Lecture 23:
Heisenberg Uncertainty Principle

Phy851 Fall 2009

Heisenberg Uncertainty Relation

- Most of us are familiar with the Heisenberg Uncertainty relation between position and momentum:

$$\Delta x \, \Delta p \geq \frac{\hbar}{2}$$

- How do we know this is true?

- Are the similar relations between other operators?

Variance

- The uncertainties are also called 'variances' defined as

$$\Delta a = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$$

$\langle A \rangle \rightarrow \langle \psi | A | \psi \rangle$

$|\psi\rangle$ is implied

- Note that the variance is state-dependent

- What does it tell us about our state?
  - Consider a distribution $P(\mathrm{a})$,
  - The average of the distribution is:

$$\bar{a} = \sum_a P(a)\, a$$

  - To estimate the ***width*** of the distribution we might consider the square of the distance from the mean: $d^2(a) = (a - \bar{a})^2$
  - The average of this quantity is

$$\overline{d^2}(a) = \sum_a P(a)\left(a^2 - 2a\bar{a} + \bar{a}^2\right)$$

$$= \overline{a^2} - 2\bar{a}^2 + \bar{a}^2$$

$$= \overline{a^2} - \bar{a}^2$$

$$d_{rms} := \sqrt{\overline{d^2}} = \sqrt{\overline{a^2} - \bar{a}^2}$$

$$\Delta a = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$$

Incompatible Observables

- For an observable *A*, the only way you can have $\Delta a=0$ is if you are in an eigenstate of *A*

- Consider two incompatible observables, A and B:

$$[A, B] = M \neq 0$$

- We cannot have $\Delta A=0$ and $\Delta B=0$ at the same time
  - Then we would have a simultaneous eigenstate of A and B
- So what is the best we can do?

- To derive the Heisenberg Uncertainty for X and P relation, let us first introduce

$$X' = X - \langle X \rangle I$$

$$P' = P - \langle P \rangle I$$

$$[X', P'] = [X, P]$$

$$\langle X'^2 \rangle = \ddot{A}x^2$$

$$\langle P'^2 \rangle = \ddot{A}p^2$$

Geometric Proof of Uncertainty Relation

- Let: $|\phi\rangle := (X' + i\lambda P')|\psi\rangle$

λ is an arbitrary real number

$|\psi\rangle$ is an arbitrary state

- For any λ and $|\psi\rangle$ we must have: $\langle\phi|\phi\rangle \geq 0$

$$\langle\psi|X'^2|\psi\rangle - i\lambda\langle\psi|P'X'|\psi\rangle + i\lambda\langle\psi|X'P'|\psi\rangle + \lambda^2\langle\psi|P'^2|\psi\rangle \geq 0$$

$$\langle P'^2\rangle\lambda^2 + \langle i[X',P']\rangle\lambda + \langle X'^2\rangle \geq 0$$

$$[X',P'] = [X,P]$$

$$\langle X'^2\rangle = \Delta x^2$$

$$\langle P'^2\rangle = \Delta p^2$$

$$\Delta p^2\lambda^2 + \langle i[X,P]\rangle\lambda + \Delta x^2 \geq 0$$

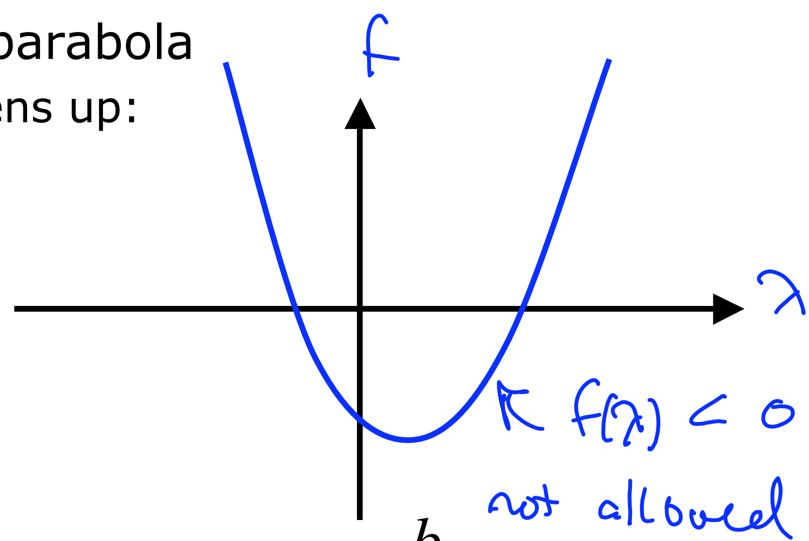
Parabolas

$$\Delta p^2 \lambda^2 + \langle i[X,P] \rangle \lambda + \Delta x^2 \geq 0$$

- Consider a general quadratic polynomial with real coefficients:

$$f(\lambda) = a\lambda^2 + b\lambda + c$$

- Its graph is a parabola
  - If a>0 it opens up:



- The minimum is at: $\lambda_{\min} = -\dfrac{b}{2a}$

- The minimum value is: $f(\lambda_{\min}) = c - \dfrac{b^2}{4a}$

- So f(λ) ≥0 requires: $f(\lambda_{\min}) \geq 0 \quad \Rightarrow ac \geq \dfrac{b^2}{4}$

$$\Delta x^2 \Delta p^2 \geq \frac{\langle i[X,P] \rangle^2}{4} \quad \Rightarrow \quad \Delta x \Delta p \geq \frac{\hbar}{2}$$

Generalized Uncertainty Relations

- Note that only at the very end did we make use of the specific form of the commutator:

$$\left[X, P\right] = i\hbar$$

- This means that our result is valid in general for any two observables:

$$\Delta a^2 \Delta b^2 \geq \frac{\left\langle i\left[A,B\right]\right\rangle^2}{4} \quad \Rightarrow \Delta a \Delta b \geq \frac{\left|\left\langle\left[A,B\right]\right\rangle\right|}{4}$$

- Consider angular momentum operators:

$$[L_x, L_y] = i\hbar L_z$$

$$\Delta l_x \Delta l_y \geq \frac{\hbar}{2}\left|\left\langle L_z\right\rangle\right|$$

- In General, the Heisenberg Lower limit depends on the state.
- *X* and *P* are special in that all states have the same limit.
- The Uncertainty relation is not as useful in the more general cases