Lecture 1: Demystifying ‘\( \hbar \)’ and ‘\( i \)’

• We are often told that the presence of \( \hbar \) distinguishes quantum from classical theories.

• One of the striking features of Schrödinger’s equation is the fact that the variable, \( \Psi \), is complex, whereas classical theories deal with real variables

\[
\frac{i \hbar}{\hbar} \frac{\partial}{\partial t} \psi(x,t) = \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right] \psi(x,t)
\]

QM:

\[
\frac{d}{dt} x(t) = \frac{\partial}{\partial p} H(x,p)
\]
\[
\frac{d}{dt} p(t) = -\frac{\partial}{\partial x} H(x,p)
\]

CM:

\[
\frac{d}{dt} \vec{E}(\vec{r},t) = \frac{1}{c^2} \vec{\nabla} \times \vec{B}(\vec{r},t)
\]
\[
\frac{d}{dt} \vec{B}(\vec{r},t) = -\vec{\nabla} \times \vec{E}(\vec{r},t)
\]

CE&M:

\[
\frac{d}{dt} \frac{\psi(x,t)}{\psi(x,t)} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \psi(x,t)
\]

• By changing units we can of course make \( \hbar \) disappear from QM

• But if it is truly fundamental, shouldn’t this same choice of units make \( \hbar \) appear then in CM?

• Q: Is \( \hbar \) necessary at all?

• If system has natural length scale and energy scale, then \( \hbar \) is needed to relate then to the natural mass scale.
•What happens to CM in these units?

\[ \frac{\partial}{\partial t} \mathbf{x} = \frac{\mathbf{p}}{m} \]
\[ \frac{\partial}{\partial t} \mathbf{p} = -\frac{\partial}{\partial \mathbf{x}} \mathbf{V} \]
\[ \mathbf{v} = V_0 u \quad m \to m \mu \]
\[ x \to x_0 \mu \quad p = p_0 \mu \quad p_0 = \frac{m_0 \mathbf{x}_0 \mathbf{V}_0}{\mu} \]
\[ t \to \frac{k}{V_0} t \]

\[ \frac{\partial}{\partial t} \mathbf{p} = \frac{\mathbf{\pi}}{\mu} \]
\[ \frac{\partial}{\partial \mathbf{r}} \mathbf{\pi} = -\frac{\partial \mathbf{u}(r)}{\partial \mathbf{r}} \]

•Same mass scale makes CM dimensionless as well!

•Q: Are Maxwell’s Eq’s ‘classical’ or ‘quantum’?

\[ \nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{E} = 0 \]

•Apply De Broglie hypothesis to Einstein’s equation:

\[ E^2 - c^2 p^2 = m^2 c^4 \]
\[ E \to i \hbar \frac{\partial}{\partial t} \quad p \to -i \hbar \frac{\partial}{\partial x} \]
\[ \left[ -\frac{\hbar^2}{c^2} \frac{\partial^2}{\partial t^2} + \kappa^2 \nabla^2 \right] \psi = -\frac{m^2 c^2}{\hbar^2} \psi \]

•The wavefunction of a massless particle obeys the Maxwell wave equation!

•So is E just the photon wavefunction?

•‘Classical’ E&M would be ‘quantum’ if the photon had mass

My opinion:

•Maxwell’s equation is just as ‘quantum’ as Schrödinger's equation

•‘Classical’ EM is ray-optics
• Now, let's look at the ‘i’ issue:

\[ \frac{d}{dt} \psi = -\frac{1}{2\mu} p \psi + U(p) \psi \]

• Separate \( \psi \) into real and imaginary parts:

\[ \psi_1(p) = u(p) + i \psi_2(p) \]

\[ \frac{1}{2\mu} \frac{\partial}{\partial p} u - \frac{1}{2\mu} \frac{\partial}{\partial v} v = \frac{1}{2\mu} \frac{\partial}{\partial u} u'' - \frac{i}{2\mu} \frac{\partial}{\partial v} \psi^* \psi + \text{H} \]

\[ \frac{d}{dt} u(r, t) = H(u, v) v(r, t) \]

\[ \frac{d}{dt} v(r, t) = -H(u, v) u(r, t) \]

• No more ‘i’

• Structure looks familiar:
  
  • Two conjugate variables
  • Symmetric equations

CM:

\[ \frac{d}{dt} x(t) = \frac{\partial}{\partial p} H(x, p) \]

\[ \frac{d}{dt} p(t) = -\frac{\partial}{\partial x} H(x, p) \]

CE&M:

\[ \frac{d}{dt} E(r, t) = \frac{1}{c^2} \vec{V} \times \vec{B}(r, t) \]

\[ \frac{d}{dt} \vec{B}(r, t) = -\vec{V} \times \vec{E}(r, t) \]

• Can we put an ‘i’ in CM and make it look more like QM?

\[ \text{let } \zeta = \frac{1}{\sqrt{2}} (x + ip) \quad \Rightarrow \quad x = \zeta + \zeta^* \]

\[ \zeta^* = \frac{1}{\sqrt{2}} (x - ip) \quad p = \zeta - \zeta^* \]

\[ \frac{1}{\sqrt{2}} = \frac{\partial}{\partial x} \frac{1}{\sqrt{2}} + \frac{\partial}{\partial x} \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) \]

\[ \frac{1}{\sqrt{2}} = \frac{\partial}{\partial x} \frac{1}{\sqrt{2}} + \frac{\partial}{\partial p} \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) \]

\[ \frac{1}{\sqrt{2}} = \frac{\partial}{\partial x} H(x, p) \]

\[ i \frac{d}{dt} z(t) = \frac{\partial}{\partial z} H(z, z^*) \]

\[ \frac{d}{dt} \zeta = -\frac{\partial}{\partial \zeta} H(x, p) \quad \text{Newton's Second Law} \]

\[ i \psi^*(x) \left[ -\frac{\hbar^2}{2\mu} \frac{\partial^2}{\partial x^2} + U(x) \right] \psi(x) \quad \text{energy} \]

\[ i \frac{d}{dt} \psi(r, t) = \frac{\partial}{\partial \psi^*(r, t)} H(\psi, \psi^*) \]
• So what is going on?
  • The point is that QM is the correct theory
  • CM and CE&M are just approximations derived from QM
  • Thus they get their structures from QM