_f \rangle = g_J g_F \hbar\omega_0 \, m_f \tag{40}$$

The g-factors are given by

ℓ	j	g_J
1	3/2	4/3
2	3/2	4/5
2	5/2	6/5
3	5/2	6/7

j	f	g_F
3/2	1	5/4
3/2	2	3/4
5/2	2	7/6
5/2	3	5/6

so that the net g-factors are

ℓ	j	f	$g_J g_F$
1	3/2	1	5/3
1	3/2	2	1
2	3/2	1	1
2	3/2	2	3/5
2	5/2	2	7/5
2	5/2	3	1
3	5/2	2	1
3	5/2	3	5/7

The sublevels and degeneracies for the $j = 5/2$, $f = 3$ level are

ΔE_Z $[\hbar\omega_0]$	d	m$_f$	ℓ
21/7	1	3	2
15/7	1	3	3
4/7	1	2	2
10/7	1	2	3
7/7	1	1	2
5/7	1	1	3
0	2	0	2,3
-5/7	1	1	3
-7/7	1	1	2
-10/7	1	2	3
-14/7	1	2	2
-15/7	1	3	3
-21/7	1	3	2

For the $j = 5/2$ and $f = 2$ level, they are

ΔE_Z [$\hbar\omega_0$]	d	m$_f$	ℓ
14/5	1	2	2
10/5	1	2	3
7/5	1	1	2
5/5	1	1	3
0	2	0	2,3
-5/5	1	1	3
-7/5	1	1	2
-10/5	1	2	3
-14/5	1	2	2

For the $j = 3/2$, $j = 2$ level, the sublevels and degeneracies are

ΔE_Z [$\hbar\omega_0$]	d	m$_f$	ℓ
10/5	1	2	1
6/5	1	2	2
5/5	1	1	1
3/5	1	1	2
0	2	0	1,2
-3/5	1	1	2
-5/5	1	1	1
-6/5	1	2	2
-10/5	1	2	1

Lastly, for the $j = 3/2$, $f = 1$ level, we have

ΔE_Z [$\hbar\omega_0$]	d	m$_f$	ℓ
5/3	1	1	1
3/3	1	1	2
0	2	0	1,2
-3/3	1	1	2
-5/3	1	1	1